Fast narrow bounds on the value of Asian options

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Abstract

We consider the problem of finding bounds on the value of fixed-strike and floating-strike Asian options. A good lower bound for both types was derived in Rogers & Shi (1995). We provide an alternative derivation, which leads to a simpler expression for the bound, and also to the bound given by Curran (1992) for fixed-strike options; we derive an analogous bound for floating-strike options. Combining these results with a new upper bound allows the accurate valuation of fixed-strike and floating-strike Asian options for typical parameter values.

Keywords: Brownian motion, Asian option, fixed-strike, floating-strike.

1 Introduction

A fixed-strike Asian (call) option with strike $K > 0$ and maturity T on an asset with price process $\{S_t\}$ is a contract with value $(\frac{1}{T}\int_0^T S_t\,dt - K)^+$ at time T. We assume that $T = 1$ year and that the asset price follows a geometric Brownian motion $S_t = S\exp(\sigma B_t + ct)$ where c is a constant, S and σ are positive constants and B_t is a Brownian motion. The arbitrage-free time-0 value of this option is then equal to $\exp(-\rho)\mathbb{E}[(\int_0^1 S\exp(\sigma B_t + \alpha t)\,dt - K)^+]$, where ρ is the risk-free interest rate (assumed constant) and $\alpha = \rho - \frac{1}{2}\sigma^2$ (see Baxter & Rennie (1996) for an introduction to arbitrage-free valuation). Thus the problem of valuing the option boils down to calculating $\mathbb{E}[(S\int_0^1 \exp(\sigma B_t + \alpha t)\,dt - K)^+]$. Another type of Asian option, the floating-strike option, pays out $(\int_0^1 S_t\,dt - S_1)^+$, and leads to the consideration of $\mathbb{E}[(\int_0^1 \exp(\sigma B_t + \alpha t)\,dt - \exp(\sigma B_1 + \alpha))^+]$.

Several approaches to the problem of valuing Asian options have been tried. Carverhill & Clewlow (1990) use a convolution method to compute the distribution of $\int_0^1 S_t\,dt$, and work by Yor (1992) and Geman & Yor (1993) has lead to a formula for the price as a triple integral. Approximate formulae and bounds in the form of single and double integrals by Levy (1992), Levy & Turnbull (1992), Curran (1992) and Rogers & Shi (1995) seem to be faster to evaluate however, and the methods of this paper fall into this category. Using intuition and simple optimization we derive bounds on the value of fixed-strike and floating-strike Asian options which can be computed quickly, and which are accurate for typical parameter values. The method generalizes to other options on sums of lognormal assets: discretely monitored Asians, currency basket options, and swaptions in Gaussian HJM models, for example.

The plan of this paper is as follows: in Section 2 we state some useful facts about covariances related to Brownian motion, and in Section 3 present the impressive lower bound of Rogers & Shi (1995) for fixed-strike and floating-strike options, and the approximation to the fixed-strike bound given by Curran (1992). This approximation is notable since it is very close to the bound of Rogers & Shi (1995) and is much easier to compute, involving only one-dimensional integrals rather than a troublesome two-dimensional integral. We give an alternative derivation of the bounds of Rogers & Shi (1995), leading to an expression involving only one-dimensional integrals; we also provide a generalization Curran's approximation to floating-strike options.

In Section 4 we derive upper bounds to complement the lower bounds, and in Section 5 we present a numerical comparison of the various bounds discussed here, using the parameter values from Curran (1992).

2 Useful covariances

Like most of the approximation formulae in the literature, we will exploit the high correlation between $\int_0^1 \exp(\sigma B_t + \alpha t)\, dt$ and $\int_0^1 B_t\, dt$ for the values of σ and α met in practice, and the fact that the second integral is Gaussian. We will need the covariance matrix of the bivariate Gaussian random variable $(B_t, \int_0^1 B_s\, ds)$:

$$\mathbb{E}\left(\left(B_t, \int_0^1 B_s\, ds\right)^T \left(B_t, \int_0^1 B_s\, ds\right)\right) = \begin{pmatrix} t & t(1-t/2) \\ t(1-t/2) & \frac{1}{3} \end{pmatrix}.$$

Thus the conditional distribution of B_t given $\int_0^1 B_s\, ds = z$ is normal with mean $3t(1-t/2)z$ and variance $t - 3t^2(1-t/2)^2$, and the conditional distribution of $\int_0^1 B_s\, ds$ given $B_t = x$ is normal with mean $(1-t/2)x$ and variance $\frac{1}{3} - t(1-t/2)^2$. We will also need the covariance matrix of $(B_t, \int_0^1 B_s\, ds - B_1)$:

$$\mathbb{E}\left(\left(B_t, \int_0^1 B_s\, ds - B_1\right)^T \left(B_t, \int_0^1 B_s\, ds - B_1\right)\right) = \begin{pmatrix} t & -t^2/2 \\ -t^2/2 & \frac{1}{3} \end{pmatrix},$$

so, given $\int_0^1 B_s\,ds - B_1 = x$, the conditional distribution of B_t is normal with mean $-3t^2x/2$ and variance $t - 3t^4/4$.

3 Lower bounds

In this section we present two derivations of the bound of Rogers & Shi (1995): their own, and an alternative, which yields a simpler expression for the bound. It also leads to the bound of Curran (1992) for fixed-strike options. We show how a similar bound for floating-strike options may be derived.

The derivation of Rogers & Shi (1995) exploits the inequality:

$$\mathbb{E}\left[A^+\right] = \mathbb{E}\left[\mathbb{E}\left(A^+ \,\middle|\, C\right)\right] \geq \mathbb{E}\left[\left(\mathbb{E}\left(A \,|\, C\right)\right)^+\right]$$

which holds for any random variables A and C.

For fixed-strike Asian option they choose $A = \int_0^1 S_t\,dt - K$ and $C = \int_0^1 B_t\,dt$. Since the inner expectation is $\int_0^1 \mathbb{E}\left(S\exp\left(\sigma B_t + \alpha t\right) \,\middle|\, \int_0^1 B_s\,ds\right)\,dt - K$ and, conditional $\int_0^1 B_t\,dt = z$, B_t is normal with mean $3t(1-t/2)z$ and variance $t - 3t^2(1-t/2)^2$, we have the bound

$$V_{\text{fixed}} \geq e^{-\rho}\int_{-\infty}^{\infty}\sqrt{3}\phi\left(\sqrt{3}z\right)\left[\int_0^1 Se^{3\sigma t(1-t/2)z+\alpha t+\frac{1}{2}\sigma^2\left(t-3t^2(1-t/2)^2\right)}\,dt - K\right]^+ dz. \quad (3.1)$$

To bound a floating-strike option they use $A = \int_0^1 S_t\,dt - S_1$, $C = \int_0^1 B_t\,dt - B_1$, and get

$$V_{\text{floating}} \geq e^{-\rho}\int_{-\infty}^{\infty}\sqrt{3}\phi\left(\sqrt{3}z\right)\left[\int_0^1 S\exp\left(-3\sigma t^2 z/2 + \alpha t + \tfrac{1}{2}\sigma^2\left(t - 3t^4/4\right)\right)dt - S\exp\left(-3\sigma z/2 + \alpha + \sigma^2/8\right)\right]^+ dz.$$

Both of these formulae are slightly tricky to evaluate since the outer integration has a non-smooth integrand.

An alternative approach is to approximate the event that the option eventually ends in-the-money with something more tractable. Let $\mathcal{A} = \{\omega : \int_0^1 S_t\,dt > K\}$, and note that

$$\mathbb{E}\left[\left(\int_0^1 S\exp\left(\sigma B_t + \alpha t\right)\,dt - K\right)^+\right] = \int_0^1 \mathbb{E}\left[\left(S\exp\left(\sigma B_t + \alpha t\right) - K\right)I(\mathcal{A})\right]\,dt. \quad (3.2)$$

If we replace $\mathcal{A}$ by some other event $\mathcal{A}'$, we no longer have equality in (3.2); the right hand side is now a lower bound. We will use $\mathcal{A}' = \{\int_0^1 B_t\,dt > \gamma\}$. This Gaussian form allows the expectation to be written as a Black-Scholes type formula once γ has been determined, and just leaves us with a one-dimensional integral of a smooth integrand, which should be very fast.

To determine the optimal value of γ, let $N_t = \sigma B_t + \alpha t + \log S$ and note that for any random variable X with density $f_X(x)$

$$\frac{\partial}{\partial \gamma} \int_0^1 \mathbb{E}\left(\exp(N_t) - K; X > \gamma\right) dt = \int_0^1 \mathbb{E}\left(\exp\left(N_t\right) - K \mid X = \gamma\right)\left(-f_X\left(\gamma\right)\right) dt.$$

Thus the optimal value of γ, γ^* satisfies

$$\int_0^1 \mathbb{E}(\exp(N_t) \mid X = \gamma^*)\, dt = K. \tag{3.3}$$

With our choice of $X = \int_0^1 B_t\, dt$, we conclude that

$$\int_0^1 S \exp\left(3\gamma^* \sigma t\left(1 - t/2\right) + \alpha t + \tfrac{1}{2}\sigma^2\left(t - 3t^2\left(1 - t/2\right)^2\right)\right) dt = K, \tag{3.4}$$

which determines γ^* uniquely. We now have the bound

$$V_{\text{fixed}} \geq e^{-\rho} \int_0^1 \mathbb{E}\left[\left(Se^{\sigma B_t + \alpha t} - K\right) I\left(\int_0^1 B_s\, ds > \gamma^*\right)\right] dt,$$

and it remains to calculate the expectation. Fix $t \in (0, 1)$ and let $N_1 = \sigma B_t + \alpha t + \log S$ and $N_2 = \int_0^1 B_s\, ds - \gamma^*$. Write $\mu_i = \mathbb{E}(N_i)$, $\sigma_i^2 = \mathbb{V}\mathrm{ar}(N_i)$ and $c = \mathbb{C}\mathrm{ov}(N_1, N_2)$, then using

$$\mathbb{E}\left[\left(e^{N_1} - K\right) I\left(N_2 > 0\right)\right] = e^{\mu_1 + \frac{1}{2}\sigma_1^2} \Phi\left(\frac{\mu_2 + c}{\sigma_2}\right) - K\Phi\left(\frac{\mu_2}{\sigma_2}\right),$$

where Φ is the normal distribution function, and substituting $\mu_1 = \alpha t + \log S$, $\mu_2 = -\gamma^*$, $\sigma_1^2 = \sigma^2 t$, $\sigma_2^2 = \frac{1}{3}$, $c = \sigma t(1 - t/2)$, we have

$$V_{\text{fixed}} \geq e^{-\rho}\left[\int_0^1 Se^{\alpha t + \frac{1}{2}\sigma^2 t} \Phi\left(\frac{-\gamma^* + \sigma t\left(1 - t/2\right)}{1/\sqrt{3}}\right) dt - K\Phi\left(\frac{-\gamma^*}{1/\sqrt{3}}\right)\right].$$

Integrating this numerically is significantly easier than integrating (3.1).

To see that this bound gives the same answers as that of Rogers & Shi (1995), let $Y = \int_0^1 S_t\, dt$, $Z = \int_0^1 B_t\, dt$ and note that $\mathbb{E}[(\mathbb{E}(Y - K|Z))^+] = \mathbb{E}[(\mathbb{E}(Y - K|Z))I(\mathbb{E}(Y - K|Z) > 0)]$, and since from (3.3) and (3.4), $\mathbb{E}(Y - K|Z)$ is strictly increasing in Z, we have: $\mathbb{E}(Y - K|Z) > 0$ if and only if $Z > \gamma^*$. Thus γ^* satisfies $\mathbb{E}[(\mathbb{E}(Y - K|Z))^+] = \mathbb{E}[(\mathbb{E}(Y - K|Z))I(Z > \gamma^*)]$ which is just $\mathbb{E}[(Y - K)I(Z > \gamma^*)]$.

The bound of Curran (1992) arises from solving (3.4) approximately, using the following method: let $f(\gamma) = \mathbb{E}(\int_0^1 S \exp(\sigma B_t + \alpha t)\, dt | \int_0^1 B_s\, ds = \gamma)$, and note that a reasonable approximation to f is $\tilde{f}(\gamma) := S \exp(\gamma\sigma + \alpha/2)$, obtained by interchanging the orders of integration and exponentiation. Recall that we seek $\gamma^* = f^{-1}(K)$ and observe that if $f \approx \tilde{f}$

then $f^{-1}(x) \approx \tilde{f}^{-1}(2x - f \circ \tilde{f}^{-1}(x))$, the approximation being exact if $f - \tilde{f}$ is constant. Thus $\gamma^* \approx \tilde{f}^{-1}(2K - f \circ \tilde{f}^{-1}(K))$, giving

$$\gamma^* \approx \sigma^{-1} \left[\log \left(\frac{2K}{S} - \int_0^1 e^{3(\log(K/S)-\alpha/2)t(1-t/2)+\alpha t+\frac{1}{2}\sigma^2(t-3t^2(1-t/2)^2)}\, dt \right) - \alpha/2 \right],$$

which is the continuous limit of the bound given by Curran (1992).

For the floating-strike option, let $\mathcal{A} = \{\omega : \int_0^1 S_t\, dt > S_1\}$ and use an approximation to $\mathcal{A}$ of the form $\mathcal{A}' = \{\int_0^1 B_t\, dt - B_1 > \gamma\}$. With this choice, γ^*, the optimal value of γ, satisfies

$$\mathbb{E}\left[\int_0^1 \left(S \exp\left(\sigma B_t + \alpha t\right) - S \exp\left(\sigma B_1 + \alpha\right)\right) dt \,\middle|\, \int_0^1 B_s\, ds - B_1 = \gamma^* \right] = 0$$

giving

$$\exp\left(3\gamma^*\sigma/2 - \alpha - \sigma^2/8\right) \int_0^1 \exp\left(-3\gamma^*\sigma t^2/2 + \alpha t + \tfrac{1}{2}\sigma^2\left(t - 3t^4/4\right)\right) dt = 1, \qquad (3.5)$$

which has a unique solution.

Our lower bound for the floating-strike case is thus

$$V_{\text{floating}} \geq e^{-\rho} \int_0^1 \mathbb{E}\left[\left(S \exp\left(\sigma B_t + \alpha t\right) - S \exp\left(\sigma B_1 + \alpha\right)\right) I\left(\int_0^1 B_s\, ds - B_1 > \gamma^* \right) \right] dt,$$

which reduces to

$$V_{\text{floating}} \geq e^{-\rho} \left[\int_0^1 S e^{\alpha t + \frac{1}{2}\sigma^2 t} \Phi\left(\frac{-\gamma^* - \sigma t^2/2}{1/\sqrt{3}} \right) dt - S e^{\alpha + \frac{1}{2}\sigma^2} \Phi\left(\frac{-\gamma^* - 1/2}{1/\sqrt{3}} \right) \right].$$

Again this bound gives the same answers as that of Rogers & Shi (1995). To see this, let $Y = \int_0^1 S_t\, dt$, $Z = \int_0^1 B_t\, dt$, and note first that

$$\frac{\partial}{\partial \gamma} \mathbb{E}\left(Y - S_1 \,|\, Z - B_1 = \gamma\right) = \int_0^1 (-3\sigma t^2/2) \exp(A(t,\gamma))\, dt + (3\sigma/2)\exp(A(1,\gamma)),$$

where $A(t,\gamma) = \log S - 3\gamma\sigma t^2/2 + \alpha t + \frac{1}{2}\sigma^2\left(t - 3t^4/4\right)$. Since γ^* solves $\mathbb{E}(Y - S_1 | Z - B_1 = \gamma^*) = 0$, we have $\int_0^1 \exp(A(t,\gamma^*))\, dt = \exp(A(1,\gamma^*))$, so

$$\left.\frac{\partial}{\partial \gamma}\right|_{\gamma=\gamma^*} \mathbb{E}\left(Y - S_1 \,|\, Z - B_1 = \gamma\right) = \int_0^1 (3\sigma(1-t^2)/2) \exp(A(t,\gamma^*))\, dt > 0.$$

Thus $\mathbb{E}\left(Y - S_1 \,|\, Z - B_1 = \gamma\right) > 0$ if and only if $\gamma > \gamma^*$. A similar argument to the fixed-strike case completes the proof.

We can now generalize Curran's formula to the case of floating-strike options by solving (3.5) approximately. Let $f(\gamma) = e^{3\gamma\sigma/2-\alpha-\sigma^2/8}\mathbb{E}(\int_0^1 e^{\sigma B_t + \alpha t} | \int_0^1 B_t\, dt - B_1 = \gamma)$ and let

$\tilde{f}(\gamma) = \exp(\sigma\gamma - \alpha/2)$ be the approximation to f obtained by interchanging the orders of integration and exponentiation. Then the solution to (3.5) is given approximately by $\gamma^* \approx \tilde{f}^{-1}(2 - f \circ \tilde{f}^{-1}(1))$, giving

$$\gamma^* \approx \sigma^{-1}\left[\alpha/2 + \log\left(2 - e^{-\alpha/4-\sigma^2/8}\int_0^1 \exp\left(\alpha t - 3\alpha t^2/4 + \tfrac{1}{2}\sigma^2\left(t - 3t^4/4\right)\right) dt\right)\right].$$

4 Upper bounds

In this section we derive a new upper bound on the value of fixed-strike and floating-strike Asian options in the form of a double integral. Rogers & Shi (1995) obtained an upper bound by considering the error made by their lower bound (see Section 3). As an indication of their relative accuracy, if $\sigma = 0.3$, $\rho = 0.09$, $S = 100$, and $T = 1\,\text{year}$ and $k = 100$, the lower bounds of Section 3 are both 8.8275 and the upper bound of this section is 8.8333. By comparison, the upper bound given by Rogers & Shi (1995) is 9.039.

The inequality underlying the new bound is the following: let X be a random variable, and let $f_t(\omega)$ be a random function with $\int_0^1 f_t(\omega)\,dt = 1$ for all ω. Then

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1 S\exp(\sigma B_t + \alpha t)\,dt - X\right)^+\right] &= \mathbb{E}\left[\left(\int_0^1 (S\exp(\sigma B_t + \alpha t) - X f_t)\,dt\right)^+\right] \\ &\le \mathbb{E}\left[\int_0^1 (S\exp(\sigma B_t + \alpha t) - X f_t)^+\,dt\right] \qquad (4.1) \\ &= \int_0^1 \mathbb{E}\left[(S\exp(\sigma B_t + \alpha t) - X f_t)^+\right]\,dt.\end{aligned}$$

For both the fixed-strike and floating-strike cases we will use $f_t = \mu_t + \sigma(B_t - \int_0^1 B_s\,ds)$, where μ_t is a deterministic function satisfying $\int_0^1 \mu_t\,dt = 1$; and derive an expression for μ_t which is approximately optimal in each case. As the bounds have a very similar derivation we will concentrate on the fixed-strike option and give the appropriate modifications for the floating-strike case at the end of the section.

Take $X = K$ in (4.1) and first consider the choice $f_t = \mu_t$. To choose μ_t we will minimize the right hand side of (4.1) over the set of deterministic functions f_t such that $\int_0^1 f_t\,dt = 1$. Let $L(\lambda, \{f_t\}) = \mathbb{E}[\int_0^1 (S\exp(\sigma B_t + \alpha t) - K f_t)^+\,dt - \lambda(\int_0^1 f_t\,dt - 1)]$ be the Lagrangian, and consider stationarity with respect to $\{f_t\}$ for the unconstrained problem. This gives the condition

$$\int_0^1 \left(-K\mathbb{P}\left(S\exp\left(\sigma B_t + \alpha t\right) \ge K f_t\right) - \lambda\right)\epsilon_t\,dt = 0,$$

where $\{\epsilon_t\}$ is some small deterministic perturbation. Thus we see that $\mathbb{P}(S\exp(\sigma B_t + \alpha t) \ge K f_t)$ must be independent of t. Equivalently we have $\log(K f_t/S) - \alpha t = \gamma\sigma\sqrt{t}$ for some constant γ. Thus the optimal choice for f_t is

$$f_t = (S/K)\exp\left(\sigma\gamma\sqrt{t} + \alpha t\right),$$

where the constant γ is chosen so that $\int_0^1 f_t\,dt = 1$. Since $\int_0^1 f_t\,dt$ is monotone increasing in γ, the correct value for γ is easy to estimate numerically.

If instead $f_t = \mu_t + \sigma(B_t - \int_0^1 B_s\,ds)$, the condition for stationarity with respect to small deterministic perturbations is

$$\mathbb{P}\left[S\exp(\sigma B_t + \alpha t) \geq K\left(\mu_t + \sigma\left(B_t - \int_0^1 B_s\,ds\right)\right)\right] = \lambda, \quad \text{for all } t, \tag{4.2}$$

but this cannot easily be re-arranged to give the dependence of μ_t on λ. Instead we will use the approximation $\exp(\sigma B_t) \approx 1 + \sigma B_t$ which should be reasonable for small σ. This leads to the condition $\mathbb{P}(S\exp(\alpha t)(1+\sigma B_t) \geq K f_t) = \lambda$ for all t. Letting $N_t = S\exp(\alpha t) + (S\exp(\alpha t)\sigma - K\sigma)B_t + K\sigma\int_0^1 B_s\,ds$, we conclude that $\mathbb{P}(N_t \geq K\mu_t)$ must be independent of t. Using the facts about the joint distribution of $(B_t, \int_0^1 B_s\,ds)$ given in Section 2, we deduce that

$$\mu_t = \frac{1}{K}\left(S\exp\left(\alpha t\right) + \gamma\sqrt{v_t}\right), \tag{4.3}$$

where

$$v_t = \mathbb{V}\mathrm{ar}(N_t) = c_t^2 t + 2(K\sigma)c_t t(1 - t/2) + (K\sigma)^2/3, \tag{4.4}$$

$$c_t = S\exp(\alpha t)\sigma - K\sigma. \tag{4.5}$$

Imposing $\int_0^1 \mu_t\,dt = 1$ gives

$$\gamma = \left(K - S(e^\alpha - 1)/\alpha\right) \Big/ \int_0^1 \sqrt{v_t}\,dt. \tag{4.6}$$

We estimate the integral $\int_0^1 \sqrt{v_t}\,dt$ numerically. We now know the constant γ and hence the function μ_t for our upper bound:

$$V_{\text{fixed}} \leq e^{-\rho}\int_0^1 \mathbb{E}\left[\left(S\exp\left(\sigma B_t + \alpha t\right) - K\left(\mu_t + \sigma\left(B_t - \int_0^1 B_s\,ds\right)\right)\right)^+\right] dt.$$

Conditioning on $B_t = x$, this becomes

$$V_{\text{fixed}} \leq e^{-\rho}\int_0^1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)\mathbb{E}\left[(a(t,x) + b(t,x)N)^+\right]\,dx\,dt, \tag{4.7}$$

where N has a $N(0,1)$ distribution, and the functions a and b are given by

$$a(t,x) = S\exp(\sigma x + \alpha t) - K(\mu_t + \sigma x) + K\sigma(1-t/2)x, \tag{4.8}$$

$$b(t,x) = K\sigma\sqrt{\tfrac{1}{3} - t\,(1-t/2)^2}. \tag{4.9}$$

The calculation of $\mathbb{E}[(a+bN)^+]$ is straightforward and gives $a\Phi(a/b)+b\phi(a/b)$. In the form of (4.7) the integrand is badly behaved near $(0,0)$ so we perform the change of variables $v=\sqrt{t}$, $w=x/\sqrt{t}$, giving

$$V_{\text{fixed}} \le e^{-\rho}\int_0^1\int_{-\infty}^{\infty} 2v\phi(w)\left[a(t,x)\Phi\left(\frac{a(t,x)}{b(t,x)}\right) + b(t,x)\phi\left(\frac{a(t,x)}{b(t,x)}\right)\right] dw\,dv.$$

This expression, combined with (4.3), (4.4), (4.5), (4.6), (4.8) and (4.9), constitutes the upper bound in the fixed-strike case.

For the case of a floating-strike option, we now take $X = S_1$. Setting $Z = \int_0^1 B_t\,dt$, the condition for stationarity with respect to small deterministic perturbations analogous to (4.2) is

$$\mathbb{E}\left[-S\exp(\sigma B_1+\alpha); S\exp(\sigma B_t+\alpha t) \ge (\mu_t + \sigma\,(B_t - Z))S\exp(\sigma B_1+\alpha))\right] = \lambda, \quad \forall t.$$

We approximate this by

$$\mathbb{P}[S\exp(\sigma B_t+\alpha t) \ge (\mu_t + \sigma(B_t - Z))S\exp(\sigma B_1+\alpha)] = \lambda', \quad \forall t,$$

and further, using the approximation $\exp(\sigma(B_t - B_1)) \approx 1+\sigma(B_t-B_1)$, by

$$\mathbb{P}[\exp(\alpha(t-1))(1+\sigma(B_t-B_1) \ge \mu_t + \sigma(B_t-Z))] = \lambda'', \quad \forall t.$$

This implies that for some γ,

$$\mu_t = \exp\left(\alpha\,(t-1)\right) + \gamma\sqrt{v_t}$$

where $v_t = \mathbb{V}\mathrm{ar}[\exp(\alpha(t-1))(1+\sigma(B_t-B_1)) - \sigma(B_t - Z)]$. Since $\int_0^1 \mu_t\,dt = 1$ we must have $\gamma = (1-(1-\exp(-\alpha))/\alpha)/\int_0^1\sqrt{v_t}\,dt$. Our bound is thus

$$\begin{aligned} V_{\text{floating}} &\le e^{-\rho}\int_0^1 \mathbb{E}\left[\left(Se^{\sigma B_t+\alpha t} - (\mu_t + \sigma\,(B_t-Z))\,Se^{\sigma B_1+\alpha}\right)^+\right] dt \\ &= e^{-\rho}\int_0^1 \mathbb{E}\left[\left(e^{N_1(t)} - N_2(t)e^{N_3(t)}\right)^+\right] dt, \end{aligned}$$

where $N_1(t) = \sigma B_t + \alpha t + \log S$, $N_2(t) = \mu_t + \sigma(B_t - \int_0^1 B_s\,ds)$ and $N_3(t) = \sigma B_1 + \alpha + \log S$. If we condition on $N_2(t) = x$ we can perform the remaining expectation analytically. Let

$\mu_i(t) = \mathbb{E}(N_i(t))$, $\sigma_{ij}(t) = \mathbb{C}\mathrm{ov}(N_i(t), N_j(t))$ and denote by tildes the conditional distributions given $N_2(t) = x$: $\tilde{\mu}_i(t,x) = \mu_i + (x - \mu_2)\sigma_{i2}/\sigma_{22}$ and $\tilde{\sigma}_{ij} = \sigma_{ij} - \sigma_{i2}\sigma_{j2}/\sigma_{22}$. Finally let $v^2 = \mathbb{V}\mathrm{ar}(N_1(t) - N_3(t)|N_2(t) = x) = \tilde{\sigma}_{11} - 2\tilde{\sigma}_{13} + \tilde{\sigma}_{33}$.

Our upper bound on the price of a floating-strike Asian option is then

$$\begin{aligned} V_{\text{floating}} \leq e^{-\rho} \int_0^1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_{22}}} \phi\left(\frac{x - \mu_2}{\sqrt{\sigma_{22}}}\right) \Bigg[& e^{\tilde{\mu}_1 + \frac{1}{2}\tilde{\sigma}_{11}} \Phi\left(\frac{\tilde{\mu}_1 - \tilde{\mu}_3 - \log(x) + \tilde{\sigma}_{11} - \tilde{\sigma}_{13}}{v}\right) \\ & - x e^{\tilde{\mu}_3 + \frac{1}{2}\tilde{\sigma}_{33}} \Phi\left(\frac{\tilde{\mu}_1 - \tilde{\mu}_3 - \log(x) + \tilde{\sigma}_{13} - \tilde{\sigma}_{33})}{v}\right) \Bigg] \, dx \, dt, \end{aligned}$$

where we take $\log(x) = -\infty$ for $x \leq 0$.

5 Numerical Results

In Table 1 we consider fixed-strike options and show the upper bound of Rogers & Shi (1995), the upper and lower bounds derived in Sections 3 and 4, the approximation of Curran (1992) and the Monte-Carlo results of Levy & Turnbull (1992). All calculations assume $\rho = 0.09$, an initial stock price of $S = 100$ and an expiry time of 1 year. For the lower bound and the new upper bound, the approximate time taken (on an HP 9000/730) is parenthesized; for the Monte-Carlo studies, the estimated standard error is bracketed beneath.

In Table 2 we show how the upper and lower bounds of Rogers & Shi (1995) compare to the upper bound of Section 4 and the generalization to floating-strike options of Curran's lower bound, described in Section 3. The approximate time taken is parenthesized.

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Volatility σ	Strike K	Curran lower	R–S lower	M–C result	Upper bound	R–S upper
0.05	95	8.8088 (0.00016)	8.8088 (0.0019)	8.81 [0.00]	8.8089 (0.013)	8.821
	100	4.3082 (0.00012)	4.3082 (0.0011)	4.31 [0.00]	4.3084 (0.019)	4.318
	105	0.9583 (0.00012)	0.9583 (0.0011)	0.95 [0.00]	0.9585 (0.019)	0.968
0.10	95	8.9118 (0.00016)	8.9118 (0.0018)	8.91 [0.00]	8.9130 (0.019)	8.95
	100	4.9150 (0.00023)	4.9150 (0.0017)	4.91 [0.00]	4.9155 (0.020)	5.10
	105	2.0699 (0.00023)	2.0699 (0.0018)	2.06 [0.00]	2.0704 (0.021)	2.34
0.30	90	14.9827 (0.00023)	14.9827 (0.0019)	14.96 [0.01]	14.9929 (0.024)	15.194
	100	8.8275 (0.00023)	8.8275 (0.0019)	8.81 [0.01]	8.8333 (0.024)	9.039
	110	4.6949 (0.00023)	4.6949 (0.0018)	4.68 [0.01]	4.7027 (0.028)	4.906
0.50	90	18.1829 (0.00023)	18.1829 (0.0019)	18.14 [0.03]	18.2208 (0.028)	18.57
	100	13.0225 (0.00023)	13.0225 (0.0018)	12.98 [0.03]	13.0569 (0.063)	13.69
	110	9.1179 (0.00023)	9.1179 (0.0018)	9.10 [0.03]	9.1561 (0.064)	9.97

Table 1 Comparison of various bounds on fixed-strike Asian option prices for $S = 100$, $\rho = 0.09$, and an expiry time of 1 year. Parenthesized numbers are computation times in seconds, bracketed numbers are estimates of standard errors (from Curran (1992)).

Volatility σ	Interest Rate ρ	Generalized	R–S lower	Upper bound	R–S upper
0.1	0.05	1.2454 (0.00018)	1.2454 (0.0036)	1.2457 (0.027)	1.355
	0.09	0.6992 (0.00020)	0.6992 (0.0054)	0.6997 (0.025)	0.825
	0.15	0.2516 (0.00020)	0.2516 (0.0078)	0.2525 (0.024)	0.415
0.2	0.05	3.4044 (0.00017)	3.4044 (0.0024)	3.4064 (0.031)	3.831
	0.09	2.6216 (0.00020)	2.6216 (0.0032)	2.6237 (0.033)	3.062
	0.15	1.7098 (0.00020)	1.7098 (0.0044)	1.7124 (0.028)	2.187
0.3	0.05	5.6246 (0.00017)	5.6246 (0.0018)	5.6318 (0.028)	6.584
	0.09	4.7382 (0.00020)	4.7382 (0.0022)	4.7456 (0.033)	5.706
	0.15	3.6085 (0.00020)	3.6085 (0.0032)	3.6166 (0.033)	4.604

Table 2 Comparison of bounds on floating-strike Asian option prices for $S = 100$ with an expiry time of 1 year. Parenthesized numbers are computation times in seconds. The parameter values are those used in Rogers & Shi (1995).