Measuring interconnectedness between financial institutions with Bayesian time-varying vector autoregressions


Marco Valerio Geraci ¹,²  Jean-Yves Gnabo ²

¹ECARES, Université libre de Bruxelles

²CeReFiM, University of Namur

13 September 2016
Motivation

The financial crisis highlighted the importance of **interconnectedness**

“A bank’s **systemic impact** is likely to be positively related to its **interconnectedness** vis-à-vis other financial institutions.”

- Basel Committee on Banking Supervision (2013)

Knowing how firms are interconnected can help identify potential channels of **contagion**

**Problem:**

1. We do not observe true connections given by the **network of direct and indirect spillovers** (Adrian and Brunnermeier, 2016)
   - Direct spillovers:
     - Contractual obligations, asset & liability exposures, derivatives
   - Indirect spillovers:
     - Common portfolio holdings, fire sales

2. Connections are **time varying**
Motivation

The financial crisis highlighted the importance of interconnectedness

“A bank’s systemic impact is likely to be positively related to its interconnectedness vis-à-vis other financial institutions.”

- Basel Committee on Banking Supervision (2013)

Knowing how firms are interconnected can help identify potential channels of contagion

Problem:

1. We do not observe true connections given by the network of direct and indirect spillovers (Adrian and Brunnermeier, 2016)
   - Direct spillovers:
     - Contractual obligations, asset & liability exposures, derivatives
   - Indirect spillovers:
     - Common portfolio holdings, fire sales

2. Connections are time varying
Motivation

The financial crisis highlighted the importance of interconnectedness

“A bank’s systemic impact is likely to be positively related to its interconnectedness vis-à-vis other financial institutions.”

- Basel Committee on Banking Supervision (2013)

Knowing how firms are interconnected can help identify potential channels of contagion

Problem:

1. We do not observe true connections given by the network of direct and indirect spillovers (Adrian and Brunnermeier, 2016)
   - Direct spillovers:
     - Contractual obligations, asset & liability exposures, derivatives
   - Indirect spillovers:
     - Common portfolio holdings, fire sales

2. Connections are time varying
Motivation

The financial crisis highlighted the importance of interconnectedness

“A bank’s systemic impact is likely to be positively related to its interconnectedness vis-à-vis other financial institutions.”

- Basel Committee on Banking Supervision (2013)

Knowing how firms are interconnected can help identify potential channels of contagion

Problem:

1. We do not observe true connections given by the network of direct and indirect spillovers (Adrian and Brunnermeier, 2016)
   - Direct spillovers:
     - Contractual obligations, asset & liability exposures, derivatives
   - Indirect spillovers:
     - Common portfolio holdings, fire sales

2. Connections are time varying
Motivation

The financial crisis highlighted the importance of interconnectedness

“A bank’s systemic impact is likely to be positively related to its interconnectedness vis-à-vis other financial institutions.”

- Basel Committee on Banking Supervision (2013)

Knowing how firms are interconnected can help identify potential channels of contagion

Problem:

1. We do not observe true connections given by the network of direct and indirect spillovers (Adrian and Brunnermeier, 2016)
   - Direct spillovers:
     - Contractual obligations, asset & liability exposures, derivatives
   - Indirect spillovers:
     - Common portfolio holdings, fire sales

2. Connections are time varying
Motivation

The financial crisis highlighted the importance of **interconnectedness**

“A bank’s **systemic impact is likely to be positively related to its interconnectedness vis-à-vis other financial institutions.**”

- Basel Committee on Banking Supervision (2013)

Knowing how firms are interconnected can help identify potential channels of **contagion**

**Problem:**

1. We do not observe true connections given by the **network of direct and indirect spillovers** *(Adrian and Brunnermeier, 2016)*
   - Direct spillovers:
     - Contractual obligations, asset & liability exposures, derivatives
   - Indirect spillovers:
     - Common portfolio holdings, fire sales

2. Connections are **time varying**
Goal

Develop a framework to estimate interconnectedness that can account for time-varying connections
Previous Studies

Market-based measures of interconnectedness use **stock price data** and measures of **statistical association**

- Contemporaneous dependencies: (e.g. correlation, tail dependence)
  - Adams et al. (2014); Acharya et al. (2012, 2010); Adrian and Brunnermeier (2016); Brownlees and Engle (2016); Balla et al. (2014); Dungey et al. (2013); Hautsch et al. (2015); Peltonen et al. (2015)

- Temporal dependencies: (e.g. Granger causality, vector autoregressions)
  - Barigozzi and Brownlees (2016); Barigozzi and Hallin (2015); Billio et al. (2012); Diebold and Yılmaz (2009, 2014)

We propose a framework to model both contemporaneous and temporal dependencies
Previous Studies

Market-based measures of interconnectedness use **stock price data** and measures of **statistical association**

- **Contemporaneous dependencies**: (e.g. correlation, tail dependence)
  - Adams et al. (2014); Acharya et al. (2012, 2010); Adrian and Brunnermeier (2016); Brownlees and Engle (2016); Balla et al. (2014); Dungey et al. (2013); Hautsch et al. (2015); Peltonen et al. (2015)

- **Temporal dependencies**: (e.g. Granger causality, vector autoregressions)
  - Barigozzi and Brownlees (2016); Barigozzi and Hallin (2015); Billio et al. (2012); Diebold and Yılmaz (2009, 2014)

We propose a framework to model both contemporaneous and temporal dependencies
Market-based measures of interconnectedness use **stock price data** and measures of **statistical association**

- **Contemporaneous dependencies**: (e.g. correlation, tail dependence)
  - Adams et al. (2014); Acharya et al. (2012, 2010); Adrian and Brunnermeier (2016); Brownlees and Engle (2016); Balla et al. (2014); Dungey et al. (2013); Hautsch et al. (2015); Peltonen et al. (2015)

- **Temporal dependencies**: (e.g. Granger causality, vector autoregressions)
  - Barigozzi and Brownlees (2016); Barigozzi and Hallin (2015); Billio et al. (2012); Diebold and Yılmaz (2009, 2014)

*We propose a framework to model both contemporaneous and temporal dependencies*
Previous Studies

Previous studies have used time-invariant measures of statistical association to infer interconnectedness

- e.g. Billio et al. (2012) use Granger causality

Granger causality is an in-sample test, based on $T$ observations

- If the strength/direction of causality changes in $[0, T]$, the test inference is affected

Simple solution: adopt rolling windows but this is subject to limitations

- Reduces degrees of freedom $\Rightarrow$ costly in high-dimensional systems
- Susceptible to outliers (Zivot and Wang, 2006)
- Window size $\Rightarrow$ trade-off bias vs. precision (Clark and McCracken, 2009)

We propose a framework that accounts for time-varying nature of connections
Previous Studies

Previous studies have used time-invariant measures of statistical association to infer interconnectedness

- e.g. Billio et al. (2012) use Granger causality

Granger causality is an in-sample test, based on $T$ observations

- If the strength/direction of causality changes in $[0, T]$, the test inference is affected

Simple solution: adopt rolling windows but this is subject to limitations

- Reduces degrees of freedom $\Rightarrow$ costly in high-dimensional systems
- Susceptible to outliers (Zivot and Wang, 2006)
- Window size $\Rightarrow$ trade-off bias vs. precision (Clark and McCracken, 2009)

We propose a framework that accounts for time-varying nature of connections
Previous Studies

Previous studies have used **time-invariant** measures of statistical association to infer interconnectedness

- e.g. *Billio et al. (2012)* use **Granger causality**

Granger causality is an in-sample test, based on $T$ observations

- If the strength/direction of causality changes in $[0, T]$, the test inference is affected

Simple solution: adopt **rolling windows** but this is subject to **limitations**

- Reduces degrees of freedom $\Rightarrow$ costly in high-dimensional systems
- Susceptible to outliers (*Zivot and Wang, 2006*)
- Window size $\Rightarrow$ trade-off bias vs. precision (*Clark and McCracken, 2009*)

We propose a framework that accounts for time-varying nature of connections
We propose a market-based framework for measuring interconnectedness

1. The framework accounts for the \textbf{time-varying} nature of connections
   - Does not rely on rolling windows

2. The framework models both contemporaneous and temporal dependencies

3. Our TVP-VAR model accounts for the properties of asset returns
   - heteroskedasticity, fat-tails and skewness of asset returns
Main findings

- Assess TVP framework in simulation exercises against the classical approach of Granger causality testing on rolling windows (GC+RW)
  - Our TVP framework performs well vs. GC+RW
    - In terms of the precision in estimating connection strength
    - In terms of determining the presence/absence of a connection

- Estimate interconnectedness for the US financial system between 1990-2014
  - At the aggregate level: between banks, broker-dealers, insurers, real estate companies
  - At the disaggregated level: between 20 systemically important financial institutions (SIFIs)
Main findings

Estimate interconnectedness for the US financial system between 1990-2014

1. Measures of connectivity and centrality computed using the TVP framework are less volatile than the rolling window approach
   - The rolling window approach is more sensitive to extreme observations

2. Banks were the largest contributors to financial spillovers
   - Whereas real estate companies were the most influenced

3. The time-varying parameter framework produces stable rankings
   - More stable than rankings produced by the rolling window approach
   - More stable than rankings produced by other market-based measures (e.g. Marginal expected shortfall (MES), Beta)
   - More reactive than book-value measures (e.g. Leverage)

4. Key financial institutions were identified
   - American International Group, Goldman Sachs, and Merrill Lynch among largest propagators
   - Bear Stearns among the largest receivers
Empirical methodology

Estimating networks by Classical Granger Causality

We parallel measures of interconnectedness based on Granger causality testing (Billio et al., 2012)

Let \( R_t = [r_{1,t}, \ldots, r_{N,t}] \) be a vector of returns

- Draw a directional edge \((i \rightarrow j)\) if \( r_i \) Granger causes \( r_j \)

Granger causality can be tested by running the VAR

\[
R_t = c + \sum_{s=1}^{p} B_s R_{t-s} + u_t,
\]

and testing

\[
H_0 : b_{1}^{(j,i)} = b_{2}^{(j,i)} = \cdots = b_{p}^{(j,i)} = 0.
\]

This is a conditional Granger causality test (Geweke, 1984)
Empirical methodology

We adopt the TVP-VAR framework (Primiceri, 2005)

\[ R_t = c_t + \sum_{s=1}^{p} B_{s,t} R_{t-1} + u_t \equiv X_t' \theta_t + u_t, \quad u_t \sim t_\nu (0, \Xi_t), \]

where \( X_t' = I_N \otimes [1, R_{t-1}', \ldots, R_{t-1}'] \)

\[ \theta_{t+1} = \theta_t + \nu_{t+1}, \quad \nu_t \sim \mathcal{N} (0, Q_t), \]

- The off-diagonal elements of \( \Xi_t \) capture the time-varying contemporaneous dependencies
- The elements of \( B_{1,t}, \ldots, B_{p,t} \) capture the time-varying temporal dependencies
Empirical methodology

We adopt the TVP-VAR framework (Primiceri, 2005)

\[ R_t = c_t + \sum_{s=1}^{p} B_{s,t} R_{t-1} + u_t \equiv X_t' \theta_t + u_t, \quad u_t \sim t_{\nu}(0, \Xi_t), \]

where \( X_t' = I_N \otimes [1, R_{t-1}', \ldots, R_{t-1}'] \)

\[ \theta_{t+1} = \theta_t + \nu_{t+1}, \quad \nu_t \sim \mathcal{N}(0, Q_t), \]

- The off-diagonal elements of \( \Xi_t \) capture the time-varying **contemporaneous** dependencies
- The elements of \( B_{1,t}, \ldots, B_{p,t} \) capture the time-varying **temporal** dependencies
Empirical methodology

We adopt the TVP-VAR framework \textbf{(Primiceri, 2005)}

\[
R_t = c_t + \sum_{s=1}^{p} B_{s,t} R_{t-1} + u_t \equiv X'_t \theta_t + u_t, \quad u_t \sim t_{\nu}(0, \Xi_t),
\]

where \( X'_t = I_N \otimes [1, R'_{t-1}, \ldots, R'_{t-1}] \)

\[
\theta_{t+1} = \theta_t + \nu_{t+1}, \quad \nu_t \sim \mathcal{N}(0, Q_t),
\]

We assume \textbf{stochastic volatility} for the diagonal of \( \Xi_t \)

- We allow for a \textbf{leverage effects} between shocks to \textbf{stochastic volatility} and shocks to asset returns \( u_t \)
- This allows for skewness in the asset returns
Using **Bayes factor**, we evaluate the time-dependent hypothesis of no link between $i$ and $j$ at $t$

$$H_{0,t} : b_{1,t}^{(ji)} = b_{2,t}^{(ji)} \cdots = b_{p,t}^{(ji)} = 0.$$ 

We draw a **time-dependent** directional edge $(i \rightarrow_t j)$ if, given the posterior distribution of $B_t$, there is sufficient evidence against $H_{0,t}$.
Empirical analysis

We collected stock prices at monthly close for 155 financial institutions:
- banks, insurers, broker/dealers and real estate companies - SEC codes 6000 to 6799
- components of the S&P 500 between Jan 1990 and Dec 2014

We define monthly stock returns for firm $i$ at month $t$ as

$$r_{i,t} = \log\left(\frac{p_{i,t} + d_{i,t}}{p_{i,t-1}}\right),$$

We estimated the financial network at the aggregate level and at the disaggregated level:

- Aggregate level: four-variable TVP-VAR with sector indices
- Disaggregated level: pairwise bi-variate TVP-VARs between stock returns of 20 SIFIs
Results at the aggregate level: the sectorial network

Network density

Aggregate level: four-variable TVP-VAR(1) with sector indices

- **Network density** is smoothly varying rather than abruptly changing

\[
\text{Density}_t = \frac{1}{N(N-1)} \sum_{i=1}^{N_t} \sum_{j \neq i} (i \rightarrow_t j) \cdot |b_{ij}^{(i)}|,
\]

with \(i, j \in \{\text{Banks, Brokers, Insurers, Real Estate}\}\) and \(i \neq j\), where \(b_{ij}^{(i)}\) is the cross coefficient connecting \(i\) to \(j\), in period \(t\), in the TVP-VAR, and where, in this case, \(N = 4\)
Results at the aggregate level: the sectorial network

Network density

*Sectorial Network Density*

**Bold solid** = TVP; **Blue** = RW 36M
Results at the aggregate level: the sectorial network

Network density

Sectorial Network Density

Bold solid = TVP; Blue = RW 36M; Red = RW 24M
Results at the Disaggregated level: the SIFI network

Degree centrality

Disaggregated level: pairwise bi-variate TVP-VARs between 20 SIFIs

- SIFIs selected from FSB and Diebold and Yılmaz (2014)
- We compute in-degree and out-degree measures

\[
\text{In-Degree}_{i,t} = \frac{1}{(N_t - 1)} \sum_{j \neq i} (j \rightarrow_t i) \cdot |b_{t}^{(i,j)}|,
\]

\[
\text{Out-Degree}_{i,t} = \frac{1}{(N_t - 1)} \sum_{j \neq i} (i \rightarrow_t j) \cdot |b_{t}^{(j,i)}|,
\]

where, in this case, \( N_t \leq 20 \)

- We identified key players during the crisis
- We studied interconnectedness based rankings
Results at the Disaggregated level: the SIFI network

Ranking stability

We ranked firms according to their interconnectedness

- $Z_{i,t}^{in}$ is the ranking of institution $i$ at time $t$ in terms of **in-degree**
- $Z_{i,t}^{out}$ is the ranking of institution $i$ at time $t$ in terms of **out-degree**

The ranking can be used for monitoring and policy action

- e.g. the Financial Stability Board (FSB) and the BCBS ranks financial institutions according to their systemic importance
- The ranking is used to determine additional loss absorbency requirements
Results at the Disaggregated level: the SIFI network

Ranking stability

Rankings are unhelpful if they are prone to frequent unmotivated changes
- Daníelsson et al. (2015) and Dungey et al. (2013)

We computed measures of ranking stability

\[
SI^Q_{in} = \sqrt{\frac{1}{N_t(T-1)} \sum_{i=1}^{N_t} \left( Z_{i,t}^{in} - Z_{i,t-1}^{in} \right)^2}, \quad SI^A_{in} = \frac{1}{N_t(T-1)} \sum_{i=1}^{N_t} \left| Z_{i,t}^{in} - Z_{i,t-1}^{in} \right|
\]
Results at the Disaggregated level: the SIFI network

Ranking stability

<table>
<thead>
<tr>
<th></th>
<th>Stability Indicators</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>quadratic</td>
</tr>
<tr>
<td></td>
<td>$SI_Q^{in}$</td>
</tr>
<tr>
<td>Rolling windows</td>
<td>2.4</td>
</tr>
<tr>
<td>Time-varying parameter</td>
<td>1.2</td>
</tr>
</tbody>
</table>

**Average stability measures 1994-2014**

- Rankings based on rolling windows were more unstable
## Results at the Disaggregated level: the SIFI network

### Ranking stability

Average stability measures across all $t$

<table>
<thead>
<tr>
<th></th>
<th>$SI_Q$</th>
<th>$SI_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SRisk</strong></td>
<td>1.3</td>
<td>0.8</td>
</tr>
<tr>
<td><strong>Marginal expected shortfall</strong></td>
<td>3.1</td>
<td>2.3</td>
</tr>
<tr>
<td><strong>Leverage</strong></td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td><strong>Market beta</strong></td>
<td>3.1</td>
<td>2.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$SI_Q^{in}$</th>
<th>$SI_Q^{out}$</th>
<th>$SI_A^{in}$</th>
<th>$SI_A^{out}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rolling windows</strong></td>
<td>2.5</td>
<td>2.7</td>
<td>1.7</td>
<td>1.8</td>
</tr>
<tr>
<td><strong>Time-varying parameter</strong></td>
<td>1</td>
<td>1.1</td>
<td>0.6</td>
<td>0.7</td>
</tr>
</tbody>
</table>

*Average stability measures 2000-2014*

- Rankings based on TVP were more stable than MES and Beta (market data)
- Rankings based on TVP were less stable than Lev. (book value data)
Conclusion

Develop a market-based measure of interconnectedness

- Relies on **Bayesian estimation of time-varying parameter VARs**
  - Accounts for time-varying nature of connections
  - Models both temporal and contemporaneous dependencies
  - Accomodates many of the properties of asset returns (heteroskedasticity, skewness, heavy tails)

- Compared to classical rolling window approach
  - Less susceptible to extreme observations
  - Offers greater **flexibility**
  - Performs well in simulations

- Empirical analysis reveals limitations of rolling window approach
  - Rolling window connectivity and centrality measures are susceptible to outliers
  - Provide unstable interconnectedness rankings
Conclusion

Develop a market-based measure of interconnectedness

- Relies on **Bayesian estimation of time-varying parameter VARs**
  - Accounts for time-varying nature of connections
  - Models both temporal and contemporaneous dependencies
  - Accommodates many of the properties of asset returns (heteroskedasticity, skewness, heavy tails)

- Compared to classical rolling window approach
  - Less susceptible to extreme observations
  - Offers greater **flexibility**
  - Performs well in simulations

- Empirical analysis reveals limitations of rolling window approach
  - Rolling window connectivity and centrality measures are susceptible to outliers
  - Provide unstable interconnectedness rankings

Thank you
References I


References II


Appendix: Simulations

The Granger Causal Network (Seth, 2010)
Appendix: Simulations

\[ x_{1,t} = \alpha_{1,t} + \beta_{1,1,t}x_{1,t-1} + \epsilon_{1,t} \]
\[ x_{2,t} = \alpha_{2,t} + \beta_{2,1,t}x_{1,t-1} + \beta_{2,2,t}x_{2,t-1} + \epsilon_{2,t} \]
\[ x_{3,t} = \alpha_{3,t} + \beta_{3,1,t}x_{1,t-1} + \beta_{3,3,t}x_{3,t-1} + \epsilon_{3,t} \]
\[ x_{4,t} = \alpha_{4,t} + \beta_{4,1,t}x_{1,t-1} + \beta_{4,4,t}x_{1,t-1} + \beta_{4,5,t}x_{5,t-1} + \epsilon_{4,t} \]
\[ x_{5,t} = \alpha_{5,t} + \beta_{5,4,t}x_{4,t-1} + \beta_{5,5,t}x_{5,t-1} + \epsilon_{5,t} \]

where, \([\epsilon_{1,t} \ldots \epsilon_{5,t}]' = \epsilon_t \sim \mathcal{N}(0, R)\) and \(R = cI_5\) where \(c\) was set to 0.01.
Appendix: Experiment 1 - constant linkages

For the first experiment, we fix all regression parameters to constants drawn at the beginning of each simulation.

\[
\alpha_{i,t} = a_i \quad \forall t \in [0, T] \\
\beta_{i,j,t} = b_{i,j} \quad \forall t \in [0, T]
\]

where \(a_i\) and \(b_{i,j}\) are drawn from a \(U(0, 1)\) at the beginning of each simulation

\[
\forall (i, j) \in \{(2, 1), (3, 4), (3, 5), (4, 1), (4, 5), (5, 4)\} \cup \{i = j \mid i = 1, \ldots, 5\}
\]
Appendix: Experiment 1 - constant linkages

Pairwise testing

**MSE**

**ROC**

**PR Curve**

**Bold solid** = TVP; light dashed = rolling windows
Appendix: Experiment 1 - constant linkages

Conditional testing

**MSE**

**ROC**

**PR Curve**

**Bold solid** = TVP; light dashed = rolling windows

Go back
Appendix: Experiment 2 - markov switching linkages

For only the cross terms $i, j \in \{(2, 1), (3, 4), (3, 5), (4, 1), (4, 5), (5, 4)\}$

$$\beta_{i,j,t} = \begin{cases} 0 & s_{i,j}^t = 0 \\ b_{i,j} & s_{i,j}^t = 1 \end{cases}$$

Let $s_{i,j}^t$ follow a first order Markov chain with the following transition matrix:

$$P = \begin{bmatrix} P(s_{i,j}^t = 0 | s_{i,j}^{t-1} = 0) & P(s_{i,j}^t = 1 | s_{i,j}^{t-1} = 0) \\ P(s_{i,j}^t = 0 | s_{i,j}^{t-1} = 1) & P(s_{i,j}^t = 1 | s_{i,j}^{t-1} = 1) \end{bmatrix} = \begin{bmatrix} p_{00} & p_{10} \\ p_{01} & p_{11} \end{bmatrix}$$

where we set $p_{00} = 0.95$ and $p_{11} = 0.90$
Appendix: Experiment 2 - markov switching linkages

Pairwise testing

**MSE**

**ROC**

**PR Curve**

**Bold solid** = TVP; light dashed = rolling windows
Appendix: Experiment 2 - markov switching linkages

Conditional testing

MSE

ROC

PR Curve

**Bold solid** = TVP; light dashed = rolling windows
Appendix: Experiment 3 - random walk law of motion

\[ \alpha_{i,t+1} = \alpha_{i,t} + \omega_{i,t} \quad \omega_{i,t} \sim \mathcal{N}(0, c^2) \]

\[ \beta_{t+1} = \beta_{t} + \zeta_{t} \quad \zeta_{t} \sim \mathcal{N}(0, \tau_{i,j}^2) \]

where,

\[ \tau_{i,j}^2 = \begin{cases} 
3 \times c^2 & \text{if } i \neq j \\
2 \times c^2 & \text{if } i = j 
\end{cases} \]
Appendix: Experiment 3 - random walk law of motion

Pairwise testing

MSE

ROC

PR Curve

**Bold solid** = TVP; light dashed = rolling windows
Appendix: Experiment 3 - random walk law of motion

Conditional testing

**MSE**

**ROC**

**PR Curve**

**Bold solid** = TVP; **light dashed** = rolling windows
Assume the usual **lower triangular factorization** for the variance-covariance matrix,

\[ \Xi_t = A_t H_t A'_t \]

and let,

\[
H_t \equiv \begin{bmatrix}
  h_{1,t} & 0 & \cdots & 0 \\
  0 & h_{2,t} & \cdots & \vdots \\
  \vdots & \vdots & \ddots & 0 \\
  0 & \cdots & 0 & h_{n,t}
\end{bmatrix},
A_t \equiv \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  \alpha_{21,t} & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & 0 \\
  \alpha_{n1,t} & \cdots & \alpha_{n\,n-1,t} & 1
\end{bmatrix}
\]
Then \( h_t = [h_{1,t}, \ldots, h_{n,t}]' \) and \( \alpha_t = [\alpha_{1,2,t}, \ldots, \alpha_{n,n-1,t}]' \) evolve according to

\[
\ln h_t = \ln h_{t-1} + \eta_t \\
\alpha_t = \alpha_{t-1} + \tau_t
\]

This allows for stochastic volatility and time-varying contemporaneous dependencies in the shocks to returns.
The error term of the measurement equation is composed of two components,

$$ u_t = \Sigma_t \sqrt{\lambda_t} z_t $$

where

- $\nu / \lambda_t \sim \chi^2\nu$ and
- $z \sim N(0, I_n)$

It follows that,

$$ u_t \sim t_\nu(0, \Sigma_t), $$

The errors $[\varepsilon_t, \eta_t, \omega_t, \tau_t]'$ are jointly normal with mean zero and variance-covariance matrix $V$.
$$V = \begin{bmatrix} I & \Omega & 0 & 0 \\ \Omega & Z_\eta & 0 & 0 \\ 0 & 0 & Z_\omega & 0 \\ 0 & 0 & 0 & S \end{bmatrix}$$

where,

$$\Omega = \begin{bmatrix} \rho_1 \sigma_1 & 0 & \cdots & 0 \\ 0 & \rho_2 \sigma_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \rho_n \sigma_N \end{bmatrix}, Z_\eta = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix}, \text{ and}$$

$$Z_\omega = \begin{bmatrix} \sigma_{\omega,1} & 0 & \cdots & 0 \\ 0 & \sigma_{\omega,2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{\omega,N \cdot (1+N)} \end{bmatrix}$$

\(\Omega\) allows \(\varepsilon_t\) and \(\eta_t\), to be contemporaneously correlated row-by-row.