Filling in the Blanks: Interbank Linkages and Systemic Risk

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Motivation

- Interbank contagion is central, but bilateral linkages often unknown
- Standard: estimate counterparty exposures by maximum entropy
- Yet spreading exposures as evenly as possible can be misleading:
  - Conceals “true” structure of linkages in network analysis
  - Diversification assumption causes bias in systemic stress tests
- This short paper proposes opposite benchmark: minimum density
- Produces a highly concentrated sparse network that
  - Retains some of the original network structure, and
  - Provides useful robustness bounds for systemic stress tests
### Actual Data

#### True Network

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>A_i</th>
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**Density 33%**

### Estimated Networks

#### Maximum Entropy Solution

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**Density 62%**

#### Minimum Density Solution

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**Density 21%**
Roadmap

- Part I: The minimum density approach (MD)
- Part II: Network features of estimated benchmarks (ME, MD) versus the "true" interbank network
- Part III: Performance in a systemic stress test
Part I: Maximum Entropy vs Minimum Density

- Notation:
  - Interbank network: \( \mathbf{X} \in [0, \infty)^{N \times N} \)
  - Bilateral exposures: \( X_{ij} \geq 0 (X_{ii} = 0) \)
  - Interbank assets: \( A_i = \sum_j X_{ij} \)
  - Interbank liabilities \( L_i = \sum_j X_{ji} \) = "marginals"

- Suppose we only know the marginals \( A_i \) and \( L_i \) for each of the \( N \) banks
  \( \rightarrow \) estimate max entropy \( \mathbb{E} \) and min density solution \( \mathbb{Z} \) on marginals

- **Entropy**: find matrix \( \mathbb{E} \) that satisfies the marginals, given "prior" \( Q_{ij} = A_i L_j \):

\[
\min_{\mathbb{E}} \sum_{i,j} E_{ij} \log \frac{E_{ij}}{Q_{ij}} \quad \text{s.t.} \quad \sum_j E_{ij} = A_i \quad \text{and} \quad \sum_j E_{ji} = L_i
\]
The Maximum Entropy solution $\mathbb{E}$

- **Advantages:**
  - Implementable using a standard iterative algorithm (RAS)
  - Yields a unique solution for $\mathbb{E}$

- **Disadvantages:**
  - Is optimal only if *nothing else* is known about a network
  - Completeness contradicts facts of real interbank networks:
    - Sparsity (Bech Atalay 2010), Tiering (Craig von Peter 2014), relationships (Cocco et al 2009), with disassortative features.
  - When diversification reduces contagion, entropy underestimates systemic risk (Mistrulli 2011, Markose 2012)
  - $\rightarrow$ Case for an alternative benchmark
The Minimum Density method

- **Premise**: Network linkages are costly and based on relationships.

- **Efficiency**: Minimally connected network s.t. satisfying marginals:

$$\min_{\mathbb{Z}} \sum_{i=1}^{N} \sum_{j \neq i}^{N} c \times 1_{[Z_{ij} > 0]} \quad \text{s.t.}$$

$$\sum_{j=1}^{N} Z_{ij} = A_i \quad \forall i = 1, 2, \ldots, N$$

$$\sum_{j=1}^{N} Z_{ij} = L_i \quad \forall i = 1, 2, \ldots, N$$

$$Z_{ij} \geq 0 \quad \forall i, j$$

- Analogous to transport network design problems: NP-hard (O’Kelly 2012)
Approach guided by two main ideas

- Robust choice under uncertainty $\rightarrow$ multinomial logit function

$$\max_{p \in \Delta} \left[ v'p - \delta v(p, q) \right] \quad \Rightarrow \quad p^*_i = \frac{q_i e^{v_i/\delta}}{\sum_{j \in \mu} q_i e^{v_j/\delta}}$$

- Prior $\sim$ economic incentives: focus on disassortative relationships matching large surpluses with small deficits and v.v.

$$Q_{ij} \propto \max \left\{ \frac{AD_i}{LD_j}, \frac{LD_j}{AD_i} \right\} \quad \forall i, j \in \mu.$$  

- $i \rightarrow j$ if big lender to small borrower, or small lender to big borrower
- Algorithm identifies probable links to load to the maximum extent.
The Minimum Density Algorithm

Complexity rises exponentially \(2^N\) even before allocating value → algorithm

1) Compute current deficits \(AD_i = (\Sigma_j Z_{ij} - A_i), \ LD_i = (\Sigma_j Z_{ji} - L_i)\)

2) Select link \((i, j)\) according to probability \(Q_{ij} \propto \max\left\{\frac{AD_i}{LD_j}, \frac{LD_j}{AD_i}\right\} \ \forall i, j\)

3) Load exposure \(Z_{ij} = \lambda \times \min\{AD_i, LD_j\}\) with \(\lambda = 1, \) or less*

If \(V(Z' = Z + Z_{ij}) \geq V(Z)\), then accept link \(Z_{ij}\)

4) Update set of priors \(Q_{ij}\) as in steps 1-2)

5) Iterate until 100% volume is allocated \(\Sigma \Sigma Z_{ij} = \Sigma \Sigma X_{ij} = \Sigma A_i\)
   - Interbank assets matched: \(\Sigma_j Z_{ij} = A_i \ \forall i\)
   - Interbank liabilities matched: \(\Sigma_j Z_{ji} = L_i \ \forall i\)

* We can generate “low density” solutions using \(\lambda < 1.\)
Part II: Comparing benchmarks with the original network

- The “true” interbank network constructed from Bundesbank data
  - “Gross- und Millionenkreditstatistik” between 2000+ banks
  - All large (≥€ 1.5m) or concentrated (>10% K) exposures
  - Consolidated at the bank holding company level (“Konzern”) and excluding cross-border linkages

- Basic features: large market ($N = 1779$), considerable volume (>1 trillion), sparse (density=0.59%) core-periphery structure

- Maximum Entropy (ME) blurs network structure (density 93%)

- Minimum Density (MD) solution is “too” efficient (density 0.1%)
Figure 2: The figure shows the concentration of value on the largest links for the different networks. The x-axis ranks bilateral linkages (in descending order of size) and expresses the first n links as a share of the total number of links in the original network X (18624). The y-axis shows the cumulative share of value allocated to the largest n links, relative to total interbank volume. The dots indicate at which point 100% of volume has been reached. For X this is at unity, for Z this occurs at 0.185, whereas E needs 158 times the number of links in X before reaching 100% of interbank volume.
MD preserves *some* structural features – ME fails to do so

<table>
<thead>
<tr>
<th>Network Characteristic</th>
<th>Max Entropy</th>
<th>True Network</th>
<th>Min Density</th>
<th>Low Density</th>
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<tbody>
<tr>
<td></td>
<td>E</td>
<td>X</td>
<td>Z</td>
<td>Y</td>
</tr>
<tr>
<td>Density, in %</td>
<td>92.8</td>
<td>0.59</td>
<td>0.11</td>
<td>0.61</td>
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<tr>
<td>Degree (average)</td>
<td>1649</td>
<td>10.5</td>
<td>1.94</td>
<td>10.9</td>
</tr>
<tr>
<td>Degree (median)</td>
<td>1710</td>
<td>6</td>
<td>1</td>
<td>4</td>
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<tr>
<td>Assortativity</td>
<td>-0.03</td>
<td>-0.53</td>
<td>-0.40</td>
<td>-0.32</td>
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<tr>
<td>Dependence when borrowing, %</td>
<td>12.2</td>
<td>84.7</td>
<td>97.3</td>
<td>93.4</td>
</tr>
<tr>
<td>Dependence when lending, %</td>
<td>7.2</td>
<td>45.1</td>
<td>97.4</td>
<td>87.2</td>
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<td>Clustering local average, %</td>
<td>99.8</td>
<td>33.4</td>
<td>0.03</td>
<td>7.62</td>
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<tr>
<td>Core size, % banks</td>
<td>92.6</td>
<td>2.5</td>
<td>1.1</td>
<td>2.1</td>
</tr>
<tr>
<td>Error score, % links</td>
<td>14.6</td>
<td>9.2</td>
<td>41.2</td>
<td>35.7</td>
</tr>
</tbody>
</table>

Table 1: Comparing basic network features of benchmark estimates with those of the original German interbank network.
Figure 3: The figure displays the degree distribution in its cumulative form, showing the number of banks with degree greater than the number shown on the x-axis, on a double log scale. A straight line would indicate a Pareto cumulative indicative of a power law distribution. The degree distribution of the original network X has been smoothed to preserve the confidentiality of individual bank data and shows averages at the end-points instead.
Figure 4: This figure shows three network features for 65 different low density solutions $\mathcal{Y}$. The implementation here sets $\lambda = 0.5$ for the first $k$ links being filled by the algorithm and $\lambda = 1$ thereafter, with $k$ raised from 0 to 100,000 in 65 (unequally spaced) steps. The first realization (at $k = 0$) is the minimum density network $\mathcal{Z}$ with the network features shown as red dots (as in Table 1). The black circles indicate the values for the original network $\mathcal{X}$, plotted at the point where a comparable low density network $\mathcal{Y}$ reaches a density similar to $\mathcal{X}$ (at $k=16,000$).
Part III: Performance in systemic stress tests

- Run stress tests to compare ME, MD with “true” network in practice

- Standard simulation methodology – ingredients:
  - Trigger: single bank failure (+ a capital shock)
  - Mechanism: (1) sequential default algorithm, and
  - (2) Eisenberg-Noe clearing vector (endogenous LGD + cost $\beta$)

- Let each of 1779 banks fail 1x1 and measure contagion (if any):
  - # banks in default as a consequence of contagion (excluding $i$)
  - System assets affected, and deadweight loss (€ bn)
  - Report average over $i$’s, and repeat for higher LGD or costs $\beta$.

- Evaluate how close contagion in $\mathbb{E}, \mathbb{Z}$ is to “true contagion” in $\mathbb{X}$.
Test 1: Sequential default algorithm

Contagious defaults (#)

System assets affected (€ bn)

Deadweight loss (€ bn)

Loss Given Default (%)
Test 2: Clearing vector methodology (Eisenberg-Noe)
Interpretation

- Results are similar across contagion methodologies...
- ...but differ across interbank networks $\mathbb{E}, \mathbb{Z}, \mathbb{X}$ (the inputs)

- Max entropy $\mathbb{E}$ underestimates systemic stress substantially:
  - Diversified exposures, smaller losses can be absorbed
- Min density $\mathbb{Z}$ overstates contagion for most of parameter space
  - + Concentrated exposures, failure may kill the counterparty
  - − Sparsity: fewer conduits for the propagation of losses
  - Former dominates due to negative assortativity in $\mathbb{Z}$ and $\mathbb{X}$

- In line with earlier findings on bias (Mistrulli 2011)... 
- ...but no tipping point in sight (Nier et al 2007, Gai et al 2011).
Conclusion

- The paper had a simple goal: to provide a meaningful alternative to the maximum entropy benchmark for estimating counterparty exposures.
- Min density solution uses information theory and economic rationale.
- The solution retains more structural features of the original network.
- In stress testing applications:
  - ME understates contagion, whereas MD generally overstates it.
  - Using ME & MD jointly delivers a useful confidence interval, and
  - MD also allows for many interior outcomes (low density solutions).

The broad interval shows: pattern of linkages matters for systemic risk!

Thanks for your attention.