Bounds on the value of barrier options with curved boundaries

G. W. P. Thompson
Centre for Financial Research,
Judge Institute of Management,
University of Cambridge

Abstract

In this paper we consider the problem of pricing European options which contain a knock-out clause involving one or two curved boundaries. Using a change of measure and Jensen's inequality we derive upper and lower bounds in the form of double integrals, and demonstrate their accuracy with relevant numerical examples.

Keywords: Brownian motion, Barrier option

1 Introduction

A barrier option is a type of path-dependent or 'exotic' option, which like most such options cannot be priced analytically except in a few special cases. A simple example of a barrier option is a knock-out call option on a stock price; this is a European call option with an extra clause canceling the contract if the stock price hits a specified boundary. More complex is the 'double barrier' option, with both an upper and a lower knock-out boundary. Alternatively the contract might be 'knock-in', permitting exercise only if the stock price has hit the boundary before the option's expiry.

In this paper we will restrict our attention to single and double barrier knock-out options (the value of the corresponding knock-in can be found by subtracting the value of the knock-out from the value of the normal European option). Other types of barrier option which we will not consider include 'protected' barrier options, where the barrier clause is only effective for part of the time, 'rainbow' barrier options, where the barrier clause refers to the price of a second stock, and 'Parisian' options, where cancellation only occurs after the time spent above the boundary exceeds a threshold.

For the case of a constant boundary or an 'exponential' boundary (a boundary whose logarithm is a linear function of time), a simple analytic solution exists (see Goldman, Sosin

& Gatto (1979) or Musiela & Rutkowski (1997) for example) but except for these cases, explicit formulae cannot normally be found.

One reason for considering options with more general boundaries is to allow us to extend the Black-Scholes model to allow for a deterministic but time dependent volatility and interest-rate. The valuation of the standard European call option is no harder in this new model than before, but valuing even the simplest barrier option generally becomes intractable.

Numerical methods based on trees and lattices are very popular approaches to the problem of pricing barrier options, but these can require extensive computation to obtain accurate results (see Boyle & Lau (1994) for a discussion) unless a more ingenious approach is employed (see Rogers & Zane (1997) and Rogers & Stapleton (1997) for very successful methods).

Less numerical methods, aiming to leave the answer as an infinite series or as a single or double integral can also be very successful. In Kunitomo & Ikeda (1992) the case of two exponential boundaries is considered and a formula in the form of a rapidly convergent infinite series is derived. Another approach via Laplace transforms is pursued in Geman & Yor (1996), for the case of two constant boundaries. Methods applicable to problems with more general boundaries include that of Roberts & Shortland (1997) using hazard rate bounds, and Lo (1997) who uses a clever modification of the method of images to yield a simple explicit formula for the approximate value, together with a bound on the error. In principle, the method of Lo (1997) can be used to produce formulae with arbitrary precision. It also has the advantage of giving a *simple* pricing formulae rather than leaving the answer as an integral, though her method requires judicious parameter choices to achieve very narrow bounds. The method derived in this paper give a much more complex pricing formula, but is simple to use and also gives narrow bounds.

2 Notation

We assume that the stock price S_t follows a Black-Scholes model $S_t = S_0 \exp(\sigma B_t + \alpha t)$, where B_t is a standard Brownian motion, the volatility σ, the initial stock price S_0, and the interest-rate ρ, are positive constants and w.l.o.g. $\alpha = \rho - \frac{1}{2}\sigma^2$. The arbitrage-free value of the option is then $e^{-\rho T}\mathbb{E}[(S_T - K)_+ \, \mathrm{I}(\mathcal{A})]$ where T is the option's expiry time, K the strike price, $\mathrm{I}(\mathcal{A})$ denotes the indicator function of the event that the option is not knocked-out and $(\,\cdot\,)_+$ denotes $\max(\,\cdot\,, 0)$.

Among the various flavours of one-sided options, we will concentrate on the 'up-and-out' call option. This has a single knock-out boundary F_t, which satisfies $F_0 > S_0$; thus the contract is cancelled if $S_t \geq F_t$ for some $0 \leq t \leq T$. It is more convenient to have the boundary expressed in terms of the underlying Brownian motion B_t, so we define $f_t = \sigma^{-1}(\log(F_t/S_0) - \alpha t)$. With this definition, the up-and-out option is cancelled if and only if B_t ever hits f_t. The 'down-and-out' option, where the knock-out boundary is initially below the stock price can be handled similarly.

Turning to the two-sided knock-out option, we have two boundaries $G_t < F_t$, with $G_0 < S_0 < F_0$, and must cancel the contract if ever $S_t \notin (G_t, F_t)$. Defining f_t as above and g_t by $g_t = \sigma^{-1}(\log(G_t/S_0) - \alpha t)$, we cancel the contract if $B_t \notin (g_t, f_t)$ for some $0 \leq t \leq T$.

As mentioned previously, the 'up-and-in' variant option where the contract is worthless unless the boundary is hit, poses no additional problem since the sum of an up-and-in call and an up-and-out call is equivalent to a normal European call. We will denote by X the time-T value of the standard European call $(S_0 \exp(\sigma B_T + \alpha T) - K)_+$.

The only technical assumption we will need is that f_t, and g_t be twice-differentiable, in particular, they are continuous.

3 One-sided barrier options

In this section we consider the up-and-out call option, whose time-T value is $(S_0 \exp(\sigma B_T + \alpha T) - K)_+ \,\mathrm{I}(B_s < f_s, 0 \leq s \leq T)$, and will derive upper and lower bounds on the time-0 value in the form of double integrals.

Recall the definition $X = (S_0 \exp(\sigma B_T + \alpha T) - K)_+$ as the time-T value of the standard European call option, and that the time-0 value of the up-and-out option is $e^{-\rho T}\mathbb{E}[X \,\mathrm{I}(B_s < f_s, 0 \leq s \leq T)]$; we will attempt to bound $\mathbb{E}[X \,\mathrm{I}(B_s < f_s, 0 \leq s \leq T)]$. Let $\tilde{B}_t = B_t - f_t + f_0$ and define the probability measure $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} = \exp\left(\int_0^T f_t' \, dB_t - \frac{1}{2}\int_0^T (f_t')^2 \, dt\right)$. By the Cameron-Martin-Girsanov Theorem we have

$$\mathbb{E}\left[X \,\mathrm{I}(B_s < f_s, 0 \leq s \leq T)\right] = \tilde{\mathbb{E}}\left[e^{-\int_0^T f_t' \, d\tilde{B}_t - \frac{1}{2}\int_0^T (f_t')^2 \, dt} X \,\mathrm{I}(\tilde{B}_s < f_0, 0 \leq s \leq T)\right]$$

where $\tilde{B}$ is a $\tilde{\mathbb{P}}$-Brownian motion. Integrating $f_t' \, d\tilde{B}_t$ by parts we have

$$\mathbb{E}\left[X \,\mathrm{I}(B_s < f_s, 0 \leq s \leq T)\right] = e^{-\frac{1}{2}\int_0^T (f_t')^2 \, dt} \tilde{\mathbb{E}}\left[e^{\int_0^T f_t'' \tilde{B}_t \, dt - f_T' \tilde{B}_T} X \,\mathrm{I}(\tilde{B}_s < f_0, 0 \leq s \leq T)\right]. \tag{3.1}$$

Now recall Jensen's inequality for $\exp(y)$: for any random variable Y, $\mathbb{E}\exp(Y) \geq \exp(\mathbb{E}Y)$, and $\int_0^1 \exp\left(a(t)\right) dt \geq \exp\left(\int_0^1 a(t) \, dt\right)$ with equality if and only if Y (respectively a) is almost surely constant. Notice that if β is a non-negative random variable with $\mathbb{E}\beta > 0$ we have $\mathbb{E}(\exp(Y)\beta) \geq (\mathbb{E}\beta)\exp\bigl(\mathbb{E}(Y\beta)/(\mathbb{E}\beta)\bigr)$ and if $T > 0$ then $T^{-1}\int_0^T \exp\left(Ta(t)\right) dt \geq$

$\exp\left(\int_0^T a(t)\,dt\right)$. Using these forms of Jensen's inequality we can bound the expectation on the right-hand-side of (3.1) as follows

$$\left(\tilde{\mathbb{E}}\beta\right) e^{\left(\int_0^T \tilde{\mathbb{E}}(\gamma_t\beta)\,dt\right)/\left(\tilde{\mathbb{E}}\beta\right)} \leq \tilde{\mathbb{E}}\left(e^{\int_0^T \gamma_t\,dt}\beta\right) \leq \frac{1}{T}\int_0^T \tilde{\mathbb{E}}\left(e^{T\gamma_t}\beta\right)\,dt, \tag{3.2}$$

where $\gamma_t = f_t''\tilde{B}_t$ and $\beta = \exp\big(-f_T'\tilde{B}_T\big)X\,\mathrm{I}(\tilde{B}_s < f_0, 0 \leq s \leq T)$. Note that both of these bounds are exact if and only if $f_t'' = 0$ for almost all $t \in [0,T]$, i.e. when f is linear.

Defining

$$G(a, S_0, \sigma, K, u, t) = \mathbb{E}\left[e^{aW_t}\left(S_0 e^{\sigma W_t} - K\right)_+ \mathrm{I}(W_s < u, 0 \leq s \leq t)\right] \tag{3.3}$$

for a $\mathbb{P}$-Brownian motion W, we see that $\tilde{\mathbb{E}}\beta = G\left(-f_T', S_0\exp\left(\alpha T + \sigma(f_T - f_0)\right), \sigma, K, f_0, T\right)$. The function G can be written in terms of the normal distribution function Φ, as

$$\begin{aligned}
&G(a, S_0, \sigma, K, u, t) = \\
&\quad \mathrm{I}(l < u)\,\mathrm{I}(u > 0)S_0 e^{\frac{1}{2}(a+\sigma)^2 t}\left\{\left[\Phi\left(\frac{u-(a+\sigma)t}{\sqrt{t}}\right) - \Phi\left(\frac{l-(a+\sigma)t}{\sqrt{t}}\right)\right] + \right.\\
&\qquad\qquad \left. e^{2(a+\sigma)u}\left[\Phi\left(\frac{u+(a+\sigma)t}{\sqrt{t}}\right) - \Phi\left(\frac{2u-l+(a+\sigma)t}{\sqrt{t}}\right)\right]\right\} \\
&\quad - Ke^{\frac{1}{2}a^2 t}\left\{\left[\Phi\left(\frac{u-at}{\sqrt{t}}\right) - \Phi\left(\frac{l-at}{\sqrt{t}}\right)\right] + e^{2au}\left[\Phi\left(\frac{u+at}{\sqrt{t}}\right) - \Phi\left(\frac{2u-l+at}{\sqrt{t}}\right)\right]\right\}
\end{aligned}$$

where $l = \sigma^{-1}\log(K/S_0)$. We now consider the expressions $\tilde{\mathbb{E}}(\gamma_t\beta)$ and $\tilde{\mathbb{E}}(\exp(T\gamma_t)\beta)$. Conditioning on $\tilde{B}_t = x$ we have

$$\begin{aligned}
\tilde{\mathbb{E}}(\gamma_t\beta) &= \tilde{\mathbb{E}}\left[f_t''\tilde{B}_t e^{-f_T'\tilde{B}_T}X\,\mathrm{I}(\tilde{B}_s < f_0, 0 \leq s \leq T)\right] \\
&= f_t''\int_{-\infty}^{f_0} x\tilde{\mathbb{P}}\big(\tilde{B}_s < f_0, 0 \leq s \leq t, \text{ and } \tilde{B}_t \in dx\big) \\
&\qquad\qquad \times \tilde{\mathbb{E}}\big[e^{-f_T'\tilde{B}_T}X\,\mathrm{I}\big(\tilde{B}_s < f_0, t \leq s \leq T\big)\,\big|\,\tilde{B}_t = x\big],
\end{aligned} \tag{3.4}$$

and for $\tilde{\mathbb{E}}[\exp(T\gamma_t)\beta]$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}\big(e^{T\gamma_t}\beta\big) &= \tilde{\mathbb{E}}\big[e^{Tf_t''\tilde{B}_t}e^{-f_T'\tilde{B}_T}X\,\mathrm{I}(\tilde{B}_s < f_0, 0 \leq s \leq T)\big] \\
&= \int_{-\infty}^{f_0} e^{Txf_t''}\tilde{\mathbb{P}}\big(\tilde{B}_s < f_0, 0 \leq s \leq t, \text{ and } \tilde{B}_t \in dx\big) \\
&\qquad\qquad \times \tilde{\mathbb{E}}\big[e^{-f_T'\tilde{B}_T}X\,\mathrm{I}\big(\tilde{B}_s < f_0, t \leq s \leq T\big)\,\big|\,\tilde{B}_t = x\big].
\end{aligned} \tag{3.5}$$

To evaluate the integrands, we use the well-known result that

$$\tilde{\mathbb{P}}\big(\tilde{B}_s < f_0, 0 \leq s \leq t, \text{ and } \tilde{B}_t \in dx\big) = \mathrm{I}(x < f_0)\frac{1}{\sqrt{t}}\left[\phi\left(\frac{x}{\sqrt{t}}\right) - \phi\left(\frac{2f_0 - x}{\sqrt{t}}\right)\right]dx, \tag{3.6}$$

and observe that we can write the expectation on the right-hand side of (3.4) and (3.5) as

$$\tilde{\mathbb{E}}\left[e^{-f_T'(x+\tilde{W}_{T-t})}\left(S_0 e^{\sigma\left(x+\tilde{W}_{T-t}+f_T-f_0\right)+\alpha T}-K\right)_+ \mathrm{I}(\tilde{W}_s < f_0 - x, 0 \le s \le T-t)\right]$$
$$= e^{-xf_T'} G\left(-f_T', S_0 e^{\alpha T+\sigma(x+f_T-f_0)}, \sigma, K, f_0 - x, T-t\right)$$

where $\tilde{W}_s = \tilde{B}_{s+t} - \tilde{B}_t, 0 \le s \le T-t$ is a $\tilde{\mathbb{P}}$-Brownian motion, giving bounds in the form of a double integral.

4 Two-sided barrier options

Deriving bounds on the value of double-barrier options is slightly more complicated than the single-barrier case.

Recall that if X is the time-T value of a standard European call option, then the time-0 value of the two-sided knock-out option is given by $e^{-\rho T}\mathbb{E}[X\,\mathrm{I}(g_s < B_s < f_s, 0 \le s \le T)]$; we will try to bound $\mathbb{E}[X\,\mathrm{I}(g_s < B_s < f_s, 0 \le s \le T)]$. We start by transforming B to give a process with unit volatility, and under which the knock-out boundaries become constant (similar to the approach of Rogers & Zane (1997)). Define the process $Y_t = (B_t - g_t)/(f_t - g_t)$ so

$$dY_t = \frac{dB_t}{f_t - g_t} - \frac{dt}{f_t - g_t}\left[g_t' + Y_t\left(f_t' - g_t'\right)\right],$$

the time change τ_t, by $\tau_0^{-1} = 0$, $\left(\tau_s^{-1}\right)' = (f_s - g_s)^{-2}$ and finally set $Z_t = Y_{\tau_t}$ and $\tilde{T} = \tau_T^{-1}$. The process Z_t is then a diffusion with SDE

$$dZ_t = dW_t + (\zeta_t + \xi_t Z_t)\,dt, \qquad Z_0 = -g_0/(f_0 - g_0),$$

for some $\mathbb{P}$-Brownian motion W, where, writing $s = \tau_t$ we have

$$\zeta_t = -(f_s - g_s)g_s',$$
$$\xi_t = -(f_s - g_s)(f_s' - g_s').$$

We now follow the same path as in Section 3. Define the probability measure $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} = \exp\left(-\int_0^{\tilde{T}}(\zeta_t + \xi_t Z_t)\,dW_t - \frac{1}{2}\int_0^{\tilde{T}}(\zeta_t + \xi_t Z_t)^2\,dt\right)$ and use the Cameron-Martin-Girsanov Theorem to give

$$\mathbb{E}[X\,\mathrm{I}(g_s < B_s < f_s, 0 \le s \le T)] = \tilde{\mathbb{E}}\left[e^{\int_0^{\tilde{T}}(\zeta_t + \xi_t Z_t)\,dZ_t - \frac{1}{2}\int_0^{\tilde{T}}(\zeta_t + \xi_t Z_t)^2\,dt} X\,\mathrm{I}(0 < Z_u < 1, 0 \le u \le \tilde{T})\right]$$

where $Z_t - Z_0$ is now a Brownian motion under $\tilde{\mathbb{P}}$. From Itô's Lemma we have $d(\zeta_t Z_t) = \zeta_t' Z_t\,dt + \zeta_t\,dZ_t$ and $d(\xi_t Z_t^2) = \xi_t' Z_t^2\,dt + 2\xi_t Z_t\,dZ_t + \xi_t\,dt$ so

$$\mathbb{E}\left[X\,\mathrm{I}(g_s < B_s < f_s, 0 \le s \le T)\right] = e^{-\zeta_0 Z_0 - \frac{1}{2}\xi_0 Z_0^2 - \frac{1}{2}\int_0^{\tilde{T}}(\zeta_t^2+\xi_t)\,dt} \times \tilde{\mathbb{E}}\left[e^{\int_0^{\tilde{T}}\gamma_t\,dt}\beta\right], \tag{4.1}$$

where, again writing $s = \tau_t$ we have

$$\begin{aligned}\gamma_t &= -\left(\zeta_t' + \zeta_t\xi_t\right)Z_t - \tfrac{1}{2}\left(\xi_t' + \xi_t^2\right)Z_t^2\\ &= (f_s - g_s)^3\left(g_s'' Z_t + \tfrac{1}{2}(f_s'' - g_s'')Z_t^2\right),\\ \beta &= \exp\left(\zeta_{\tilde{T}} Z_{\tilde{T}} + \tfrac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2\right) X\,\mathrm{I}(0 < Z_u < 1, 0 \le u \le \tilde{T}).\end{aligned} \tag{4.2}$$

We now use inequalities (3.2), replacing T with $\tilde{T}$, to bound the expectation on the right-hand side of (4.1). From (4.2) we see that these bounds are exact if and only if both f and g are linear.

It remains to compute $\tilde{\mathbb{E}}\beta$, $\tilde{\mathbb{E}}(\gamma_t\beta)$ and $\tilde{\mathbb{E}}(\exp(T\gamma_t)\beta)$. To do this we need the result that if W is a Brownian motion and

$$P(l,u,t,x) = \mathbb{P}\left(W_t \in dx, \text{ and } l < W_s < u, 0 \le s \le t\right)$$

then P is given by

$$P(l,u,t,x) = \\ \mathrm{I}(l<0<u)\,\mathrm{I}(l<x<u)\frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty}\left[\phi\left(\frac{x+2n(u-l)}{\sqrt{t}}\right) - \phi\left(\frac{x-2u+2n(u-l)}{\sqrt{t}}\right)\right]dx \tag{4.3}$$

(see Revuz & Yor (1994), page 106 for example). (It is worth remarking that this infinite series converges very rapidly.) Since X can be expressed as

$$X = \left[S_0\exp\left(\alpha T + \sigma g_T + \sigma(f_T - g_T)Z_{\tilde{T}}\right) - K\right]^+,$$

we can write $\tilde{\mathbb{E}}\beta = \bar{G}\left(\zeta_{\tilde{T}}, \frac{1}{2}\xi_{\tilde{T}}, S_0\exp(\alpha T + \sigma g_T), \sigma(f_T - g_T), K, \tilde{T}, Z_0\right)$, where $\bar{G}$ is defined by

$$\bar{G}(a,\bar{a},S_0,\sigma,K,t,w_0) = \mathbb{E}\left(e^{aW_t + \bar{a}W_t^2}\left(S_0 e^{\sigma W_t} - K\right)_+ \mathrm{I}(0 < W_s < 1, 0 \le s \le t)\,|\,W_0 = w_0\right) \tag{4.4}$$

for $W - w_0$ a $\mathbb{P}$-Brownian motion. Using (4.3) we have

$$\bar{G}(a, \bar{a}, S_0, \sigma, K, t, w_0) = \mathrm{I}(m < 1)\,\mathrm{I}(0 < w_0 < 1) \times$$
$$\frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} \Big(S_0[H(m, \bar{a}, a+\sigma, -2n + w_0, 2t) - H(m, \bar{a}, a+\sigma, 2(1-w_0) - 2n + w_0, 2t)] -$$
$$K[H(m, \bar{a}, a, -2n + w_0, 2t) - H(m, \bar{a}, a, 2(1-w_0) - 2n + w_0, 2t)] \Big) \tag{4.5}$$

where $m = \max(\sigma^{-1}\log(K/S_0), 0)$ and the function H is defined by

$$\begin{aligned} H(a, \bar{a}, b, c, d) &= \frac{1}{\sqrt{(2\pi)}} \int_a^1 e^{bx + \bar{a}x^2 - (x-c)^2/d} \\ &= \sqrt{D} e^{2D[(b+\bar{a}c)c/d + b^2/4]} \left[\Phi\left(\sqrt{D}\left(\frac{2c}{d} + b\right) - \frac{a}{\sqrt{D}} \right) - \Phi\left(\sqrt{D}\left(\frac{2c}{d} + b\right) - \frac{1}{\sqrt{D}} \right) \right] \end{aligned} \tag{4.6}$$

and $D = \frac{1}{2}d/(1 - \bar{a}d)$, if $D > 0$, which was the case for our numerical examples. (If $D < 0$ a similar expression could be given involving Dawson's integral: $\int_0^x \exp(u^2)\,du$.)

We can calculate $\tilde{\mathbb{E}}(\gamma_t \beta)$ and $\tilde{\mathbb{E}}(\exp(T\gamma_t)\beta)$ by conditioning on $B_t = x$, as we did in Section 3. Thus for $\tilde{\mathbb{E}}(\gamma_t \beta)$ we have

$$\begin{aligned} &\tilde{\mathbb{E}}\Big[(f_s - g_s)^3 \big(g_s'' Z_t + \tfrac{1}{2}(f_s'' - g_s'')Z_t^2\big) e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}(0 < Z_u < 1, 0 \le u \le \tilde{T})\Big] = \\ &\qquad (f_s - g_s)^3 \int_{g_0}^{f_0} \big(g_s'' x + \tfrac{1}{2}(f_s'' - g_s'')x^2\big)\, \tilde{\mathbb{P}}\,(0 < Z_u < 1, 0 \le u \le t, \text{ and } Z_t \in dx) \\ &\qquad\qquad \times \tilde{\mathbb{E}}\left(e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}(0 < Z_u < 1, t \le u \le \tilde{T}) \;\Big|\; Z_t = x \right), \end{aligned}$$

which combined with (4.3)–(4.6), give a double integral, and for $\tilde{\mathbb{E}}\,(\exp(T\gamma_t \beta))$ we have

$$\begin{aligned} &\tilde{\mathbb{E}}\Big[e^{(f_s - g_s)^3 \left(g_s'' Z_t + \frac{1}{2}(f_s'' - g_s'')Z_t^2\right)} e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}(0 < Z_u < 1, 0 \le u \le \tilde{T})\Big] = \\ &\qquad \int_{g_0}^{f_0} e^{(f_s - g_s)^3 \left(g_s'' x + \frac{1}{2}(f_s'' - g_s'')x^2\right)} \tilde{\mathbb{P}}\,(0 < Z_u < 1, 0 \le u \le t, \text{ and } Z_t \in dx) \\ &\qquad\qquad \times \tilde{\mathbb{E}}\left(e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}(0 < Z_u < 1, t \le u \le \tilde{T}) \;\Big|\; Z_t = x \right). \end{aligned}$$

In each case, we can write the integrand in terms of $\bar{G}$ and P, since

$$\tilde{\mathbb{P}}(0 < Z_u < 1, 0 \le u \le t, \text{ and } Z_t \in dx) = P\big(-Z(0), 1 - Z(0), t, x - Z(0)\big)$$

and

$$\begin{aligned} &\tilde{\mathbb{E}}\big[e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}(0 < Z_u < 1, t \le u \le \tilde{T}) \,\big|\, Z_t = x\big] \\ &\qquad = \bar{G}\big(\zeta_{\tilde{T}}, \tfrac{1}{2}\xi_{\tilde{T}}, S_0 \exp(\alpha T + \sigma g_T), \sigma(f_T - g_T), K, \tilde{T} - t, x\big). \end{aligned}$$

5 Numerical examples

We consider a selection of one-sided and two-sided barrier problems. The integrals were evaluated using routines from the NAG library. Further details related to the numerical results and the source code are available from `http://www-cfr.jims.cam.ac.uk`.

5.1 One-sided barrier options

Roberts & Shortland (1997) consider an up-and-in call option with a constant knock-in boundary F and expiry time 1, on a stock with constant volatility but a non-constant interest-rate. Specifically, they assume that r_t has been perturbed from some equilibrium level r_∞, to which it returns via an exponential decay: $r_t = r_\infty + (r_0 - r_\infty)\exp(-ct)$, for some constant c.

The time-0 value of the option is

$$e^{-\int_0^1 r_u\,du}\mathbb{E}\left[\left(Se^{\sigma B_1+\int_0^1 r_u\,du-\frac{1}{2}\sigma^2}-K\right)_+ \mathrm{I}(B_t \le f_t, 0 \le t \le 1)\right]$$

where

$$\begin{aligned} f_t &= \sigma^{-1}\left(\log(F/S) - \int_0^t r_u\,du + \tfrac{1}{2}\sigma^2 t\right) \\ &= \sigma^{-1}\left[\log(F/S) - \left(r_\infty t + (r_0 - r_\infty)\left(1-e^{-ct}\right)/c - \tfrac{1}{2}\sigma^2 t\right)\right] \end{aligned}$$

The problem which Roberts & Shortland (1997) examine has the parameters $S_0 = 10, \sigma = 0.1, K = 11, r_0 = 0.15, r_\infty = 0.1, c = 1, F = 12$. As remarked earlier, the price of an up-and-in option is just the difference between the price of a standard European option and an up-and-out option with the same boundary.

Our bounds on the corresponding knock-out option evaluate to $[0.0781, 0.0791]$, taking about 0.17 seconds on an HP workstation. Subtracting these from the price of the standard European call option (in this case 0.595389) we deduce bounds of $[0.516289, 0.517289]$ on the value of the up-and-in option, with a width of 0.19%. A comparison of the method described here with the methods of Roberts & Shortland (1997) and Lo (1997) is shown in Table 1.

5.2 Two-sided barrier options

We now see how our bounds on two-sided knock-out options compare to the results of the two-sided barrier problems contained in Rogers & Zane (1997), Geman & Yor (1996), Kunitomo & Ikeda (1992) and Rogers & Stapleton (1997). Three types of knock-out boundaries are considered: (i) constant boundaries, $L < S_t < U$; (ii) exponential boundaries,

Method	Lower bound	Upper bound	Width
Roberts & Shortland	0.516758	0.517968	0.23%
Lo	0.516243	0.517556	0.25%
New	0.516289	0.517289	0.19%

Table 1 Bounds on the value of a one-sided barrier option.

$\log(L) + \delta_L t < \log(S_t) < \log(U) + \delta_U t$ and (iii) linear boundaries, $L + \delta_L t < S_t < U + \delta_L t$. Problems (i) and (ii) have analytic formulae in the form of infinite series (see Kunitomo & Ikeda (1992), or in terms of an inverse Laplace transform (see Geman & Yor (1996)), and as remarked above, our bounds are also exact in these cases. To solve these problems by performing the double integral presented here is much more time-consuming than using the alternative methods. It is the final case we are more interested in, where no analytic solution exists.

The results are presented in Tables 2–7. The methods of Rogers & Zane (1997) (RZ) and Rogers & Stapleton (1997) (RS) use a lattice, with size N, and become increasingly accurate as $N \to \infty$; the figures presented here correspond to the largest value of N they considered ($N = 3200$).

For the type (i) problems, the computation time of our bounds was between 0.18 and 0.24 seconds.

In Table 3 Kunitomo & Ikeda (1992) give a figure of 10.86 for problem (ii-8), which is not consistent with our bounds. Our implementation of their method gives 10.831. For the type (ii) problems, the numerical integration is more troublesome and we work to fewer significant figures. Here computation time is between 0.22 and 0.47 seconds.

In Table 7 we consider type (iii) problems, including those considered by Rogers & Zane (1997). The figures reported in Rogers & Zane (1997) for problems (iii-4) and (iii-5) are significantly different from our bounds. We implemented the algorithm of Rogers & Zane (1997) as described in their paper as closely as possible and report the results in the final column. We also consider some more extreme numerical examples, to demonstrate that the bounds are not always so accurate. The computation time for type (iii) problems is approximately 1–2 seconds in each case, with cases (iii-1) and (iii-2) taking longer than the others.

Problem	σ	ρ	K	L	U	S	T
(i-1)	0.2	0.02	2	1.5	2.5	2	1
(i-2)	0.5	0.05	2	1.5	3	2	1
(i-3)	0.5	0.05	1.75	1	3	2	1
(i-4)	0.2	0.02	100	75	125	100	1

Table 2 Parameter values for type (i) double-barrier problems (constant boundaries).

Problem	RZ	RS	GY	KI	Lower	Upper
(i-1)	0.041079	N/A	0.0411	0.041089	0.041088	0.041089
(i-2)	0.017837	N/A	0.0178	0.017856	0.017856	0.017857
(i-3)	0.076147	N/A	0.07615	0.076172	0.076171	0.076173
(i-4)	2.0539	2.0558	2.055	N/A	2.0544	2.0545

Table 3 Numerical results for type (i) double-barrier problems (constant boundaries).

Problem	σ	ρ	T	S	K	δ_U	δ_L	L	U
(ii-1)	0.2	0.05	0.5	1000	1000	0.1	-0.1	500	1500
(ii-2)	0.2	0.05	0.5	1000	1000	0.1	-0.1	600	1400
(ii-3)	0.2	0.05	0.5	1000	1000	0.1	-0.1	700	1300
(ii-4)	0.2	0.05	0.5	1000	1000	0.1	-0.1	800	1200
(ii-5)	0.2	0.05	0.5	1000	1000	-0.1	0.1	500	1500
(ii-6)	0.2	0.05	0.5	1000	1000	-0.1	0.1	600	1400
(ii-7)	0.2	0.05	0.5	1000	1000	-0.1	0.1	700	1300
(ii-8)	0.2	0.05	0.5	1000	1000	-0.1	0.1	800	1200
(ii-9)	0.25	0.1	1	95	100	0.1	-0.1	90	160

Table 4 Parameter values for type (ii) double-barrier problems (exponential boundaries)

Problem	RZ	RS	KI	Lower	Upper
(ii-1)	67.7834	N/A	67.78	67.71	67.85
(ii-2)	64.6401	N/A	64.63	64.56	64.70
(ii-3)	55.1992	N/A	55.20	55.14	55.26
(ii-4)	34.5713	N/A	34.58	34.54	34.62
(ii-5)	62.7532	N/A	62.75	62.68	62.82
(ii-6)	52.5021	N/A	52.50	52.44	52.55
(ii-7)	33.4429	N/A	33.45	33.41	33.49
(ii-8)	10.8217	N/A	10.86	10.82	10.85
(ii-9)	5.3680	5.3672	5.3679	5.362	5.374

Table 5 Numerical results for type (ii) double-barrier problems (exponential boundaries).

Problem	σ	ρ	T	S	K	U	L	δ_U	δ_L
(iii-1)	0.25	0.1	1	95	100	160	90	20	-20
(iii-2)	0.25	0.1	1	95	100	160	90	15	-15
(iii-3)	0.25	0.1	1	95	100	160	90	10	-10
(iii-4)	0.25	0.1	1	95	100	160	90	5	-5
(iii-5)	0.25	0.1	1	95	100	160	90	-5	5
(iii-6)	0.25	0.1	1	95	100	160	90	-10	10
(iii-7)	0.25	0.1	1	95	100	160	90	-15	15
(iii-8)	0.25	0.1	1	95	100	160	90	-20	20

Table 6 Parameter values for type (iii) double-barrier problems (linear boundaries).

Problem	RZ	Lower	Upper	Approximate
(iii-1)	N/A	6.402	6.603	6.40999
(iii-2)	N/A	5.751	5.784	5.75352
(iii-3)	N/A	5.036	5.040	5.03683
(iii-4)	4.3438	4.267	4.269	4.26779
(iii-5)	2.5438	2.637	2.638	2.63747
(iii-6)	N/A	1.831	1.832	1.83117
(iii-7)	N/A	1.090	1.091	1.09001
(iii-8)	N/A	0.490	0.493	0.49069

Table 7 Numerical results for type (iii) double-barrier problems (linear boundaries).

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