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Hedging European and Barrier Options Using Stochastic Optimization

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Abstract

We hedge European and Barrier options in a discrete time and discrete space setting by using stochastic optimization to minimize the mean downside hedge error under transaction costs. Scenario trees are generated using a method which ensures the absence of arbitrage and which matches the mean and variance of the underlying asset price in the sampled scenarios to those of a given distribution. The stochastic optimization based strategy is benchmarked to the method of delta hedging for the case where the underlying asset price follows a discretized geometric Brownian motion and implemented for the case where the underlying asset price is driven by a discretized Variance Gamma process.

1 Introduction

By definition, in a *complete* market every derivative can be hedged perfectly using a replicating trading strategy. However, in reality unpriced uncertainties and market frictions make most markets *incomplete* and the replication of some derivatives impossible. This makes a derivative genuinely risky and can make the problem of hedging it very challenging.

Methods based on the *greeks* (see e.g. Hull [11]) probably remain the most common form of hedging. In these methods a hedging strategy is sought which makes the combined position of the derivative and the hedging portfolio immune to small changes in factors such as the underlying asset price and volatility. While these methods are intuitive and practical, it can be difficult to devise a hedging strategy that accounts for all relevant underlying factors and market frictions such as transaction costs. Other methods of hedging derivatives in incomplete markets existing in the literature include superreplication (see e.g. El Karoui and Quenzez [4]), move based methods (see e.g. Martellini and Priaulet [16]), quantile hedging (Follmer and Leukert [5]), efficient hedging (Follmer and Leukert [6]) and utility based methods (Monoyios [17]). Naturally, each of these methods has its pros and cons.

In this paper we derive hedging strategies for European and Barrier options using *stochastic optimization*. This framework allows for different measures of the hedge error and thus different attitudes to risk and can easily handle various market frictions such as transaction costs and position limits. In addition, due to the simulation based nature of the methodology nearly arbitrary models for the underlying asset stochastics can be accommodated. We generate scenario trees using a method which ensures the absence of arbitrage and which matches the mean and variance of the underlying asset price in the sampled scenarios to those of a given distribution. We then benchmark the stochastic optimization based hedging strategy to delta hedging for the case where the underlying asset price follows a discretized geometric Brownian motion and implement it for the case where the underlying asset price is driven by a discretized Variance Gamma process.

Follmer and Schweitzer [7] and Bertsimas, Kogan and Lo [1] also develop optimization based hedging strategies, but both papers use a different measure of the hedge error, work within a frictionless market and use different solution techniques. This paper also complements the work of King [12] who formulates a superreplicating strategy using stochastic optimization, Dempster and Thompson [3] who use stochastic programming to track a portfolio of European options with a subset of the target portfolio and Gondzio et al. [9] who use stochastic optimization to maximize the "utility" of the hedge error under stochastic volatility and transaction costs. In the last paper there are few numerical examples and in particular only European options are studied.

2 Stochastic Optimization Framework

2.1 Set Up

There are $T+1$ times, indexed by $t = 0, \dots, T$, where T corresponds to the *maturity* of the derivative. Trading is allowed at all times except for maturity. Below we restrict the hedging strategy to positions in the underlying asset and cash, but the extension to an arbitrary set of hedging securities is straight forward (see Dempster and Thompson [3]). Uncertainty is represented by a finite set of *scenarios* denoted by Ω, which can in turn be represented in the form of a *scenario tree.* Let $pr(\omega)$ denote the probability of scenario $\omega \in \Omega$. An example scenario tree is given in Figure 1 for $T = 2$ and $|\Omega| = 4$.

By numbering the *nodes* (vertices in the scenario tree) as in Figure 2 the set of nodes can be indexed by $n = 0, \dots, N$ where $n = 0$ corresponds to the node at $t = 0$, and $n = N$

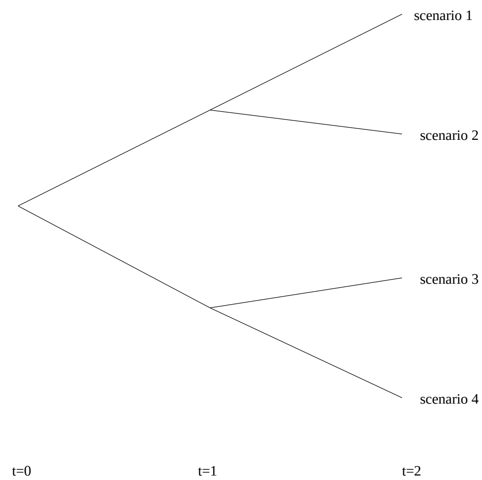

Figure 1: Example hedging scenario tree

corresponds to the bottom node at $t = T$. Let N_t denote the set of nodes at $t = 0, \ldots, T$ and a_n denote the ancestor of node $n \in N_t$ for $t = 1, \ldots, T$. The *ancestor* of node $n \in N_t$ is the unique node in N_{t-1} connected to node n. Let the set of children of node n in N_t for $t = 0, \ldots, T-1$ be denoted by c_n with the i^{th} child denoted by c_n^i. The *children* of node $n \in N_t$ are the set of nodes in N_{t+1} that are connected to n. Let V_n denote the set of scenarios that visit node n and let pr_n denote the unconditional probability of node n given by $\sum_{\omega \in V_n} pr(\omega)$. Let:

- $S_n = (S_n^0, S_n^1)'$ denote the vector of *asset prices* where S_n^0 and S_n^1 denote respectively the prices of cash and the underlying asset in node $n \in N_t$ for $t = 0, \ldots, T$. For simplicity we assume that the cash returns are constant.

- $X_n = (X_n^0, X_n^1)'$ denote the vector of *positions* where X_n^0 and X_n^1 denote respectively the number of units of cash and the underlying asset held between node $n \in N_t$ and each node in c_n for $t = 0, \ldots, T-1$.

- $Xb_n = (Xb_n^0, Xb_n^1)'$ denote the vector of *buy variables* where Xb_n^0 and Xb_n^1 denote respectively the number of units of cash and the underlying asset bought at node

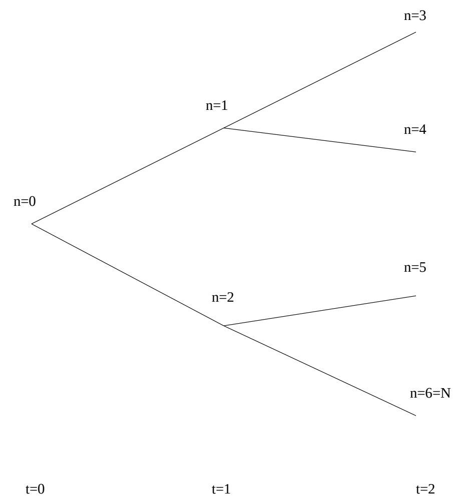

Figure 2: Nodal representation

$n \in N_t$ for $t = 0, \ldots, T-1$. The buy (and following sell) variables are used to account for a *proportional transaction cost* c on buying and selling the underlying asset.

- $Xs_n = (Xs_n^0, Xs_n^1)'$ denote the vector of *sell variables* where Xs_n^0 and Xs_n^1 denote respectively the number of units of cash and the underlying asset sold at node $n \in N_t$ for $t = 0, \ldots, T-1$.

A *hedging strategy* is given by $\{X_n, Xb_n, Xs_n : n \in N_t, t = 0, \ldots, T-1\}$. For European options we only define the derivative price for $n \in N_T$, i.e. at maturity, where the derivative price is given by its payoff function. While the derivative price can be defined at other nodes using a pricing model, this would make the hedging strategy dependent on the choice of pricing model which we wish to avoid. Likewise for knock-out Barrier options we only define the derivative price at the nodes where the option terminates. For a given scenario this will be the node at maturity or the node in which the barrier is first hit, which ever comes first. In either case we let I denote the set of nodes over which the derivative price is defined, f_n for $n \in I$ denote the *derivative price* and h_n for $n \in I$ denote the *hedge error*.

2.2 Problem Formulation

One advantage of the stochastic optimization based hedging strategy is its ability to incorporate different measures of the hedge error and thus different attitudes to risk. In this paper we use a linear downside measure of the hedge error. Another advantage of the methodology is the ease in which market frictions such as transaction costs, position limits and turnover constraints can be explicitly modeled. Here we explicitly model proportional transaction costs.

We assume that no money can be added to the hedging strategy between $t = 0$ and $t = T$ and that it begins with an *initial wealth* w_0 in cash corresponding to the price of the derivative at $t = 0$. This corresponds to the case where the writer has sold the derivative and then uses the premium received as the initial wealth of the hedging strategy. The resulting hedging problem becomes:

$$\min E[\sum_{t=1}^{T} \boldsymbol{h_t}] \tag{1}$$

$$\text{s.t. } \boldsymbol{e_t^+} - \boldsymbol{e_t^-} = \boldsymbol{X'_{t-1}} \boldsymbol{S_t} - \boldsymbol{f_t}; t = 1, \ldots, T \tag{2}$$

$$\boldsymbol{e_t^+}, \boldsymbol{e_t^-} \geq 0; t = 1, \ldots, T \tag{3}$$

$$\boldsymbol{h_t} = \boldsymbol{e_t^-} \geq 0; t = 1, \ldots, T \tag{4}$$

$$Xb_0 - Xs_0 = X_0 \tag{5}$$

$$\boldsymbol{X_{t-1}} + \boldsymbol{Xb_t} - \boldsymbol{Xs_t} = \boldsymbol{X_t}; t = 1, \ldots, T-1 \tag{6}$$

$$Xs_0^0 S_0^0 + (1-c) Xs_0^1 S_0^1 - Xb_0^0 S_0^0 - (1+c) Xb_0^1 S_0^1 + w_0 \geq 0 \tag{7}$$

$$\boldsymbol{Xs_t^0} S_t^0 + (1-c) \boldsymbol{Xs_t^1} \boldsymbol{S_t^1} - \boldsymbol{Xb_t^0} S_t^0 - (1+c) \boldsymbol{Xb_t^1} \boldsymbol{S_t^1} \geq 0; t = 1, \ldots, T-1 \tag{8}$$

$$\boldsymbol{Xb_t}, \boldsymbol{Xs_t} \geq 0; t = 0 \ldots, T-1. \tag{9}$$

For convenience we work with the *nodal form* of this problem which is given by:

$$\min \sum_{n \in I} pr_n h_n \tag{10}$$

$$\text{s.t. } e_n^+ - e_n^- = X'_{a_n} S_n - f_n; n \in I \tag{11}$$

$$e_n^+, e_n^- \geq 0; n \in I \tag{12}$$

$$h_n = e_n^-; n \in I \tag{13}$$

$$Xb_0 - Xs_0 = X_0 \tag{14}$$

$$X_{a_n} - Xs_n + Xb_n = X_n; n > 0, n \notin N_T \tag{15}$$

$$Xs_0^0 S_0^0 + (1-c) X s_0^1 S_0^1 - X b_0^0 S_0^0 - (1+c) X b_0^1 S_0^1 + w_0 \geq 0 \tag{16}$$

$$Xs_n^0 S_n^0 + (1-c) X s_n^1 S_n^1 - X b_n^0 S_n^0 - (1+c) X b_n^1 S_n^1 \geq 0; n > 0, n \notin N_T \tag{17}$$

$$Xb_n, Xs_n \geq 0; n \notin N_T. \tag{18}$$

The hedging problem is a linear *dynamic stochastic program* (DSP) and can be solved using techniques such as *simplex* (see e.g. Luenberger [14]), *interior point* (see e.g. Wright [20]) and *decomposition* methods (see e.g. Birge and Louveaux [2]). The objective (10) is the minimization of the *mean downside hedge error* where the downside hedge error of a node is defined by (11) - (13). The difference in the values of the derivative and hedging portfolio in node $n \in I$ is split into upside and downside components, e_n^+ and e_n^- respectively, and the downside hedge error is given by e_n^-. (14) and (15) are *inventory balance constraints* which give the position in each asset at each node. (16) and (17) are *cash balance constraints* which ensure that no money is added to the hedging strategy, and (18) requires the buy and sell variables to be nonnegative.

The uncertainty in the problem is a result of the uncertainty in $\boldsymbol{S}$. Because the values of S_n for $n \neq 0$ are usually generated using simulation techniques a wide range of stochastic models for $\boldsymbol{S}$ including jumps and stochastic volatilities can be accommodated.

2.3 Implementation

In practice a separate problem would be solved at each trading time and only the first stage solution would be implemented. Specifically, at each trading time $t = 0, \ldots, T-1$ a $T-t$ stage hedging DSP would be constructed and solved. The first stage solution of this problem would give the time t hedging positions, X_t. One reason for this method of implementation is that the realized values of the variables are unlikely to coincide with any of the simulated values in the scenario trees. If this were to be the case then the optimal hedging strategy would be undefined. Another reason for solving a new problem at each time is that this allows the stochastic model of $\boldsymbol{S}$ to be updated using market information observed from the previous time.

3 Scenario Tree Generation

The effectiveness of the stochastic optimization based hedging strategy depends crucially on an accurate representation of the uncertainty and thus on an accurate scenario tree. In this paper we employ a method of scenario tree generation which ensures the absence of arbitrage and matches the mean and variance of the underlying asset price in the sampled scenarios to those implied by a given stochastic model, a process referred to as *moment matching*. As discussed in Villaverde [19], moment matching can help to suppress *sampling error*, i.e. differences in the distribution of the sampled scenarios and the distribution the scenarios are sampled from. An *arbitrage* is a trading strategy that is guaranteed not to lose money and is expected to make money. Since such opportunities are usually assumed not to exist in practice, generating scenario trees which are arbitrage free is consistent with reality.

This method of scenario tree generation employs the following algorithm at each node n. First the children of n are generated using random sampling. Let $\{\hat{S^1_m} : m \in c_n\}$ denote the randomly generated children. The following optimization problem is then solved:

$$\min_{S_m^1 : m \in c_n} \sum_{m \in c_n} (S_m^1 - \hat{S_m^1})^2 \tag{19}$$

$$\text{s.t. } E[S^1] = \mu_n \tag{20}$$

$$Var[S^1] = \sigma_n^2 \tag{21}$$

$$\frac{S_1^1}{S_1^0} \geq S_0^1 + \alpha \tag{22}$$

$$\frac{S_2^1}{S_1^0} \leq S_0^1 - \alpha, \tag{23}$$

where μ_n and σ_n^2 denote the mean and variance of the underlying asset price implied by the given stochastic model at node n and α is a small strictly positive number. This problem finds new children $\{S_m^1 : m \in c_n\}$ which are close to the original randomly generated children (in the least squares sense) subject to mean, variance and arbitrage constraints. (20) and (21) ensure a mean and variance consistent with the given stochastic model and (22) and (23) enforce the absence of arbitrage. A derivation of the arbitrage constraints is given in Appendix A. This problem has nonlinear constraints but can be solved with *sequential quadratic programming* (see e.g. Gill [8]).

4 Benchmarking to Delta Hedging

In this section we benchmark the stochastic optimization based hedging strategy to *delta hedging* with respect to the mean downside hedge error obtained by applying the two hedging methods to simulated *test scenarios*. We consider the case of no transaction costs in detail as well as providing a description of how the mean hedge error of the two hedging strategies change with increasing transaction costs.

We consider four stage problems for European and Barrier call options with weekly rebalancing where the underlying asset prices are assumed to follow a *discretized geometric*

Brownian motion (GBM) of the form:

$$\frac{\boldsymbol{S_t^1} - S_{t-1}^1}{S_{t-1}^1} = \mu + \sigma \boldsymbol{\epsilon_t}, \tag{24}$$

where the $\boldsymbol{\epsilon_t}$ terms are independent standard normal random variables and the time step is one week. The underlying asset of the European option was assumed to be the S&P 500 index [1] and the underlying asset of the Barrier option was assumed to be the price of 1 GBP in USD (UK/US exchange rate)) [2].

Each option had a maturity of four weeks and was assumed to be at-the-money initially. The Barrier option was assumed to be a down-and-out call [3]. The initial wealths of the hedging strategies were given by the Black-Scholes prices of the options[4].

We first generated 1000 test scenarios for each underlying asset by simulating (24). We then applied each hedging method to the test scenarios under a range of proportional transaction costs. The stochastic optimization based hedging strategy was implemented as in Section 2.3 with scenario trees generated using (24) and the mean and variance matching arbitrage free method described in Section 3 [5]. A description of the implementation of the delta hedging strategies is given in Appendix B. The hedge error for the European option was always calculated at maturity while the hedge error for Barrier option was calculated either at maturity or when the barrier was first hit, which ever came first.

[1] $\mu = .0028$ and $\sigma = .0189$ per week.

[2] $\mu = .0001$ and $\sigma = .0146$ per week.

[3] The Barrier option was written on 1000 GBP and had a barrier equal to 98% of the strike price.

[4] The return on cash was assumed to be constant at 5% per annum.

[5] A branching factor of 20 was used.

4.1 European Call Option

Table 1 gives the mean, standard deviation, minimum and maximum of the hedge error distribution over the 1000 test scenarios for each hedging method with no transaction costs. The results for both methods are similar but the stochastic optimization strategy, which is formulated to minimize the mean hedge error, produces a better mean while the delta strategy produces a better standard deviaion and maximum.

Method	**Mean**	**St. Dev.**	**Min**	**Max**
Delta	2.76	4.73	0.00	34.20
SP	2.35	5.89	0.00	64.75

Table 1: European option-GBM results

Figure 3 shows the that the hedge error histograms of the two methods for the no transaction costs problem are similarly shaped.

Figure 4 shows the mean downside hedge errors for the two methods using transaction costs ranging from 0% to 1% in .2% increments. The graph corresponding to the stochastic optimization strategy lies below that of the delta strategy which implies that it produces a lower mean hedge error at all levels of transaction costs. In addition, the mean hedge error of the delta hedge increases faster than that of the stochastic optimization based hedge so that the gap between the two strategies widens as the transaction costs increase.

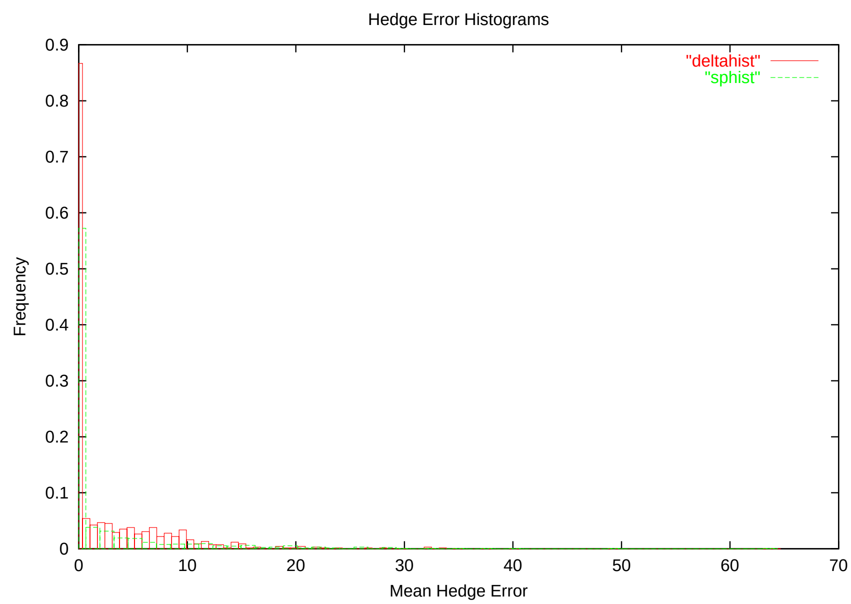

Figure 3: European option-GBM downside hedge error histograms with 0% transaction costs

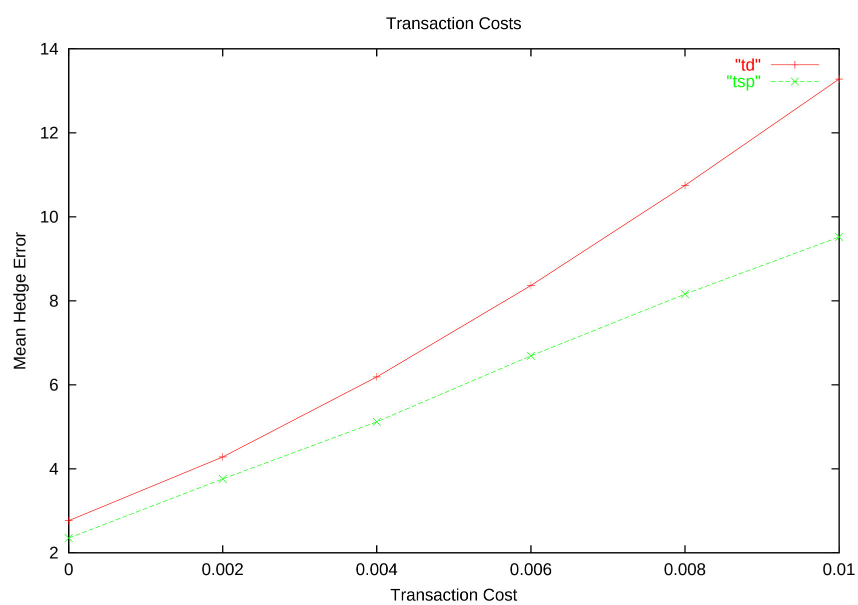

Figure 4: European option-GBM mean downside hedge error vs. transaction costs

4.2 Barrier Call Option

Table 2 gives the mean, standard deviation, minimum and maximum of the hedge error distribution over the 1000 test scenarios for each hedging method with no transaction costs. As with the European option the stochastic optimization strategy produces a better mean and the delta hedge produces a better standard deviation and maximum.

Method	**Mean**	**St. Dev.**	**Min**	**Max**
Delta	5.57	5.73	0.00	30.56
SP	4.76	7.65	0.00	57.63

Table 2: Barrier option-GBM results

As with the European option, Figure 5 shows the hedge error histograms of the two methods for the no transaction costs problem are similar in shape.

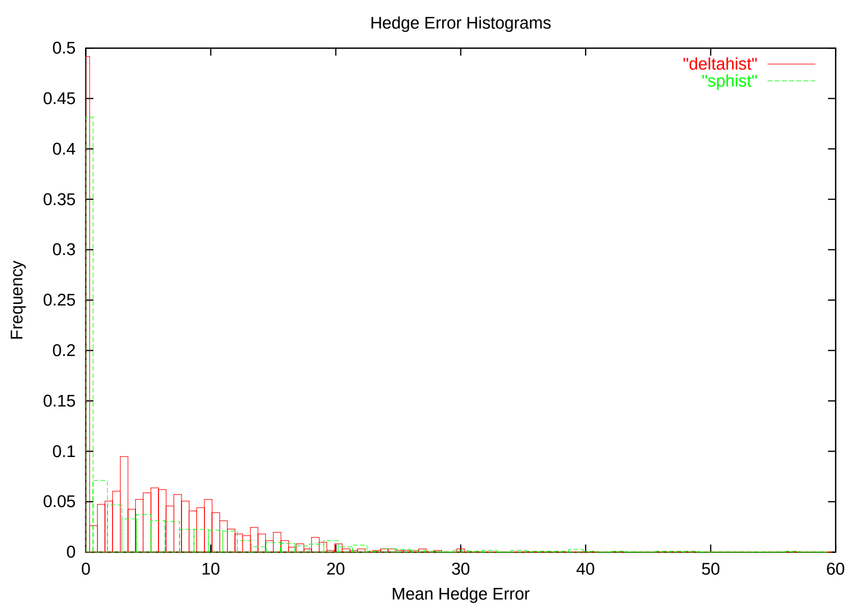

Figure 5: Barrier option-GBM downside hedge error histograms with 0% transaction costs

Figure 6 shows the mean downside hedge errors for the two methods using transaction costs ranging from 0% to 1% in .2% increments. As with the European option, the stochastic optimization based hedging strategy produces a lower mean hedge error at all levels of transaction costs and the gap widens as the transaction costs increase.

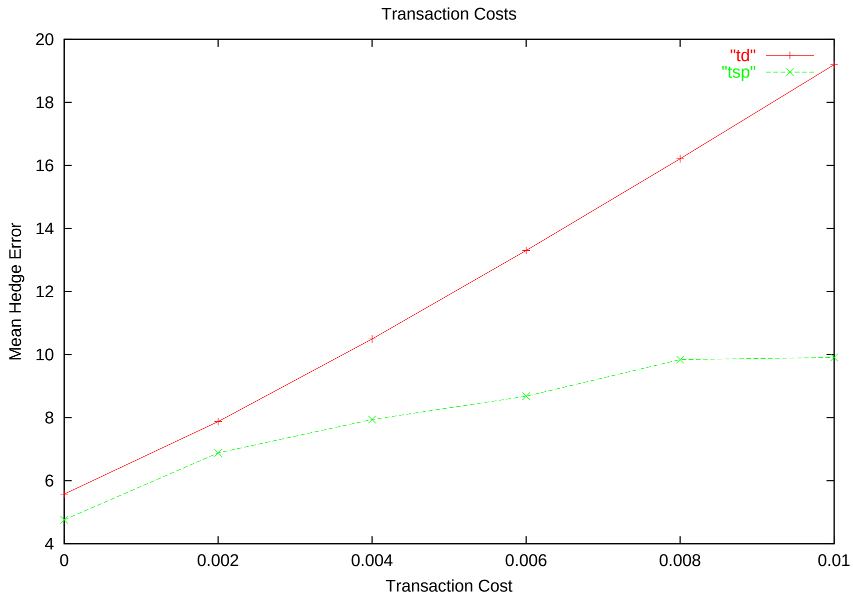

Figure 6: Barrier option-GBM mean downside hedge error vs. transaction costs

5 Variance Gamma Process

In this section we implement the stochastic optimization based hedging strategy on the European and Barrier option problems of the previous section for the case where the underlying asset price is driven by a discretized Variance Gamma (VG) process. The VG process is a Levy process that was first introduced by Madan and Seneta [15] as a model for stock returns. Because it is a pure jump process, this process is able to take into account excess skewness and kurtosis.

The underlying asset price is assumed to evolve according to:

$$\boldsymbol{S_t^1} = S_0^1 e^{\boldsymbol{x_t}}, \tag{25}$$

where $\boldsymbol{x_t}$ is a *discrete time Variance Gamma process* $VG(\sigma, \nu, \theta)$. This process is described in detail in Appendix C [6].

As described in (see e.g. Schoutens [18]), derivative pricing formulas are not easily obtained with the VG model which is problematic for the delta hedging strategy. However, a hedging strategy is readily obtained using stochastic optimization since this method does not depend on a pricing formula. All that is needed is a method for simulating the VG process. To simulate the VG process we use the fact that it can be represented as the difference of two Gamma processes and simulate each Gamma process using *Johnk's Gamma Generator* (see e.g. Schoutens [18]).

[6] To keep the first two moments of the VG processes similar to those of the GBMs of the previous section we chose (σ, ν, θ) equal to $(.0028, .0189, 1)$ for the underlying of the European option and $(.00017, .0146, 1)$ for the underlying of the Barrier option.

5.1 European Call Option

Table 3 gives the mean, standard deviation, minimum and maximum of the hedge error distribution over the 1000 test scenarios with no transaction costs. The results are similar but slightly worse across all statistics than the case where the underlying asset price follows a discretized GBM.

Mean	**St. Dev.**	**Min**	**Max**
3.66	9.57	0.00	90.21

Table 3: European option-VG results

Figure 7 shows the hedge error histogram for the no transaction costs problem, and Figure 8 shows the mean downside hedge errors using transaction costs ranging from 0% to 1% in .2% increments. In both cases the graphs are similar to the case where the underlying asset price follows a discretized GBM.

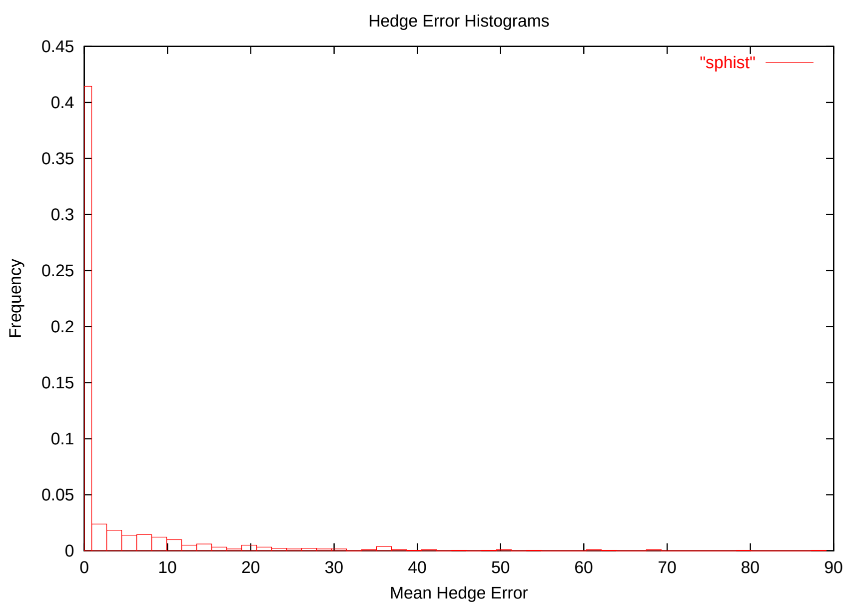

Figure 7: European option-VG downside hedge error histogram with 0% transaction costs

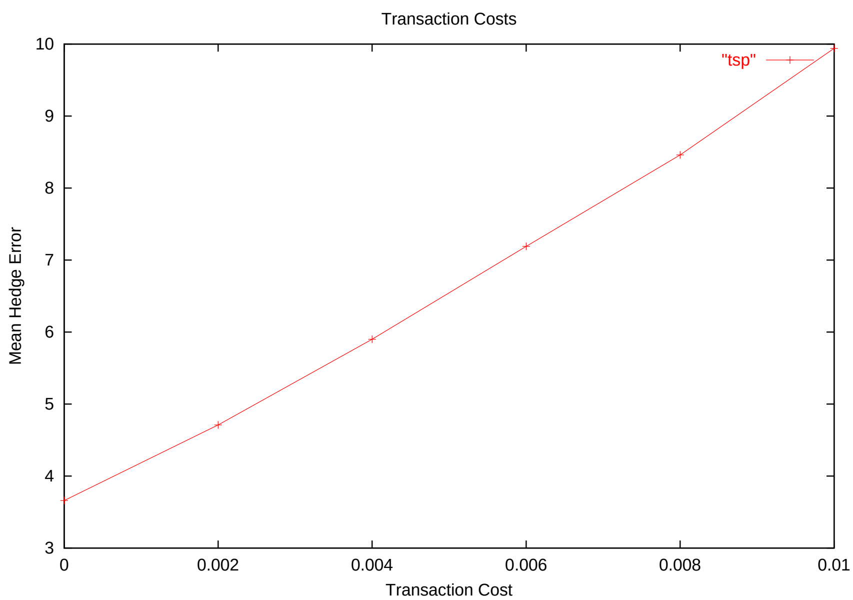

Figure 8: European option-VG mean downside hedge error vs. transaction costs

5.2 Barrier Call Option

Table 4 gives the mean, standard deviation, minimum and maximum of the hedge error distribution over the 1000 test scenarios with no transaction costs. As with the European option the results are similar to the case where the underlying asset price follows a discretized GBM.

Mean	St. Dev.	Min	Max
5.72	10.74	0.00	90.26

Table 4: Barrier option-VG results

Figure 9 shows the hedge error histogram for the no transaction costs problem, and Figure 10 shows the mean downside hedge errors using transaction costs ranging from 0% to 1% in .2% increments. As with the European option, in both cases the graphs are similar to the case where the underlying asset price follows a discretized GBM.

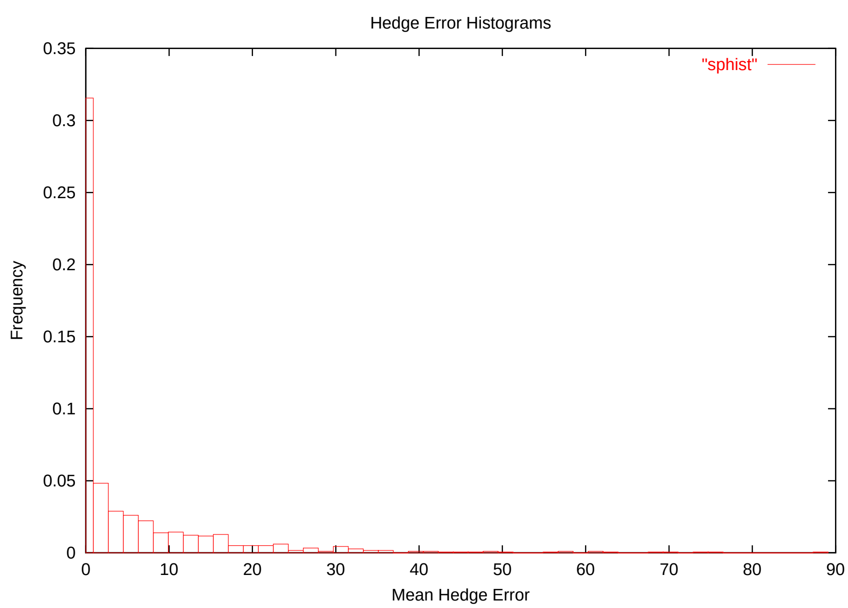

Figure 9: Barrier option-VG downside hedge error histogram with 0% transaction costs

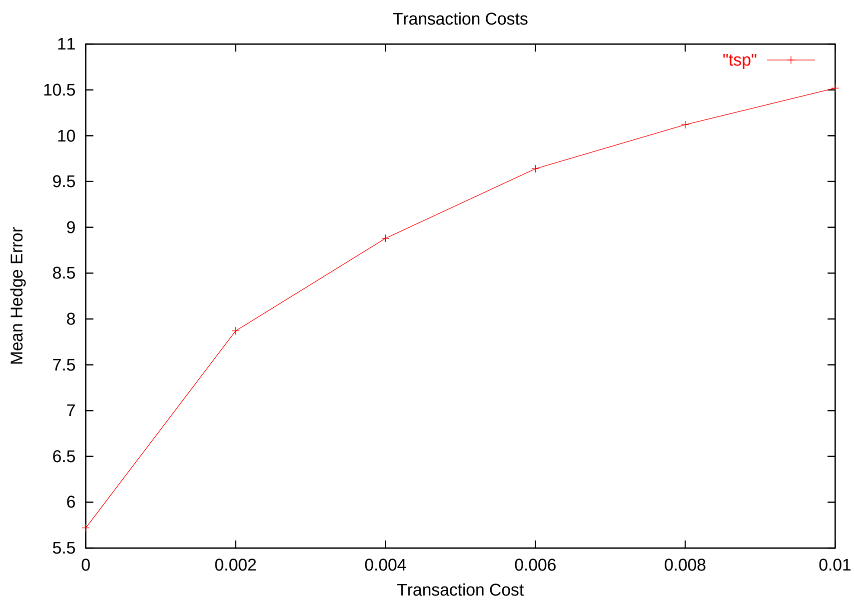

Figure 10: Barrier option-VG mean downside hedge error vs. transaction costs

6 Conclusions

This paper has presented a method of hedging derivatives in an incomplete market based on stochastic optimization. Advantages of this method include its ability to explicitly incorporate different measures of the hedge error, various market frictions and nearly arbitrary stochastic models for the underlying variables. In this paper we used a linear downside measure of the hedge error and explicitly modeled proportional transaction costs. Scenario trees were generated with a method which ensured the absence of arbitrage and which matched the mean and variance of the underlying asset price to those of a given distribution. The stochastic optimization based strategy was then benchmarked to delta hedging for European and Barrier options for the case where the underlying asset price follows a discretized GBM. These experiments showed that the stochastic optimization based method produced a lower mean downside hedge error for both types of options for a range of transaction costs. The gap between the the two strategies was shown to increase as the transaction costs increased and was more pronounced for the Barrier option than for the European option. The stochastic optimization based strategy was them implemented for the case where the underlying asset price was driven by a discretized VG process in which case delta hedging methods are not readily available. The results of these experiments were similar to the case where the underlying asset price follows a discretized GBM.

Appendix A

Without loss of generality we assume in this appendix that the cash price is always 1 or that the interest rate is zero, which implies that prices and discounted prices are the same. We also assume for simplicity that trading is allowed at maturity and that there are no transaction costs. If a scenario tree is arbitrage free under both these assumptions then it will be arbitrage free if either or both are relaxed since this can only make an investor worse off. Let P be a probability measure on the scenario tree with the transition probability from node n to node m denoted by p_n^m. Here an *arbitrage* is a trading strategy $\{\boldsymbol{X_t} : t = 0, \ldots, T\}$ such that:

$$\begin{aligned} E[\boldsymbol{S_T' X_T}] > 0 \\ S_0' X_0 = 0 \\ S_n'(X_n - X_{a_n}) = 0; n \in N_t, t = 1, \ldots, T \\ S_n' X_n \geq 0; n \in N_T. \end{aligned} \tag{26}$$

Thus, an arbitrage is a self-financing trading strategy which has a zero initial cost, a nonnegative terminal value and a positive expected terminal value. Harrison and Kreps [10] showed that there is no arbitrage if and only if there is an *equivalent martingale measure*, or a probability measure equivalent to P under which the discounted price process is a martingale. Thus, here an equivalent martingale measure Q consists of transition probabilities q_n^m such that:

$$\begin{aligned} \sum_{m \in N_t} q_n^m S_m = S_n; n \in N_u, u = 0, \ldots, T-1, t = u+1, \ldots, T \\ q_n^m > 0 \Leftrightarrow p_n^m > 0. \end{aligned} \tag{27}$$

The following Lemma shows that if there is no equivalent martingale measure for some subtree of the scenario tree then there is no equivalent martingale measure for some node in that subtree.

Lemma 1: Suppose that for some node $r \in N_t$ with $t \in \{0, \ldots, T-1\}$ and some time $u \in \{t+1, \ldots, T\}$ there is no solution to the following system:

$$\begin{aligned} \sum_{v \in N_u} q_r^v S_v^1 = S_r^1 \\ q_r^v > 0 \Leftrightarrow p_r^v > 0. \end{aligned} \tag{28}$$

Then for some node $g \in N_k$ with $k \in \{t, \ldots, u-1\}$ there is no solution to the one period system:

$$\sum_{b \in N_{k+1}} q_g^b S_b^1 = S_g^1 \tag{29}$$
$$q_g^b > 0 \Leftrightarrow p_g^b > 0.$$

Proof: We prove the contrapositive. Therefore suppose for each node in N_k for $k = t, \ldots, u-1$ there is a solution to the one period system given in (29). Then by iterative substitution:

$$\begin{aligned} S_r^1 &= \sum_{m1 \in N_{t+1}} q_r^{m1} S_{m1}^1 \\ &= \sum_{m1 \in N_{t+1}} q_r^{m1} \left(\sum_{m2 \in N_{t+2}} q_{m1}^{m2} S_{m2}^1 \right) \\ &= \sum_{m1 \in N_{t+1}} \sum_{m2 \in N_{t+2}} q_r^{m1} q_{m1}^{m2} S_{m2}^1 \\ &= \sum_{m1 \in N_{t+1}} \sum_{m2 \in N_{t+2}}, \ldots, \sum_{v \in N_u} q_r^{m1} q_{m1}^{m2}, \ldots, q_{m(u-1)}^{v} S_v^1. \end{aligned} \tag{30}$$

Thus, taking:

$$q_r^v = \sum_{m1 \in N_{t+1}} \sum_{m2 \in N_{t+2}}, \ldots, \sum_{m(u-1) \in N_{u-1}} q_r^{m1} q_{m1}^{m2}, \ldots, q_{m(u-1)}^{v} \tag{31}$$

results in a solution to (29) and the result follows.□

A *one period arbitrage* at node n in N_t with $t \in \{0, \ldots, T-1\}$ is a portfolio X_n such that:

$$\sum_{m \in c_n} (S_m' X_n) p_n^m > 0 \tag{32}$$
$$S_n' X_n = 0$$
$$S_m' X_n \geq 0; m \in c_n.$$

The following proposition shows the equivalence of arbitrage and one period arbitrage.

Proposition 1: There is arbitrage if and only if there is a one period arbitrage.

Proof:

$\Rightarrow$

Suppose there is an arbitrage. Then for some node $r \in N_t$ with $t \in \{0, \ldots, T-1\}$ and some time $u \in \{t+1, \ldots, T\}$ there is no solution to the system given in (27). By Lemma 1 this implies that there is some node $g \in N_k$ for some $k \in \{t, \ldots, u-1\}$ such that there is no solution to the one period system (29). This implies that there is a one period arbitrage at node g.

$\Leftarrow$

Suppose there is a one period arbitrage at node $g \in N_k$ with $k \in \{0, \ldots, T-1\}$ given by X_g. The following trading strategy is an arbitrage. If $k = 0$, then follow X_g at $t = 0$ and invest in cash for $t > 0$. If $k > 0$, then invest in cash for the scenarios that do not visit g. For the scenarios that do visit g, invest nothing for $t < k$, follow X_g at $t = k$ and invest in cash for $t > k$.$\Box$

Klaassen [13] notes that one way to generate an arbitrage free scenario tree is to set P equal to an equivalent martingale measure. However, while the resulting tree would be arbitrage free, this would cause all assets to have an expected return equal to that of the numeraire (cash) resulting in an unrealistic representation of future real world uncertainty.

Below we use the definition of a one period arbitrage to derive an alternative method for generating arbitrage free scenario trees. This method relies on the following proposition which gives a sufficient condition for a node to be arbitrage free. This proposition is only for the case of one stochastic asset since the only stochastic asset here is the underlying asset price.

Proposition 2: Node n in N_t with $t \in \{0, \ldots, T-1\}$ will be one period arbitrage free if in one branch the discounted price of the underlying goes up and in one branch the discounted price of the underlying goes down.

Proof: The node will be one period arbitrage free if the following SP has an objective value of 0.

$$\max \sum_{m \in c_n} (S'_m X_n) p_n^m \tag{33}$$
$$S'_n X_n = 0$$
$$S'_m X_n \geq 0; m \in c_n.$$

Since $S_n^0 = 1$ for all n this can be rewritten as:

$$\max \sum_{m \in c_n} (S'_m X_n) p_n^m \tag{34}$$
$$X_n^0 + S_n^1 X_n^1 = 0$$
$$X_n^0 + S_m^1 X_n^1 \geq 0; m \in c_n.$$

Substituting $X_n^0 = -S_n^1 X_n^1$ gives:

$$\max \sum_{m \in c_n} (S'_m X_n) p_n^m \tag{35}$$
$$X_n^1 (S_m^1 - S_n^1) \geq 0; m \in c_n.$$

If in one branch the discounted price of the underlying goes up, then $S_m^1 > S_n^1$ and the inequality implies that the optimal $X_n^1 \geq 0$. If in one branch the discounted price of the underlying goes down, then $S_m^1 < S_n^1$ and the inequality implies that the optimal $X_n^1 \leq 0$. Together this implies that optimally $X_n^1 = 0$, which implies that the corresponding X_n^0 and objective value are zero.□

Thus, an arbitrage free tree can be generated by ensuring that at each node the discounted price of the underlying goes up in one child and down in another.

To check a given scenario tree for arbitrage, the following DSP can be solved.

$$\max E[\boldsymbol{S'_T X_T}] \tag{36}$$

$$S'_0 X_0 = 0$$

$$S'_n(X_n - X_{a_n}) = 0; n \in N_t, t = 1, \ldots, T$$

$$S'_n X_n \geq 0; n \in N_T.$$

By the definition of arbitrage given in (26), if there is an arbitrage then the objective value of this DSP will be positive and will be 0 otherwise.

Appendix B

This appendix describes the implementation of the delta hedge. For a test scenario with T rebalancing times, trading takes place at $t = 0, \ldots, T-1$. The delta of the European call is given by $N(d_1)$ (see e.g. Hull [11]), where $N(\cdot)$ is the cumulative standard normal distribution function and:

$$d_1 = \frac{\ln(\frac{S}{Q}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \tag{37}$$

Here S denotes the initial underlying value, σ denotes the volatility, Q denotes the strike price and r is the risk free rate. The formula for the price of the Barrier option can be found in Hull [11]. Differentiating this with respect to the underlying gives the Barrier delta:

$$\begin{aligned} e^{-r_f T} N(x) - (e^{-r_f T} B^{2\lambda}(S^{-2\lambda+1} N'(y) y' + N(y)(-2\lambda+1) S^{-2\lambda}) \\ -X e^{-rT} B^{2\lambda-2}(S^{-2\lambda+2} N'(y - \sigma\sqrt{T}) y' + N(y - \sigma\sqrt{T})(-2\lambda+2) S^{-2\lambda+1})). \end{aligned} \tag{38}$$

Here B is the barrier level, r_f is the foreign risk free rate which is assumed to be equal to r and $N'(a)$ is the standard normal density given by:

$$N'(a) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}.$$

The entities x, λ, y and y' are given by:

$$x = \frac{\ln(\frac{S}{Q}) + (r - r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{39}$$

$$\lambda = \frac{r - r_f + \frac{\sigma^2}{2}}{\sigma^2} \tag{40}$$

$$y = \frac{\ln(\frac{B^2}{SQ})}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \tag{41}$$

$$y' = -\frac{1}{\sigma\sqrt{T}S}. \tag{42}$$

Let X_t^0 and X_t^1 denote respectively the number of units held of cash and the underlying between times t and $t+1$, and let S_t^0 and S_t^1 denote respectively the price of cash and the underlying at time t. The *delta hedging strategy* is given by:

$$X_0^1 = D_0 \tag{43}$$

$$X_0^0 = \frac{w_0 - X_0^1 S_0^1(1+c)}{S_0^0} \tag{44}$$

and

$$X_t^1 = D_t \tag{45}$$

$$X_t^0 = \frac{X_{t-1}' S_t - X_t^1 S_t^1 - |X_t^1 - X_{t-1}^1| S_t^1 c}{S_t^0} \tag{46}$$

for $t = 1, \ldots, T-1$.

Appendix C

A $VG(\theta,\nu,\sigma)$ process starts at zero, has independent and stationary increments and has increments $\boldsymbol{x_{s+t}} - \boldsymbol{x_s}$ which are distributed $VG(\sigma\sqrt{t},\nu/t,t\theta)$. The characteristic function of the $VG(\sigma,\nu,\theta)$ distribution is given by:

$$\phi(u:\sigma,\nu,\theta) = (1 - iu\theta\nu + \frac{1}{2}\sigma 2\nu u^2)^{-1/\nu}. \tag{47}$$

Let:

$$\begin{aligned} C &:= 1/\nu \qquad (48)\\ G &:= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right)^{-1} \qquad (49)\\ M &:= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} = \frac{1}{2}\theta\nu\right)^{-1}. \qquad (50)\end{aligned}$$

Then this distribution can be characterized by the Levy triplet $[\gamma, 0, \nu(dx)]$ where:

$$\gamma = \frac{-C(G(exp(-M)-1) - M(exp(-G)-1))}{MG} \tag{51}$$

$$\nu(dx) = \begin{cases} C exp(Gx)|x|^{-1}dx & x < 0 \\ C exp(-Mx)x^{-1}dx & x > 0 \end{cases}. \tag{52}$$

The first four moments of this distribution are given by:

$$\begin{aligned} mean &: \theta \qquad (53)\\ variance &: \sigma^2 + \nu\theta^2 \qquad (54)\\ skewness &: \theta\nu(3\sigma^2 + 2\nu\theta^2)/(\sigma^2+\nu\theta^2)^{3/2} \qquad (55)\\ kurtosis &: 3(1 + 2\nu - \nu\sigma^4(\sigma^2+\nu\theta^2)^{-2}). \qquad (56)\end{aligned}$$

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