**Uncorrected proof**

# Wittgenstein's Characterisation of the Inconsistency within Mathematics

*An examination of remarks made by Wittgenstein in MSS 126 and 127(referred to together as "FML")*

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Abstract

*Eine Beweisführung ist prüde, wenn die logische Zweideutigkeit ängstlich vermieden wird, grober Unsinn aber geduldet..........*

*Die Hauptunklarheit in der Mathematik ist die Unklarheit darüber, was entdeckt und was bestimmt wird.*

*Eine Beweisführung ist prüde: wenn man ängstlich die geringste logische Zweideutigkeit vermeidet. aber groben Unsinn duldet*

(Ludwig Wittgenstein *MS 126*)

Wittgenstein describes mathematicians as "prudes" and mathematics as "prudish". The language of mathematical propositions and the inconsistency of fundamental rules within the foundations of mathematics bothered Wittgenstein for as long as he wrote, spoke and thought about the subject. He thought of himself as someone who looked carefully at the foundations of mathematics and the conceptual relations between different subjects within mathematics, rather than as someone who made judgments about individual theorems. In this regard the obvious dichotomy in mathematics between what might or might not be acceptable to mathematicians in terms of language and axiom, was, for Wittgenstein, reminiscent of the actions of a prude in the most obnoxious sense.

This striking metaphor, that mathematicians are like prudes, captures much of Wittgenstein's attitude towards mathematicians (as opposed to the subject of maths). His views (published in a variety of sources)

about Cantor and Gödel, made in the decade prior to MSS 126 and 127 can be seen, for example, as part of a continuous line of thought stretching back prior to TL-P and pushing on into the richly creative years of 1929 to 1935, and then consolidated, as it were, in 1942 and 1943 in MSS 126 and 127 and other documents that were written in the later stage of his work and life.

Wittgenstein's mathematical investigations lie at the core of his work throughout the full span of his career. Ramsey allows for Wittgenstein's definition of a tautology to be distinguished in essential terms from a proposition (Ramsey 1926 *The Foundations of Mathematics*), and 16 years after Ramsey's paper was published, Wittgenstein approached and investigated the subject of mathematical semantics (such as the use of words, precepts and symbols) in propositions in a seminal work that is contained in part in MSS 126 and 127. These papers (actually small pocket notebooks) were written during 1942 and 1943. A very small proportion of which were published in RFM (section IV in the second edition and V of the third edition).

MSS 126 and 127 (which are in fact a single work), in its proper context, sits alongside a working review by Wittgenstein as exhibited through marginalia, of Hardy's iconic course book (*Course of Pure Mathematics* or "CBM") that by the early 1940's was widely accepted as the essential text for mathematics undergraduates in Great Britain.

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Introductory remarks

*"So the way out that I am recommending seems unavoidably to involve repeated appeal to the very confusion that Wittgenstein (following Kant) warned us against namely that between the self of empirical psychology and the self as metaphysical subject: I have to conceive of myself empirically as an object in the world insofar as I conceive of myself as bearing determinate relations to the languages in the hierarchy; but I must also conceive of myself metaphysically insofar as these relations are ultimately inexpressible. This is an uncomfortable conclusion to reach: it would indeed be alarming if the whole of higher mathematics depended on such confusion. But I cannot for the moment think what else to say."*

(Michael Potter *Reasons Nearest Kin. Philosophies of Arithmetic from Kant to Carnap* pp 288/289 2004 ed)

In the years since his death in 1951, the manifest truth of Wittgenstein's own view of himself as primarily a philosopher of mathematics has become increasingly more obvious. This is an uncomfortable reality for a great many scholars and researchers who are tied into the industry that Wittgenstein's work has become. To the majority of working (by which I mean academically tied) scholars, Wittgenstein is anything but a philosopher of mathematics.

In the first 30 years after his death the overwhelming preponderance of published research and analysis focused on many areas of philosophy to which Wittgenstein's work can be applied outside of mathematics.

This is not surprising. Wittgenstein has a lot to say about the Philosophy of language, of mind, of aesthetics and of course on logic (although there is a strong argument that the latter is linked inextricably with and therefore a part of Wittgenstein's work on the foundations of mathematics), and science and even theology. It is difficult in fact to find an area of study where researchers have not tried to apply or understand the impact of Wittgenstein's teaching.

Stuart Shankar is absolutely correct when he writes:

*"philosophers are energetic beings if nothing else and from the narrow concerns of a highly technical and abstruse exercise in mathematical logic have arisen thriving industries in the philosophies of language and mind."*

(Stuart Shankar *Wittgenstein's Remarks on the Significance of Godel's Theorem*)

The fact is that Wittgenstein thought of himself as a Philosopher of Mathematics and of his primary focus as the study of the foundational elements of mathematics. It is also a fact that there is a surprising paucity of work on this aspect of Wittgenstein in contrast to the "thriving industries" mentioned by Shanker.

Whilst it is not the purpose of this paper to examine the history of research on Wittgenstein and Mathematics, nor to offer a detailed examination of why such a paucity exists, there is something to be gained by a brief reminder of the very early papers and reviews still extant which were written in response to the publication by Witgenstein's literary heirs of *Remarks on the Foundations of Mathematics* in 1956 ("RFM").

In outline there is an initial period (from 1951 until the middle of the 1960's) when a number of commentators wrote negatively against Wittgenstein and his work on mathematics, often in a hostile manner. The starting point for hostility (or even disdain) towards Wittgenstein's work on mathematics is the early and trenchantly hostile review of RFM by Wittgenstein's one time student, Georg Kreisel[1] who went from Cambridge to the University of Reading where he was resident at the

[1] Published in 1958 British Journal for the Philosophy of Science, 9 (34)

time he wrote his review, to working alongside Godel at Princeton's Institute for Advanced Study.

There is a view (which I share) that Kreisel's scathingly negative review was a disincentive for many serious mathematicians or Philosophers of mathematics in pursuing and studying Wittgenstein's work on Maths. I comment a little more on Kreisel further in this paper (see below) but for now, I find myself agreeing with Mathieu Marion who writes:

*"Ever since Georg Kreisel ended his review of Wittgenstein's Remarks on the Foundations of Mathematics by saying that "it seems to me to be a surprisingly insignificant product of a sparkling mind" there has been a strong presumption that Wittgenstein did not know much about mathematics in general and, therefore, in the foundations of mathematics in particular"*

(Mathieu Marion and Paolo Mancosu *Wittgenstein's Constructivisation of Euler's Proof of the Infinity of the Primes*)

In addition to Kreisel[2], Alan Ross Anderson's similarly negative comments from the same year, and in 1959, Dummett's characterisation of Wittgenstein as a Radical Conventionalist[3] in an article that seemed to champion orthodoxy against Wittgenstein's examination of the foundations of mathematics also contributed to the creation of a long lasting negative frissance towards Wittgenstein's competence in or views about mathematics.

It is encouraging that as access to all of Wittgenstein's work has become more direct and less intermediated, and as a new generation of researchers and students have been able to benefit from work that is not dependant upon the selective availability of Wittgenstein's comments that were published by his literary heirs (or by being taught and instructed by people who in turn were limited in their interpretation of Wittgenstein through the restricted available published material), it has become possible to discern a continuous thread in his works. Wittgenstein as not only possessed a focus on mathematics, but as a philosopher, worked primarily on the foundations of mathematics. The facts, as evident from primary sources, are that

[2] There is evidence that Kreisel regretted his boorish comments. The relationship between Kreisel and Wittgenstein is complex, to say the least. I am grateful to Mathieu Marion for uncovering a letter from Kreisel to Gregori Mints dated sometime in the 1970's where Kreisel explicitly blames the "editors" for choosing comments for inclusion in RFM that "were bound to be weak". I personally find the admission to be illuminating in many ways, but I remain unconvinced about the attribution of blame to the editors (Rhees, Anscombe et al) for the simple reason that there is an abundance of evidence to show that the choice of what should and should not be included in RFM was heavily influenced and selected by Kreisel who had privileged and prior access to the various manuscript sources that the editors possessed. What is very clear in this letter, written 20yrs or so after Wittgenstein's death is that Kreisel is offering an apology of sorts for his aggressively negative comments against Wittgenstein

[3] Dummett's review was published in The Philosophical Review as "Wittgenstein's Philosophy of Mathematics".

Wittgenstein became seriously interested in mathematics during his study of aeronautical engineering in Manchester prior to his arrival in Cambridge, and maintained this focus all the way through to his lectures during the 1930's, and then in his observations in the mid to late 1940's. Researchers are no longer dependant upon the selection of comments published in RFM and other similarly edited publications[4].

I should add that a very large proportion of researchers and teachers who have looked at Wittgenstein's mathematical corpus have also tended to work on the premise that he (Wittgenstein) was somehow not a mathematician, or at least not a practicing mathematician. It is a short and easy step from this premise to the assumed consequence that Wittgenstein was technically less proficient as a mathematician than some of those who have scrutinised his work.

That this view proliferates is difficult to reconcile with the facts as they have emerged. Even if we leave aside Wittgenstein's very early work on the design of a propeller (his design was patented) that required a high degree of mathematical skill and insight, we also know that two of the most gifted mathematicians of their generation, GH Hardy and JE Littlewood, asked Wittgenstein to teach their maths students during the 1930's[5].

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[4] An accurate and balanced view of Wittgenstein's work on mathematics from the period 1929 onwards is provided by Pasquale Frascolla in his work Wittgenstein's *Philosophy of Mathematics*. A concise description of the external and internal factors that led Wittgenstein to devote such a large amount of time and effort towards mathematics can be found in Chapter 2. Frascolla also highlights the accretion of work done in formal writing (*Philosophical Remarks, Big Typescript*) along with evidence from lecture notes, and the significant volume of notes made available by Waismann of his conversations with Wittgenstein. For those who have an interest in the underlying mathematics that are being examined through the grammar and structure of their proofs, Pasquale's book reminds us that Wittgenstein's comments on the texts of Weyl, Hilbert Skolem and Hardy (see also below), and references that he makes in various letters throughout the 1930's and 1940's, are part of a continuous development, and therefore important to an accurate understanding of Wittgenstein's philosophy of mathematics.

[5] The following quote is taken from Professor Arthur Gibson's forthcoming publication on the previously unpublished Wittgenstein Skinner Archives:

**"Hardy and Littlewood facilitated Wittgenstein's teaching their mathematics' students, whose training at undergraduate level is so time-sensitive for their talents to reach their optimum harvest – that the Faculty (even now) guards against influences that are not significant contributions to that end. Wittgenstein assisted mathematics students to think mathematically and if possible some to become philosophers. They were also lectured by Littlewood, who stated in his *Elements of the Theory of Real Functions: being Notes of Lectures…*: that they were:**

> *intended to introduce third year and more advanced second year [maths students] to the modern theory of functions. The subject-matter is very abstract… The aim of the lectures, indeed, is to inculcate the proper attitude of enlightened simple-mindedness by concentrating on matters that are abstract… aimed at excluding as far as possible anything that could be called philosophy.*

MSS 126 and 127 and their relationship with RFM

Of the very significant amount of work that Wittgenstein therefore did on mathematics and the Philosophy of mathematics (a conservative estimate that is widely accepted is that well over half of all Wittgenstein's work is about mathematics with the rest divided amongst other subjects), is a relatively short but critically important manuscript written during the time that Wittgenstein was temporarily living away from Cambridge during the Second World War.

It is this document, and in fact one remark in that document, that is the focus of this paper.

In 1942 and 1943 Wittgenstein visited Cambridge, initially from Guys Hospital in London where he was working as a porter, and then, after March 1943, from Newcastle where he worked as a laboratory assistant. Whilst he was in Newcastle Wittgenstein designed a more efficient medical device employing applied mathematical skills.

During these visits Wittgenstein spent time with his then student, Georg Kreisel, and, amongst other matters, they discussed in great detail, G.H. Hardy's iconic textbook, *A Course on Pure Mathematics*.

Wittgenstein's detailed notes of his thoughts at the time are written up in the small pocket notebooks that were typically used by Wittgenstein in Cambridge.

What is also not fully clear is whether these notebooks (MSS 126 and 127) are essentially the Wittgenstein/Kreisel conversation in extended note form, or whether the conversations were a reflection of thoughts and notes that are therefore broader and deeper in scope than the Wittgenstein/Kreisel conversations. I tend to believe it is the latter. What is, however, clear, is the fact that MSS 126 and 127 is a single act of quite focused and deliberate work that, taken together, in its entirety, provides serious researchers of Wittgenstein with invaluable and unedited access to his mature thoughts on the foundations of mathematics.

Kreisel was provided with full and complete access to MSS 126 and 127

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**It is therefore a remarkable compliment, in view of Littlewood's above closing comment, that Wittgenstein was called on to teach these students. Paradoxically, this requires that Wittgenstein's teaching of philosophy for mathematicians excludes philosophy, by his making explicit what it is to be mathematics: to describe it, without excess."** (introduction to Chapter 1, *Wittgenstein-Skinner Archives* edited with commentary by Professor Arthur Gibson (forthcoming)) after Wittgenstein's death, and played an instrumental part in choosing which parts of the document (as well as other notebooks and manuscripts) were selected by the editors for publication as RFM, and in what order they (the selected notes) should appear. With respect to MSS 126 and 127, Kreisel exercised his editorial selections by extensively and thoroughly marking up a typescript thus highlighting those numbered points that should be excised and those that should be selected, as well as making certain comments and observations on the content of the typescript. A close comparison of this typescript with RFM show that all of the recommendations made by Kreisel (with respect to MSS 126 and 127) were followed and accepted by Wittgenstein's literary heirs.

For ease of reference, I refer hereafter to this typescript of MSS 126 and 127 with Kreisel's marks and comments as "FML" (taken from Wittgenstein's own title for MS127 as *Mathematik und Logik*)

I will comment further on Kreisel's editorial remarks as well as his review of RFM, but at this point it would be useful to remind ourselves that for the overwhelming majority of readers of RFM, their only access to Wittgenstein's philosophy of mathematics is a book that in turn is a motley of assorted and disjunctive comments, with no explanation of context or of why paragraphs that are split or omitted.

Wittgenstein's 1942 and 1943 notebooks that I refer to as FML were demarcated as MSS 126 and 127 by Von Wright who led the project that resulted in "Remarks on the Foundations of Mathematics" being published in 1956 ("RFM"). Von Wright's co-editor was Rush Rhees, and the translation from German for the English edition was by Elizabeth Anscombe. This titling (MSS 126 and 127) gives the impression of two separate documents. As noted above, this can be misleading since the work is of a single continuous nature in a manuscript that happens to occupy two notebooks/handbooks due to relatively small size of the notebooks in question.

Approximately 35% to 40% of the comments made in MSS 126 and 127 were published as Section IV in the first edition of "Remarks on the Foundations of Mathematics". These same comments appear in Section V of the third edition of RFM published in 1978. Von Wright's numbering and labeling system accorded the prefix "1" to manuscripts as opposed to typescripts or other types of source documentation, and thus 126 and 127 have also sometimes been referred to 26 and 27 in a filing system that was used by Von Wright to accommodate manuscript notebooks and other manuscript sources that are not listed in chronological order. As noted above Wittgenstein prefaces the pocket notebook that has been labeled 127 in the Von Wright classification system with the title *Mathematik un Logik*. It is worth noting that this title does not mark any change in what appears to be a continuation of work (both in content and form) from MS126.

A full typescript copy of MSS 126 and 127 that has then been photo-copied (ie the version with extensive comments by Kreisel, and colour coded to differentiate between Kreisel's various directions as well as later highlights by Michael Nedo), is available at the Wittgenstein Trust in Cambridge. The two pocket notebooks (now frail and therefore not in a state that could withstand frequent handling) documents are held by Trinity College Cambridge in the Wren Library.  An electronic version is available through the Bergen Electronic Edition of the Wittgenstein Nachlass[6]. A concise summary of the Nachlass' distribution is noted in the following link to a webpage that mentions MSS 126 and 127 and refers to the fact that the original notebooks/handbooks were lost until the early 1990's. The link is:

http://www.nlx.com/collections/124

Further information on the manuscript collection that forms the Nachlass is contained in G H Von Wright's 1982 publication, *Wittgenstein* (University of Minnesota Press) in the (extremely useful) opening chapter *The Wittgenstein Papers*. This was originally compiled and published in 1969 (Philosophical Review, ed 78), but updated for the 1982 book[7].

Von Wright details here, amongst other things, that papers were burnt by Wittgenstein in 1949 and also provides an illuminating summary of the complicated ways in which papers and letters came to be unearthed by the literary heirs. For example Wittgenstein's correspondence with Waismann came to light 5 or 6 yrs after the publication of RFM. There is little doubt that this correspondence, had it been available in 1956, would have had an impact on the selection of material into RFM[8].

I am not aware of anything better than Von Wrights' chapter noted above for those interested in the insight that the broken and interrupted chronological flow of different documents coming to light offers into the editorial deliberations of Wittgenstein's heirs.

In any event, returning to FML and MSS 126 and 127, Von Wright's listing contains, on page 44, the following entries:

*126 Pocket Notebook. 20 October 1942 – 6 January 1943. 155pp. (Missing)*

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[6] Readers may be interested to note the announcement in May 2014 of an ambitious new project to provide open and free access to all of Wittgenstein's manuscripts held at the Wren Library through a facsmile edition. This is a joint project between Trinity College and the University of Bergen and further enhances frictionless access to source material

[7] The introduction to this book contains the following touching admission by Von Wright *"Wittgenstein influenced my intellectual development more than anyone else could have done. He did this partly by his teaching and writing, but mainly by his example"*,

[8] *Voices of Wittgenstein: The Vienna Circle* (Routledge 2003). This is a good source of the interstices that occurred when Wittgenstein and Waismann met intellectually.

*127 Pocket Notebook. "F. Mathematik und Logik." 6 January-4 April 1943: 27 February-4 March 1944; undated part. 175pp. (Missing)*

The disappearance of the manuscript notebooks for approximately 40 years further helps to explain, in part, why a work of such critical import has been so rarely referenced during the 1970's and 1980's[9]. Further, the fact that FML reflects, in part, Wittgenstein's conversations with Kreisel based on a review of an important textbook of mathematics, and then contains Kreisel's own comments, (inserted probably a dozen or so years after the notebooks were written) gives the typescript a special place in any review of Wittgenstein's work on mathematics. In fact given Kreisel's energetically negative stance towards RFM, FML occupies a rather special place in exegetic terms. I am not aware of any published author on Wittgenstein's work on the Philosophy of mathematics other than Juliet Floyd who has reviewed this typescript[10].

In summary therefore:

- MS 126 and MS 127 were written in conjunction with a review of Hardy's CPM
- Wittgenstein's review of CPM is evidenced through marginalia that he wrote on his copy of the book
- Wittgenstein's conversations with Kreisel around the same time contributed to some of the structure and content of FML
- FML is a mature work – containing many references to views that were formative over a very long period of time with respect to what we refer to as the foundations of mathematics
- FML should be read both in conjunction with a review of Wittgenstein's comments on Hardy's CPM, and in its entirety as a stand alone document
- As it is very small parts of FML, in a disjointed manner, were selected for publication in RFM

It is far too easy to fall into the trap of criticizing Wittgenstein's literary heirs. It is equally easy for comments about the choices that they (the heirs) made about what to select and what to publish in the form of themed anthologies[11], to be interpreted as criticism. I intend neither. I

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[9] Arthur Gibson's recollection is that Peter Geach returned the two notebooks (MSS 126 and 127) to Trinity in 1994 having found them in Oxford

[10] I refer below to Juliet Floyd and Felix Muhlholzer's forthcoming extended essay on Wittgenstein's marginalia on Hardy's Course on Pure Mathematics, a pre published copy of which was generously sent to me while this paper was being finalized

[11] An example of a "themed anthology" is *On Certainty,* or *On Colour,* publications that are collections created (therefore themed) and titled by his literary heirs and consisting of material that in many cases was composed and written decades apart, and with no indication to the general reader that

do, however, believe that the availability of Wittgenstein's work to everyone, in the form it was left to his heirs, is the best way for serious scholars to review Wittgenstein's work. In this context works such as RFM, which are heavily edited (or at best selectively collated) are especially vulnerable to the sort of judgement that mathematicians such as Kreisel and Bernays[12] leveled in their reviews. I don't know that anyone could have done much better, given the same set of circumstances, than Wittgenstein's heirs. Von Wright sums up just how much in awe they (the heirs) were of Wittgenstein and of the task that was handed to them, and displays an admirably balanced humility in his position in the years after various publications had been made available when he writes *"I was not able to follow him (Wittgenstein) very well in my own work – not only because my thinking cannot reach the standards he set, but also because his* ***style*** *of thought is so different from my own."*

Before turning to the comments themselves, it is important to highlight the relationship between MSS 126 and 127, and FML, with Wittgenstein's observations (marginalia) in his copy of Hardy's text book on mathematics (CPM). GH Hardy requires no introduction, however it is worth commenting on CPM. This iconic work, already in its 8th printing by the time that Wittgenstein reviewed and commented upon it in the form of marginalia in 1942, was the basic course book used to teach mathematics to undergraduates not only in Cambridge, but very widely across the United Kingdom and universities tied into the British Empire (e.g. Canada, Australia and India). CPM was aimed at the brightest students who might be preparing for the study of higher mathematics. The book therefore came to fulfill, in this regard, one of Hardy's objectives in providing an alternative method of teaching to the one he himself had encountered as a young undergraduate, and which created such a distaste on Hardy's part towards the traditional Tripos structure at Cambridge at the turn of the 20th century.

Wittgenstein was not only a contemporary of Hardy, but would have been acutely aware of and familiar with his work over an extended period of time. By the early 1940's two or more generations of undergraduates would have been instructed on the precepts contained in "Course of Pure Mathematics". This would have included undergraduates of mathematics that would have been regular attendants of Wittgenstein's classes, including Turing. Wittgenstein himself would have been aware of CPM from the time of his arrival in Britain some 30 years prior to the 1941 edition being published.

Wittgenstein's antipathy towards some of Hardy's views (but apparently not Hardy personally) has been documented relatively

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this is the case.

[12] Bernays' review, published 1959 recognises what he describes as the "fragmentary nature" of the selection, and furthermore states "in fairness to the author, it has to be admitted, however, that he would doubtless have made extensive changes in the arrangement "

extensively over the years, including Wittgenstein's comments on one of Hardy's *A Mathematician's apology*[13] which he refers to as a *"miserable book"* [14]

The 1941 edition (the 8th edition) of Hardy's *A Course of Pure Mathematics*, contains both a preface from the 7th edition of 1938 as well as extracts from the original preface from 1908. All references to page numbers within Hardy's Course Book in this paper are therefore to the 1941 8th edition of the book.

Wittgenstein's personal copy of Hardy's CPM, the one marked with his comments, has yet to surface, but a photocopy of the original marginalia was created by Rush Rhees[15] which contains 16 pages that carry comments written in Wittgenstein's hand. It is not known whether these are the entirety of all comments made by Wittgenstein[16] but they include the key (and relatively extensive) marginalia around p29 which in turn is cross referenced by the remarks from FML that are being examined in this paper (FML 40/128)[17]. I comment in more detail on the marginalia below, but it is worth noting at this stage that Wittgenstein's handwritten remarks in Hardy's CPM might be better viewed as a sort of aide memoire for further elaboration at a later date rather than completed comments that might be read by someone else. This, at least, is my view, hence my approach to try and review the marginalia together with (and part of) MSS 126 and 127.

The final part of the triumvirate of documents that make up the full suite that go together in this paper, is "Mathematical Proof", an article written by Hardy in MIND, being itself the transcript of Hardy's 1928 Rouse Ball Lecture. This lecture is readily available in the public domain, and appeared in MIND Vol XXXVIII, N.S., No 149.

Written at a time when Wittgenstein was at the height of his analytical powers, benefitting from work that he had been involved in since 1929, FML is a seminal work by a mature Wittgenstein as he examines and then remarks with acuity and great effect on the very bedrock and fabric of mathematics.

The inconsistency of mathematics leads to mathematicians becoming

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[13] *Inequalities* is more often than not regarded as Hardy's best mathematical work, whilst *A Mathematician's Apology* is certainly the most widely read broader book

[14] See Wittgenstein MS 124

[15] The full photocopy is available in the reserve collection at the Wren Library

[16] Juliet Floyd in a forthcoming paper authored with Felix Muhlholzer already referred to above, and also in a paper delivered in April 2013 at CUNY refers to 20-25pages of marginalia

[17] I would be delighted to share my copy of Wittgenstein's marginalia and notes in Hardy's Course book with anyone who is interested. Please email me on the address at the end of this paper.

prudes

***Eine Beweis fuhrung ist prude, wenn die logische zweideutigkeit angstlich vermieden wird, grober unsinn aber geduldet.........***

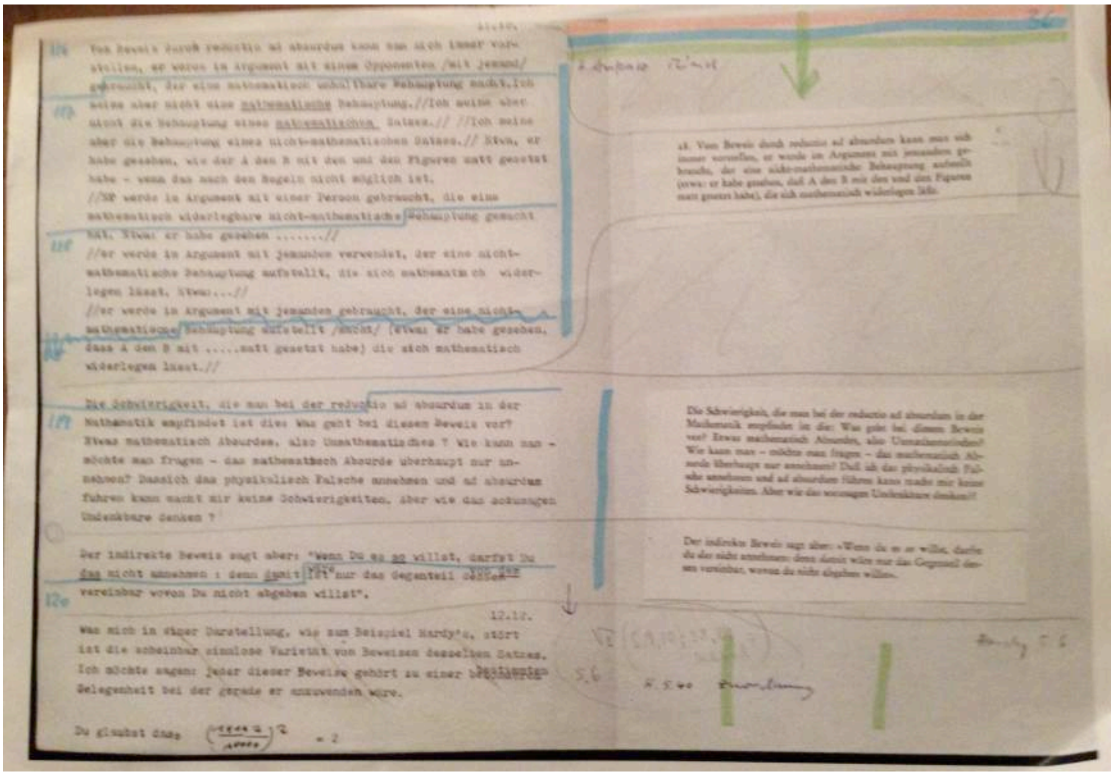
***Die Hauptunklarheit in der mathematik ist die Unklarheit daruber, was entdeckt und was bestimmt wird.***

***Eine Beweis fuhrung ist prude: wenn man angstlich die geringste logische Zweideutigkeit vermeidet. aber groben Unsinn duldet***

Wittgenstein's comments are remarkable – both for their directness and their sharp almost bitingly harsh and critical nature. These 3 sentences are a very small, but illuminating aspect of FML. One might say that Wittgenstein's remarks in FML that are being examined in this paper encapsulate much of the flavour of his approach towards mathematicians.

FML consists of 75 pages, each page consisting of a view in landscape of the original typescript on the left as viewed, and a comparison with RFM on the right. The typescript therefore affords not only a full edition of MS 126 and 127, but also exactly which parts of the MSS were selected for publication in RFM, and exactly how they appear in that publication. Where comments are published elsewhere other than RFM, these are also highlighted. The typescript also shows Kreisel's comments, and is colour coded to highlight exactly what portions have been published and which have not[18].

[18] Whilst not the focus of this paper, it would be inappropriate not to mention the scholarship of Michael Nedo of the Wittgenstein Trust who carefully collated and pieced together FML



*p 36 of the Wittgenstein Trust Typescript of MS 126, showing remarks 116, 117, 118, 119 and 120. The typescript is on the left as viewed, and on the right is remark 28 from RFM first edition IV. Also shown are Kreisels markings and further references by Michael Nedo detailing where some comment from 126 has been published other than in RFM. Whilst not quite clear on this image, remark 120 relates directly to a discussion of √2 by Hardy in CPM.*

The base typescript must have been created prior to 1956 and would appear to be used as part of an editing process that was designed to gain Kreisel's comments in the context of the overall selection of items for RFM.

As can be seen in RFM, the large majority of Kreisel's recommendations were accepted by Rhees and Von Wright. I am not aware of whether Kreisel was provided with initial drafts of selected material from RFM but will assume that he did since this would fit the process that we know was being followed at the time. This means that it is possible that Kreisel's markings are duplicatory or confirmatory of earlier conversations with Rhees and Von Wirght, as opposed to being initial or primary recommendations[19].

In addition to page numbers, each separate remark (95% of all remarks are at least a single sentence, but most often a small group of two or more sentences) is given a number. These numbers ascend from 1 to

[19] Whilst outside the scope of this paper, I have seen evidence, provided by Arthur Gibson, of Kreisel's criticism of Wittgenstein's comments on the F*oundations of Mathematics* (Lecture notes) from approximately 1938. Kreisel's comments are quite harsh, but, rather surprisingly, also contain technical errors, and are in the form of handwritten remarks on a typescript of Wittgenstein's work

155 in MS126 (the first notebook), and 1 to 217 in MS 127 (the second notebook). Although not quite clear, there are parts of MS 127 which were written after a pause, but the narrative suggests chronological continuity from 1942 and 1943.

The remarks also seem to follow a pattern discernable through themes that arise in Hardy's *Course Book*. What I mean here is that MSS 126 starts by remarks that could very easily have been prompted by the contents of Hardy's opening remarks, and this correlation, with FML following or treating subjects in the same order in which they appear in CPM continues, as far as can be tested, throughout the document. This observation remains preliminary since Wittgenstein's personal copy of Hardy's course book has been lost and I have relied on the Rhees photocopy, which in turn, I presume but cannot be sure, is complete[20].

The remarks that are the subject of discussion in this paper occur on page 40 of the document, and are situated within section 128 of the first half of the manuscript (i.e. MS 126). Within FML therefore the remarks on the prudishness of mathematicians can be identified as 40/128, which is the convention I have used.

Kreisel and Von Wright chose not to include 40/128 in RFM. No part of 40/128 is published in any edition of RFM.

Substantial parts of FML 38/125 (ie page 38 remark 125) and all of 38/126 are included in RFM, and, after excluding 40/128, half of 40/130 and a reasonable part of 41/132 are also selected for publication. A careful reading of FML from at least 36/117 through to the start of 44/141 shows that these passages are continuous and are inextricably linked. The passage selected and published in RFM as remark 31 (part IV, first edition) correlates exactly with 40/130 (half way into comment 130, and therefore a few sentences after 40/128, and on the same page of FML). Interestingly comment 30 in part IV RFM (i.e. the part immediately prior to 31 and preceding 40/128) is not even taken from the same part of FML but from a completely different passage that occurs 9 pages earlier.

Professor Felix Mulholzer makes a comment that is worth remembering at this stage.

*"Because of the heavy encroachments of its editors, RFM too must be taken with grains of salt, of course. But one cannot really doubt that even in its present,very unsatisfactory version RFM at least shows the Wittgensteinian plan for his philosophy of mathematics and his philosophical method in a sufficiently clear way."*

[20] Juliet Floyd and Felix Muhlholzer's forthcoming paper supports my supposition. As noted above I was fortunate to benefit from studying an advance copy of that paper, very generously provided by the authors

(footnote 4 of the essay 2004)

I agree with Muhlholzer's first sentence in the passage copied above, but have problems with the second sentence. In choosing to highlight problems with RFM *("heavy encroachments of its editors")* Mulholzer is, in my view being generous. FML not only shows the extent of such encroachment, by providing evidence that the exclusions were just as important as the inclusions, it also maps out the extent to which certain streams of thought are bifurcated and used out of context.

Moreover, even though I am not aware of the underlying approach that led to the final choice of remarks being chosen from FML (and here I am not making a judgement on RFM as a whole but only the part that is based on MSS126 and 127), I do believe that given the very clear context of the document and its relationship with Hardy's CPM, certain exclusions happen to create more confusion than clarity and in a small but significant way, may have contributed to the hostile reaction of mathematicians who have assumed that somehow RFM represented Wittgenstein's thoughts as he articulated them. This is clearly not correct[21].

**"Eine Beweiss fuhrung ist prude"**

Wittgenstein is deliberate in the use of the word "prudish" when describing mathematicians, and in doing so creates a precise impression of his views on the subject. There can be absolutely no confusion over what he means. We must remember that by the time that FML was being written in 1942 and 1943 Wittgenstein's cautious and at times dismissive view of Godel's position had already led to his being subject to criticism from mathematicians for a decade or so.

Wittgenstein's repeated claims that he was interested in the foundations of mathematics and not the maths itself must have been as hard to accept to some people in the 1930's as it has proved to be in the decades since his death. But in FML, by using as a reference point a book that could justifiably be described as one of the most highly regarded icons within mathematical orthodoxy, and quite possibly, the most extensively read book of advanced mathematical instruction in the western world at the time (Hardy's CPM), Wittgenstein is able to practice what he preaches by avoiding the actual maths whilst pointing out the flaws in the foundational elements of the approach upon which the maths is based. His position, as espoused in FML, retains a freshness and clarity that has not been lost over the decades.

[21] at the very least I hope that this part of RFM will at least be re-visited with the knowledge that Wittgenstein's comments should be taken into account when thinking of his reaction to Hardy's CPM. 10 years after Muhlholzer's 2004 comment quoted above, and at the time of the writing of this paper, I know that he and Floyd are in the final stages of completing a comprehensive analysis of Wittgenstein's marginalia in Hardy's CPM that I believe will be of significant assistance to providing a fuller context for that part of RFM that is rooted and based on FML

***Prudish – easily shocked (prim, stuffy, straightlaced)***

The dictionary definition can also be supplemented by recognising that in common useage the term “prude” when applied to a person also suggests something dishonest. There is no danger that a person who is described as a “prude” will mistake the description as being a compliment. The implication will be that the prude is prone to hypocrisy, and displays in his or her behaviour behaviours that are ultimately dishonest. The same word carries its common inference – or more accurately insinuation - across different contextual usages of the word. There is a family resemblance, one could say, across cultures and languages.

What is it about mathematicians that makes Wittgenstein use the word “prude” and “prudish”.

*“It's unbelievable the way in which a problem gets completely barricaded in by the misleading expressions which generations upon generation throw up for miles around it so that it becomes virtually impossible to get at it”*

(*Philosophical Grammar* 1974 p 466)

Let’s take a closer look at the very specific context of FML. Read in isolation, the notebooks (i.e. MSS 126 and 127) don’t always differentiate between what is being written or remarked upon with respect to Hardy’s *Course Book*, or where the comment is a remark upon some aspect of the philosophy of mathematics as made apparent by mathematical practice. However, when the notebooks are reviewed in conjunction with the marginalia, the correlation becomes more easily apparent. 40/128 is just such a case.

One of the pages most marked with marginalia in Hardy’s Coursebook is p29. About 80% of the way down that page Wittgenstein has highlighted, using the phrase *“...we found that the idea of a section of the natural numbers led us to a new conception of a number...”*

Wittgenstein then writes, in the margin directly adjacent to this phrase:

*“'Das Prüde an dieser Beweisführung!' (English translation – “The prudishness of this line of argument!').*

This comment, the only use of the word “prude” or “prudish” in Wittgenstein’s marginalia in the Hardy book, correlates directly with 40/128.

Compare with sentence 1 and sentence 3 of 40/128:

Sentence 1 : *Eine Beweis fuhrung ist prude, wenn die logische* *zweideutigkeit angstlich vermieden wird, grober unsinn aber geduldet..........*

Sentence 3 : *Eine Beweis fuhrung ist prude: wenn man angstlich die geringste logische Zweideutigkeit vermeidet. aber groben Unsinn duldet*

And the context suddenly comes to life. Here is Wittgenstein making his note in the margin of his copy of Hardy's Course Book and then later expanding upon the thought in his notebooks.

In the second sentence of 40/128, which ties the whole phrase together, Wittgenstein says

*'Die Hauptunklarheit in der Mathematik ist die Unklarheit darüber, was entdeckt und was bestimmt wird.'*

*(English translation : 'One of the main ambiguity/greatest confusion in mathematics is the ambiguity/confusion about what is being discovered and what is being decided').*

Are prudish people guilty of deliberately accepting the confusion that they themselves create whilst contriving to demand only absolute clarity from others ?  No wonder two generations of mathematicians have had difficulty coming to terms with Wittgenstein's direct and at times brutal honesty. We don't like being shown our shortcomings, especially not when those shortcomings are in exactly those areas that we pride ourselves on being so efficient and effective.

The marginalia on p29 of Hardy's CPM that ties in directly with 40/128, occurs in a section of the book titled Real Variables, and specifically in a sub section dealing with Dedekind's cut that is described as "Sections of Real Numbers".

The Wittgenstein who used the word "prude" is the same that was described 30yrs earlier by Bertrand Russell in the following terms

*"He doesn't want to prove this or that but to find out how things really are"* [22]

Wittgenstein's language has a poetry and grace that should not be underestimated. Von Wright, in his short monograph referred to earlier, wondered whether Wittgenstein's prose will be celebrated as a model of aesthetic excellence in writing in the years to come. In the preface to his book Von Wright states *"Wittgenstein's German has a beauty and expressiveness which easily gets lost in translation"* and later, on p33 *"it would be surprising if he were not one day ranked among the classic writers of German prose."*

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22 Russell letter to Ottoline (letter dated March 1912, Bertrand Russell Autobiography Vol 1)

Wittgenstein's creativity in the language he uses in 40/128 is highlighted in a number of ways. The subtle repetition and even alliteration of sentence 1 and sentence 3 lends a poetic quality to the whole phrase. The use of a word that cannot fail to be understood in terms of its precise meaning and allusion adds a certain bite to the phraseology (a prude would be noticed and understood in both Britain and Germany in the mid 1940's, and would have a distinct cultural identity)[23]. There is nothing accidental about the language being used here by Wittgenstein.

Prudes suppress their sexual (or true) desires for the sake of appearances. They also *"excessively concern themselves with being or appearing to be proper"* (*American Heritage Dictionary*).

Whilst Wittgenstein might not have been aiming his barb at Hardy's specific description of the Dedekind Cut (or Dedekind himself), the charge of prudishness is linked very directly to the jump that Hardy implies when he states that the idea of a section (or cut) "...led us to a new conception of a number".

Wittgenstein giving short thrift to Hardy's proscription to readers of CPM is not the only irritation that is apparent in FML. A continuing source of that same irritation was the application and use of the term "infinity" and its symbolic/notational use within mathematics.

An instructive example of the way that infinity is used in the language of mathematics is in the harmonic series and the proof through contradiction that the series is divergent (i.e. that the sum is infinite).There are a number of such proofs for this divergence[24] and we shall use a proof that makes up in straight forwardness what is lacks in mathematical 'elegance' and 'beauty'.

Our proof starts with the statement

$S= 1+ \frac{1}{2} + 1/3 + \frac{1}{4} + ....$

$= (1 + 1/3 + 1/5 + 1/7 + ....) + (1/2 + \frac{1}{4} + 1/6 + 1/8 +....)$

$= (1 + 1/3 + 1/5 + 1/7 +...) + \frac{1}{2} ( 1 + \frac{1}{2} + \frac{1}{4} + 1/6 .....)$

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23 I do not know if Wittgenstein would have been familiar with TS Eliot and his poem *Love Song of J Alfred Prufrock* but we do know of Wittgenstein's relationship with the great Cambridge English (and Russian) literature specialist F R Leavis who placed a great emphasis on Eliot's poem, claiming that its publication was an important event in English poetry and English literature. The thematic propositions that are examined with unremitting honesty and with such elegance, provide a good reference point at least in English for what the word prudish might have meant and connoted during the 1930's and early 1940's.

24 see for example Kifowit and Stamps 2006

$= (1 + 1/3 + 1/5 + 1/7 + ...) + \frac{1}{2} S$

Meaning that $1/2S = 1 + 1/3 + 1/5 + 1/7 + ....$

Which in turn leads us to the contradiction that the sum of just the even fractions is the same as the sum of 1+ the odd fractions. I won't write out the left and right sides of that equation, but it is shown incorrect on the basis of pairings $1 > \frac{1}{2}$, $1/3 > \frac{1}{4}$, $1/5 > 1/6$, 1/8 ...., meaning that S cannot exist, which means that the harmonic series must diverge.

Wittgenstein would have objected to the propositional statement in the first place, never mind the proof. But why ? He clearly would not have had any problem with the "mathematical" argument.

What he would object to in the very first place is the use of the symbol ∞ or infinity that lies innocuously (and is implied by the innocent ellipses) at the heart of the proposition being proved. Here we see a concept (infinity) being used in an otherwise precise set of definitions[25].

*"ought the word infinite to be avoided in mathematics ? Yes"*

1967 edition of RMF, I appendix 2, comment 17

Wittgenstein's deep-seated objection to attempts to make use of the infinite by mathematicians can be traced back to the *Tractatus*

*It is senseless to speak of the number of all objects* 4.1272.

And therein lies the root of the rather harsh but understandable use of the word "prudish" with Wittgenstein having mathematical proofs in mind when thinking of prudish behaviour within mathematics. A great deal of FML (not just 40/128) builds upon Wittgenstein's belief that the use of a word is the ultimate arbiter of its meaning, and therefore questioning arbitrariness in usages of the words in "real" life versus within mathematical (or other) contexts. "infinite" either as word or symbol, sits right at the core of mathematical prudishness.

There are two lines of thought that occur over an extended period of time in Wittgenstein's criticisms. The use of terminology or symbols that cannot be easily understood or applied across normal language, but tolerated in propositions created from proofs. And secondly the inconsistency or lack of "surveyability" when terms such as infinity are strictly applied.

Wittgenstein's objection to the use of the symbol ∞, or the use of what he

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[25] this is quite apart from any issues Wittgenstein had about arguments or proofs based on Reductio ad absurdum or by contradiction. He was generally cautious in this area.

considered, at best, a concept "infinity", within a mathematical proof leads naturally to a consideration of his insistence on proofs being surveyable.

*A proof in order to be surveyable must be reproducible, such reproduction must be easy, it must be certain that the reproduction has given us the same proof, that the reproduction is of the sort of that of a picture*

(RFM third edition III 1)

This remark needs little further elaboration and pertains directly to 40/128 and the cross reference to Hardy's CPM where Hardy on p29 is discussing sections of real numbers with reference to Dedekind. Wittgenstein's objection here, as exhibited through his comment in CPM[26] is not just to Hardy, but to the abstract nature of the "proofs" of Dedekind's "cut" theory.

One of the clearest of a number of expositions of Wittgenstein's approach to the issue of the surveyability of proofs is Felix Mulholzer's excellent paper *A Mathematical Proof must be surveyable - What Wittgenstein meant by this and what it implies* (2004).

Mulholzer, in this paper shows, amongst other things, that whilst there are seeming similarities between Wittgenstein and Hilbert when it comes to proofs, *"...Wittgenstein, unlike Hilbert, uses his view mainly in critical intent. He tries to undermine foundational systems in mathematics, like logistic or set theoretic ones, by stressing the unsurveyability of the proof patterns occurring in them."*

Mulholzer goes on to highlight how Wittgenstein shows that mathematics has to resort to going outside of its own foundational frameworks in order to make certain patterns surveyable. The familiar example that is used to advance this thought, used by Mulholzer, is Wittgenstein's discussions of continuity and stroke addition with respect to extensions in real numbers. The contrivance that this necessitates (my word, not Mulholzer's) on the part of mathematicians is emphasised again and again by Wittgenstein throughout his career and at all stages of his work, when he criticises the way that the word and the concept "infinity" is used by mathematicians in proofs and propositions that are thus rendered meaningless. This is the behaviour of a prude.

*"I should like to say that where surveyability is not present, i.e. where there is room for a doubt whether what we have really is a result of this substitution, the proof is destroyed"*

RFM III remark 43

[26] Wittgenstein writes in the right hand margin of p29 of CPM "Das Prude an dieser Beweisfuhrung"

On this same theme J R Brown offers the following observation in a paper

*"Wittgenstein repeatedly stressed the importance of "surveyability" – a crucial notion for him. A proof is surveyable when we can grasp it, we can take it in as a whole. The notion is not easily defined, but it is readily understood in an intuitive way and we have no trouble applying it to various examples. The standard proofs of the Pythagorean theorem or the proof of the irrationality of √2 are surveyable. The computer proof of the 4 colour theorem is not"*

*(James Robert Brown "Philosophy of Mathematics : An introduction to a world of Proofs and Pictures" 1999).*

Wittgenstein's consistency and continuity of approach to the issue of surveyability can also be seen in Lectures 3 and 4 from Lectures on the Foundations of Mathematics where there is a direct analogue to the point made by James Robert Brown. These lectures took place 3 to 4 years prior to FML and the conversation with Turing expands the footing for examining the difference between mathematical and physical proofs. I am of the view that Wittgenstein's annotations to CPM p 29 are simply a repetition of ideas that were present for some considerable period of time. In fact Wittgenstein's application and use of the description "prudish" should also be examined with reference to his approach to Cantor and set theory. Wittgenstein could be said, at times, to have railed against the inconsistency that Cantor represented in Wittgenstein's opinion.

Pasqualle Frascolla's 1994 book *Wittgenstein's Philosophy of Mathematics* (Routledge) expands upon the applicability of acquired symbols (such as the notation referred to as Null zero in set theory) and what determines their legitimacy for use in the construction of propositions. Frascolla is not the only commentator who picks up on Wittgenstein's severity when highlighting the inconsistency of mathematicians when seemingly insisting on the most disciplined boundaries (perspicuity, surveyability, reproducibility) whilst at the same time accepting symbols such as the notation for Nullzero and what it imprecisely might imply, that have not been defined and cannot be applied without crashing through the very foundational boundaries that they seek to enshrine.

It is, of course, this very inconsistency that is being described in FML as "prudish". The genteel hypocrisy of a drawing room prude is an image that needs no further explanation when being applied to mathematicians. Valerie Lynn Therrien's paper Wittgenstein and the labyrinth of actual infinity : the critique of transfinite set theory provides in a summary fashion a well selected menu of Wittgenstein's at times piquant take on this slippery but ever so essential ingredient in the armoury of mathematical proofs since Cantor's revelations of the middle of the 19th century.

*"It therefore makes no sense to speak of 'infinite numbers' or 'infinite sets'! Although, properly, it may be asserted only of rule-governed series that they may be either finite or infinite, nevertheless even then "[one] has [...] a concept of an infinite series but here that gives us at most a vague idea, a guiding light for the formation of a concept ", a mere sense that there is a process here that will not terminate"*

(Wittgenstein's comment quoted by Therrien here is from RMF II 16 1978)

Wittgenstein characterises the adoption of set-theoristic values as a *"cancerous growth"* (Monk p439) not just within mathematics, but within our culture and civilisation. In pointing out Cantor's flaws - he describes the theory (the diagonal procedure) as *"hocus pocus"* (see below), Wittgenstein offers not a solution, but, in an often quoted section of RFM (RFM third edition, II 22/23), an observation on the state of the world.

*22. the usual expression creates the fiction of a procedure, a method of ordering which, though applicable here, nevertheless fails to reach its goal because of the number of objects involved, which is greater even than the number of all cardinal numbers.*

*If it were said: "Consideration of the diagonal procedure shews you that the concept 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, are inclined to believe", that would have a good and honest sense. But just the opposite happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. I believe, and hope, that a future generation will laugh at this hocus pocus.*

*23. The sickness of a time is cured by an alteration in the mode of life of human beings, and it was possible for the sickness of philosophical problems to get cured only through a changed mode of thought and of life, not through a medicine invented by an individual.*
*Think of the use of the motor-car producing or encouraging certain sicknesses, and mankind being plagued by such sickness until, from some or other, as a result of some development or other, it abandons the habit of driving.*

(RFM third edition part II)

And here we come full circle, in many ways, to the marginalia on page 29 of Hardy's CPM where we see the use of the word "prude" applied in response to the paragraphs that deal with sections of real numbers. Daesuk Han picks up directly on this point and the issue being raised in

22 above, in his paper *Wittgenstein and the Real Numbers*[27] and his summary in the abstract is as good an encapsulation that is currently available of Wittgenstein's position. He states:

*"A real number, as an "extension" is a homeless fiction: 'homeless' in that it neither is supported by anything nor supports anything. The picture of a real number as an 'extension' is not supported by actual practice in calculus; calculus has nothing to do with 'extensions'. The extensional set-theoretic conception of a real number does not give a foundation for real analysis either. The so-called complete theory of real numbers, which is essentially an extensional approach, does not define (in any sense of the word) the set of real numbers so as to justify their completeness, despite the common belief to the contrary. The only correct foundation of real analysis consists in its being 'existential axiomatics'. And in real analysis, as existential axiomatics, a point on the real line need not be an 'extension'.*

And before we move on, a much-quoted comment of Wittgenstein that ties his remarks on prudishness into a the context being discussed by Han and Therrien:

*"we can't describe mathematics, we can only do it (and that of itself abolishes every set theory)"*

Philosophical remarks 159

Wittgenstein's withering criticism of Cantor's diagonal argument (based on the premise that the very idea that a set of "anything" could capture or represent "something" infinite is ridiculous), and his description of mathematicians as prudes in 1942 reflects a deep-seated continuity of views over many years and in some ways has contributed to the knee-jerk reaction of mathematicians towards Wittgenstein. If Cantor is a high priest of modern mathematics, then dissident views that might undermine this orthodoxy are understandably little tolerated [28].

The misuse of the concept of "infinity" by mathematicians is a recurrent theme over the course of Wittgenstein's work. I wonder what he would have made of an article published in the New Scientist last year, 60 years or more after his death. In August 2013 Amanda Gefter wrote the following as part of a very recent article titled *Infinity's end: time to ditch the never ending story*

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[27] Han's paper was published in the Aug 2010, History and Philosophy of Logic

[28] without being partisan, Floyd's current paper in draft form (available from her website) titled of the "Wittgenstein's Diagonal Argument: a variation on Cantor and Turing" is of great utility for anyone interested in the extension of Wittgenstein's technical standpoint

*"Studies of the quantum properties of black holes by Stephen Hawking and Jacob Bekenstein in the 1970s led to the development of the holographic principle, which makes the maximum amount of information that can fit into any volume of space-time proportional to roughly one quarter the area of its horizon. The largest number of informational bits a universe of our size can hold is about $10^{122}$. If the universe is indeed governed by the holographic principle, there is simply not enough room for infinity."*[29] [30]

Wittgenstein was not shy of criticizing major mathematical figures, and in addition to finding it difficult to accept Cantor's diagonal argument, Wittgenstein also took aim at Godel.

*"The philosophy of mathematics consists in an exact scrutiny of mathematical proofs. Not in surrounding mathematics with vapor"*

Philosophical Grammar 1974 p 367

By the time that Wittgenstein was writing FML he was used to being the target of criticism by mathematicians who took umbrage at the way in which he could be dismissive of Cantor and Godel. How could it be, these mathematicians would say (and still say), that anyone could fail but be overwhelmed by the authors of two of the greatest advances in modern

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[29] The article also quotes MIT's Max Tegmark who could well have been speaking from Wittgenstein's playbook when he states *"all structures that exist mathematically exist also physically"*.

[30] I wrote a letter to the New Scientist that was published a little after the article appeared. (published issue 2930)

*Dear Sirs,*

*Amanda Gefter's article (well written and well presented), reminded me of the furor that accompanied Ludwig Wittgenstein when he confronted the use of infinity and the infinite in mathematical propositions. In dissecting Cantor's theorem Wittgenstein took a view that is eerily similar to the points made by current scientists as diverse as Wildberger, Tegmark and Zeilberger who are quoted in Gefter's piece.*

*In addition, however, it occurs to me that Wittgenstein was also brave enough and big enough to question the inconsistency that sits right at the heart of so much of the philosophy of mathematics – namely that a system that demands so much rigor and empirically sustainable proof is willing to accept a concept such as infinity that has defied any and all attempts at even the most approximate proof.*

*Perhaps over 80 years after Wittgenstein made his controversial remarks, science (and some mathematicians) may well be coming around to his point of view.*

*Hilbert did indeed say "no one will drive us from the paradise which Cantor created for us". Wittgenstein's take was typically straightforward " if one person can see it as a paradise for mathematicians, why should not another see it as a joke"*

*With regards*
*Ilyas Khan*
*Fellow, University of Cambridge, Judge Business School.*

mathematics ? But to Wittgenstein, Godel's theorems were vapor, and the mathematicians who accorded such laudatory epithets to Godel's work were prudes of exactly the kind that are referred to in FML. I believe it is important to note Wittgenstein's approach and criticism as being aimed at the foundations of mathematics, and not the buildings that arose on those foundations.

Shankar makes it clear how Wittgenstein conveyed his views as being dismissive not of Godel's theorem as a theory (i.e. Wittgenstein had no interest in notational alchemy) but in the philosophical precept. Shankar's essay "Wittgenstein's Remarks on the Significance of Godel's Theorem[31]" details how Wittgenstein was being consistent when approaching Godel by wondering what the fuss was about. Stating the obvious and then codifying it was about as much as Godel (in Wittgenstein's view) had achieved. Shankar goes on in his essay to explain how Godel's overt mathematical Platonism would have been raised some further concerns on the part of Wittgenstein, and how a debate about Godel very quickly becomes a conversation about the theory's epistemological consequences. This ought not to be too surprising since Godel himself takes a similar view of his path towards the theorem[32] - indeed Wittgenstein has a lot to say in general about "impossibility" theorems.[33]

In this regard it is worth recalling Hardy's definition of two types of proof. Those that he calls "demonstration" and those that he calls "informal.[34]" It is the latter type of proofs that are extended by Godel (the essay was published in 1928 and so predates Godel's 1931 paper on the first incompleteness theorem) and which Wittgenstein is so dismissive of.

A dozen or so years before FML, Wittgenstein had written

*"Mathematics consists (entirely)[35] of calculations. In mathematics everything is algorithm, nothing meaning [nichts Bedeutung]; even when it seems there's meaning, because we appear to be speaking about mathematical things in words. What we're really doing in that case is simply constructing an algorithm with those words.*

(BT, pp. 748f.)

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[31] The essay is published in the collection "Godel's Theorem in focus" Routledge 1988 ed Stuart Shanker

[32] in particular see Hao Wang "From Mathematics to Philosophy" 1973

[33] see Mulholzer's paper on Wittgenstein and the Regular Heptagon (Grazer 2001 pp215/247)

[34] Hardy's Rouse Ball Lecture, delivered 1928. See pp 16 and 17 of the edition published in Mind vol XXXVIII, No 149

[35] I insert the parenthesis since I am aware of versions of this remark which do not include the word "entirely", removed by Wittgenstein, presumably due to the fact that not all of mathematics might be considered algorithmic

*In which sense can one say that the symbol f(a) is essentially complex ? First of all: if different rules hold for f and a, they are simply different symbols. However, if the rules relate to f(a) as a sign unity, then we do not have the right any longer to call a an independent symbol and f(a) a complex. Then, a is only a curlicue which can just as well be left out. This just comes down to saying: this case is not to be confused with the case where "a" can be replaced with a different sign. Thus f(a) is not a function of a, just as little as fox is a function of 'ox'. If one were to write this word in the form f(ox), then this in itself would be permissible but still misleading. The specification f(ox) is not a function of 'ox', is nothing other and cannot be anything other than a warning against this going astray or an indication of a grammatical difference.*

Wittgenstein 2003 pp251-253 The Voices of Wittgenstein: the Vienna Circle, London Routledge

Wittgenstein had made a similar point in his *Lectures on the Foundations of Mathematics* (p 239/40):

*'Consider Professor Hardy's article ("Mathematical Proof") and his remark that*
*"to mathematical propositions there corresponds – in some sense however sophisticated – a reality"....What is a reality ? We think of "reality" as something we can point to. ...Professor Hardy is comparing mathematical propositions to the propositions of physics. This comparison is extremely misleading'*

(See also Maddy, *Naturalism in Mathematics* – p167 - that discusses this point)

The "prudishness" of mathematicians that must have led to Wittgenstein's highlighting of Hardy on p29 is more than just a result of the lack of clarity in the use of language. Hardy may have been ambiguous, but elsewhere he is at great pains to be forcefully specific in his adherence to rigour and exactitude when crafting mathematical formulations.

**Closing Comments**

*"The ways of thought Wittgenstein is fighting are not, however, primarily the unwholesome influence on our thinking of certain lofty intellectual creations such as Cantor's Set-theory or behaviouristic psychology, but are only symptoms of a sickness not its cause. The cause is in the language-games and reflects in its turn the way of life"*

*(Von Wright, "Wittgenstein" p208)*

Amongst the increasing volumes that have been written about Wittgenstein (is there a philosopher that has attracted more research and scholarship in the past 50 years?) I still find Von Wright's austere summary of Wittgenstein in the final brief chapter of his 1980 book the most concise and convincing description of Wittgenstein's place in the pantheon of Western intellectual endeavour. Wright's comments that I have noted above, occur just before he selects a quotation of Wittgenstein from RFM that I find utterly compelling, and which helps to juxtapose the underlying thought processes that led to FML and the way that mathematicians are described as "prudes".

*"The sickness of a time is cured by an alteration on the mode of life of human beings, and it was possible for the sickness of philosophical problems to get cured only through a changed mode of thought and of life, not a medicine invented by an individual. Suppose the sue of the motor-car produces or encourages certain illnesses, and mankind is plagued by such illness until, from some cause or other, as a result of some development or other, it abandons the habit of driving"*

(RFM third edition p132)

Von Wright places Wittgenstein's work on the Philosophy of Mathematics as being a part of a fundamental observation about contemporary language and how language in turn is simply a reflection of the overall social environment in which we live. That the foundations of mathematics are awry does not lead to Wittgenstein suggesting solutions. He himself believes that change (in language) will only happen when society itself changes. The conclusion we are then led to is that the faults apparent within the grammar of propositions and proofs in mathematics are but a stark example of what is problematic within society as whole.

I started this paper by reproducing Michael Potter's comments towards the conclusion to his book *Reason's nearest kin*. Potter's admission (or is it an apology?) seems to me to bear directly on the core issue being addressed by Wittgenstein in his accusatory remarks about mathematics being prudish. If higher mathematics does indeed depend upon a basic confusion between the empirical and the metaphysical, then the forced compromises that are created through repeated rote like acceptances of precepts in an almost religious nature suggests that the label 'prudish' might be considered mild when taken in the context and time they were made.

## Acknowledgements

I wish to acknowledge my long-term debt to the pioneering work of Juliet Floyd on Wittgenstein and the Philosophy of Mathematics. She manages to combine a captivating and simple style of writing with deep-rooted scholarship and a keen eye for exactly the sort of balance that is important when reading Wittgenstein. That she is uncompromising is simply a symptom of her commitment to the sort of truth that Wittgenstein would have applauded.

Professor Arthur Gibson at the Department of Pure Mathematics and Mathematical Statistics at the University of Cambridge is a colleague of mine who was particularly gracious when looking at an early draft of this paper. His eagerly awaited book on Skinner's Wittgenstein papers will be a valuable addition to Wittgenstein Scholarship. He is a rare and valuable member of the Wittgenstein community in being a thoroughly accomplished and well-respected mathematician who happens also to be a Wittgenstein scholar. Arthur was generous with his time and his comments.

Alois Pichler at the University of Bergen is also a dearly appreciated friend and colleague to whom I am grateful for encouragement and gentle guidance on the subject matter.

Neither Arthur or Alois can be held accountable in any respect whatsoever for the opinions expressed in this paper, or any mistakes that have been made.

I am also grateful to Daesuk Han (with whom I corresponded, but have never met) for the brave stance he has taken in his own paper "Wittgenstein and the Real Numbers". I should also acknowledge Han's reference to RW Hamming's words on Hilbert and Cantor for a unique, highly acute, and entertaining glimpse of the views of a mathematician rather than a Philosopher. I would not have thought to research Hamming as seriously as I have were it not for Han's reference. Quite apart from the relevance of Hamming's legacy on the points discussed in this paper, I have to thank Han for re-introducing me afresh to this delightfully original mathematician and intellectual tour de force.

We may finally be moving away from the nonsense that Wittgenstein was not a competent enough mathematician to analyse and comment on the Philosophy of mathematics and Han represents, for me, a new generation of scholarship that will not be hampered by such views. (I am always reminded when thinking of this point, about Wittgenstein's close association and friendships with Keynes, Russell, Turing and Hardy, but above all Ramsey, none of whom found fault or anything lacking in Wittgenstein's mathematical abilities. If anything quite the reverse).

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