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WP 09/2004

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11th May 2004

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Abstract

This paper considers the estimation of parameters in a dynamic stochastic model for securities prices, where the expected rate of return is a random variable. An empirical Bayes estimator is developed from the model structure. The estimator is an improvement on other popular estimators in terms of mean squared error. The effect of reduced estimation error on accumulated wealth is analyzed for the portfolio choice model with constant relative risk aversion utility.

1 INTRODUCTION

The decision on allocation of investment capital to risky and risk free opportunities is a fundamental problem in portfolio theory. A basic input to the investment decision is the distribution of future returns on securities. The prediction of future securities returns is based on price information available at the time of the decision. Prediction errors can have a large negative impact on portfolio choice and the resulting accumulation of wealth (Loffler 2003). Errors in the prediction of mean returns are particularly damaging to wealth accumulation (Kallberg and Ziemba 1984, Chopra and Ziemba 1993).

A standard framework for the trading prices of securities is the geometric Brownian motion model (Merton 1972). With a hierarchy of stochastic differential equations the model accommodates the dynamics observed in actual prices (Chernov, et.al. 2002) The random coefficients in the stochastic differential equations are parameters in the securities price distributions, and the hierarchy generates a Bayesian model for price distributions. In the typical Bayesian approach, a noninformative or conjugate prior on parameters in the price distribution is postulated (Klein and Bawa 1976). The prior in the framework in this paper is generated by the dynamic equations for parameters. In the context of the model, the optimal estimate of the parameters given price information to date is the posterior mean or Bayes estimate. The dynamic structure provides a natural mechanism for updating estimates for parameters as more information is gathered.

One difficulty with the Bayes approach is that parameters in the prior distribution (and parameters in the sdes) are unknown. However, the data on past prices can be used to estimate those parameters and provide an empirical Bayes estimate for the posterior mean (Efron and Morris 1972, Frost and Savarino 1986). In the dynamic model format, the observed prices are points on a trajectory and the movement of prices provides the necessary information to estimate the conditional (first order) and prior (second order) parameters.

Related to the diffusion models with random parameters are the factor models of asset pricing theory (Connor and Korajczyk 1995, Ross 1976). The movements of the prices in the class of securities are driven by underlying set of common market factors. In the diffusion model, the market factors appear in the dynamics for the prior parameters. The distribution for securities prices is defined by a factor model. The covariance matrix for prices is simplified by the factor structure, and there is a reduction in the number of model parameters, which leads to improved estimates.

The impact of modeling and estimation errors on forecasts for securities prices and the resulting effect on portfolio decisions and capital accumulation have been considered in many studies. Alternative estimates for the mean return have been considered in a long series of asset prices (Grauer and Hakannson 1985, 1995), with improves results from shrinkage (Stein) estimators. The results are empirical rather than theoretical, and the structure and dynamics of price distributions is not clear.

In this paper an empirical Bayes estimate for the expected rate of return on securities is presented. The basis of the estimator is the structure of the covariance matrix for the rates of return. With a reduced dimension representation of covariance based on a truncation of eigenvectors, there are sufficient degrees of freedom to estimate all parameters in a Bayesian model for prices.. Furthermore, the structure in the covariance affects the theoretical mean squared error of estimators and facilitates comparisons. This effect carries over to portfolio decisions and accumulated wealth.

The empirical Bayes estimator is developed from the pricing model in Section 2. The mean squared error properties of the estimator are explored analytically and through simulation in Section 3. The estimator is compared to the Bayes-Stein (Jorion 1985) and maximum liklihood estimators. In Section 4 the wealth shortfall from estimation error is considered for the expected utility maximization problem. The methods in the paper are applied to portfolios of stocks from the Toronto Stock Exchange and the New York Stock Exchange.

2 EMPIRICAL BAYES ESTIMATION OF RETURNS PARAMETERS

Consider a competitive capital market where trading of securities takes place in continuous time. The distribution of prices at a point in time and the dynamics of prices over time will be analyzed with the geometric Brownian motion model (Merton, 1992).

2.1 Pricing Model

Let $P_i(t)$ equal the price of security i at time t and consider $Y_i(t) = \ln P_i(t), i = 0, ..., K$. The dynamics of price movements are defined by the stochastic differential equations

$$dY_0(t) = rdt \tag{1}$$

$$dY_i(t) = \lambda_i dt + \delta_i dV_i, i = 1, ..., K, \tag{2}$$

where $dV_i, i = 1, ..., K$, are independent Brownian motions. It is further assumed that the drift in (2) is a random variable, so that

$$\lambda_i = \mu_i + \beta_i Z_i, i = 1, ..., K, \tag{3}$$

where $Z_i, i = 1, ..., K$ are correlated Gaussian variables with ρ_{ij} the instantaneous correlation between variables $Z_i$and Z_j.

The geometric Brownian motion model in (1) - (3) is a generalization of a single stock model in Browne and Whitt (1995), where the rate of return is a random variable. This model is also used in Rogers (2001) to study parameter estimation error. There are a number of points to consider about the relevance of this somewhat specialized pricing model.

[1] This is a multi-factor diffusion model. If $U_j, j = 1, ..., m, m \leq K$, are i.i.d. standard Gaussian variables, then $Z_i = \sum_{j=1}^m \alpha_{ij} U_j$ and

$$dY_i(t) = [\mu_i + \sum_{i=1}^{m} \beta_i \alpha_{ij} U_j] dt + \delta_i dV_i, i = 1, ...K. \tag{4}$$

Typically the number of factors would be small and to have the parameters identifiable from the covariance, it is required that $m \leq (K-1)/2$. The correlation between securities prices typically reduces the number of parameters in the model, so that all parameters can be estimated.

[2] The volatilities $\delta_i, i = 1, ...K$, are non-stochastic. They represent the specific variance of each security. The securities prices are correlated, but the correlation is generated by the factors in the expected rates of return. There is evidence that stochastic volatility factors are important for capturing certain aspects of returns distributions such as heavy tails (Chernov et. al. 2002). An alternative approach to extreme returns involves adding independent shock terms to capture dramatic price changes. The dynamic equations become

$dY_i = \lambda_i dt + \delta_i dV_i + \vartheta_i dN_i(\pi_i)$,

where $dN_i(\pi_i)$ is a poisson process with intensity π_i, and shock size $\vartheta_i, i = 1, ..., K$. Between shocks, the financial market is described by the Brownian motion equations. In normal times, the random drift model is sufficient to explain the mean, variance and covariance for returns. An approach to the model with shocks is to define a conditional model given the shocks. The conditional dynamics are in the form (2), and the methods of this paper are appropriate for estimating the conditional parameters. Then the shock component is iteratively determined to minimize mean squared error.

The emphasis in this paper will be on estimation of the mean since it has the greatest impact on portfolio decisions. Chopra and Ziemba (1993) show that equal size errors in estimators for the means, variances, and covariances affect portfolio performance in the order of 20:2:1, respectively. Grauer and Hakanson (1995) also report substantial improvement in investment performance using better estimates for the mean.

[3] The rates of return in normal times may be dynamic, with defining equations $d\lambda_i(t) = \mu_i dt + \beta_i dq_i$, where dq_i,$i = 1, ..., K$, are correlated Brownian motions. So $\lambda_i(t) = \lambda_i(0) + \mu_i t + \beta_i \sqrt{t} Z_i, i = 1, ..., K$. The estimation methods discussed later are easily adapted to the estimation of parameters with dynamic stochastic rates of return.

Returning to the pricing equations, let

$$\begin{aligned} Y(t) &= (Y_1(t), ..., Y_K(t))', \\ \lambda &= (\lambda_1, ..., \lambda_K)', \\ \Delta &= diag(\delta_1^2, ..., \delta_K^2), \\ \mu &= (\mu_1, ..., \mu_K)', \\ \Gamma &= (\gamma_{ij}) = (\beta_i \beta_j \rho_{ij}). \end{aligned}$$

Without loss of generality assume $Y(0) = 0$. Given (λ, Δ), the conditional distribution of log-prices at time t is

$$(Y(t)|\lambda, \Delta) \propto N(\lambda t, t\Delta) \tag{5}$$

From (3) the rate has a prior distribution

$$\lambda \propto N(\mu, \Gamma). \tag{6}$$

It follows that the marginal distribution of log-prices is

$$Y(t) \propto N(\mu t, \Sigma_t), \tag{7}$$

with

$$\Sigma_t = t^2\Gamma + t\Delta = \Gamma_t + \Delta_t.$$

It is an important property of the model that the covariance for log-prices is partitioned into a component determined by the random drift and a component determined by the diffusion.

At any point in time it is assumed that information is available on the history of prices. Consider the data available at time t, $\{Y(s), 0 \leq s \leq t\}$, and the corresponding filtration $\Im_t^Y = \sigma\{Y(s), 0 \leq s \leq t\}$. The usual estimate for the mean log rate of return based on the data is

$$\bar{Y}_t = \frac{1}{t}Y(t) \tag{8}$$

With the prior distribution for λ in (5) and the conditional distribution for $Y(t)$ in (4), the posterior distribution for λ given $\Im_t^Y$ is

$$(\lambda|\Im_t^Y) \propto N(\hat{\lambda}_t, \Gamma_t^0), \tag{9}$$

where $\hat{\lambda}_t = \mu + (I - \Delta_t \Sigma_t^{-1})(\bar{Y}_t - \mu)$ and $\Gamma_t^0 = \frac{1}{t^2}(I - \Delta_t \Sigma_t^{-1})\Delta_t$.

It follows that the Bayes estimate for the mean rate of return at time t is the conditional expectation

$$\hat{\lambda}_t = E(\lambda|\Im_t^Y) = \mu + (I - \Lambda_t \Sigma_t^{-1})(\overline{Y_t} - \mu) \tag{10}$$

.

The Bayes estimate in (10) depends upon unknown parameters $(\lambda, \Gamma, \Delta)$. If the parameters can be estimated from the data on past returns, then replacing $(\lambda, \Gamma, \Delta)$ with estimates $(\hat{\lambda}, \hat{\Gamma}, \hat{\Delta})$ will provide an empirical Bayes estimate for the mean rate of return.

2.2 Estimation

Assume that securities have been observed at regular intervals of width $\frac{t}{n}$ in the time period $(0, t)$. The log prices at times $\frac{(s+1)t}{n}$, given the log prices at times $\frac{st}{n}$, $s = 0, ..., n$, are defined by the model as

$$Y(s) = y(s-1) + \frac{t}{n}\lambda + \sqrt{\frac{t}{n}}\Delta^{\frac{1}{2}} Z. \tag{11}$$

The first order increments process generates sample rates

$$e(s) = (Y(s) - y(s-1)) \div \frac{t}{n} = \lambda + \sqrt{\frac{n}{t}} \Delta^{\frac{1}{2}} Z, \tag{12}$$

which are stationary with covariance

$$\Sigma_{nt} = \Gamma + \frac{n}{t} \Delta = \Gamma_{nt} + \Delta_{nt},$$

and mean $E(e) = \lambda$.

$>$From the realized trajectory of prices, the observations on log-prices at times $\frac{st}{n}$, $s = 0, ..., n$, are

$$\{Y_{is}, i = 1, ..., K; s = 1, ..., n\}.$$

The corresponding sample rates are

$$\{e_{is}, i = 1, ..., K; s = 1, ...n\}.$$

With $e'_s = (e_{1s}, ..., e_{Ks})$, it follows that

$$\bar{Y}_t = \frac{1}{n} \sum_{s=1}^{n} e_s \tag{13}$$

So $\bar{Y}_t$ is the maximum liklihood estimate of λ, given the sample rates $\{e_s, s = 1, ..., n\}$. Let the covariance matrix computed from the observed rates be S_{nt}, the usual estimate of Σ_{nt}. The theoretical covariance is partitioned as $\Sigma_{nt} = \Gamma_{nt} + \Delta_{nt}$, and the objective is to reproduce that partition with the sample covariance matrix. If the eigenvalues of Γ_{nt} are $\lambda_1, ..., \lambda_K$, and $\Delta_{nt} = diag(\delta_1^2, ..., \delta_K^2)$, then the eigenvalues of Σ_{nt} are $\lambda_1 + \delta_1^2, ..., \lambda_K + \delta_K^2$. When the rank of Γ_{nt} is $m < K$, then $\lambda_{m+1} = ... = \lambda_K = 0$. Consider the spectral decomposition of S_{nt}, with the ordered eigenvalues

$$g_1, ..., g_K$$

and the corresponding eigenvectors $l_1, ..., l_K$. To generate the desired sample sample covariance structure, choose a *truncation* value $m < K$, and define the matrices

$$L_{nt} = (l_1, ..., l_m) \tag{14}$$

$$G_{nt} = L_{nt} L'_{nt} \tag{15}$$

$$D_{nt} = diag(S_{nt} - G_{nt}) = (d_1, ...d_K) \tag{16}$$

$$S^*_{nt} = G_{nt} + D_{nt} \tag{17}$$

where the eigenvectors are scaled so that

$$l'_j l_j = g_j - d_j.$$

In the theoretical covariance, it is possible that the eigenvalues$\lambda_j, j = 1, ...K$, are all positive. However, it is expected that the covariance between securities prices is generated by a small number of underlying portfolio's (factors) , and therefore the number of positive eigenvalues(m) is small relative to K. In any case m is indeterminate and an arbitrary choice introduces error. Since many of the eignvalues and eigenvectors in the above construction are discarded, the method is referred to as *truncation*. The impact of truncation error will be considered in the next section.

The matrices G_{nt}, D_{nt}, S^*_{nt} are estimates of $\Gamma_{nt}, \Delta_{nt}, \Sigma_{nt}$, respectively. Therefore, $\hat{\Gamma} = G_{nt}$ and $\hat{\Delta} = \frac{t}{n} D_{nt}$ are estimates of model parameters Γ and Δ. The estimate of Σ_t is

$$\hat{\Sigma}_t = t^2\hat{\Gamma} + t\hat{\Delta} = \hat{\Gamma}_t + \hat{\Delta}_t. \tag{18}$$

For the parameter μ, the prior mean, assumptions about the financial market can guide estimation. If it is assumed that the there is a long term equilibrium value for returns on equities, it is reasonable to say $\lambda_i, i = 1, ..., K$, have a common mean. So $\mu' = (\mu, ..., \mu)$ and the prior mean is estimated by $\hat{\mu_t}1$, where 1 is a vector of ones and

$$\hat{\mu_t} = \frac{1}{nK} \sum_i \sum_s e_{is}. \tag{19}$$

The *truncation estimator* for the conditional mean rate of return at time t is

$$\hat{\lambda}_{Tr} = \hat{\mu_t}1 + (I - \hat{\Delta}_t\hat{\Sigma}_t^{-1})(\overline{Y_t} - \hat{\mu_t}1).$$

The truncation estimator is an empirical Bayes estimator since it is in the form of the Bayes estimator, with estimates for the prior parameters. Note that the assumption of a common prior mean could be relaxed to a common mean within asset classes, or some other grouping of securities.

An alternative empirical Bayes estimator has been developed by Jorion (1986). The prior mean is estimated by a weighted grand mean

$$\tilde{\mu}_t = 1' S_{nt}^{-1} \overline{Y_t} / (1' S_{nt}^{-1} 1). \tag{20}$$

The Bayes-Stein estimate of λ is

$$\hat{\lambda}_{BS} = \tilde{\mu}_t 1 + (\frac{n}{\varphi + K})(\overline{Y_t} - \tilde{\mu}_t 1), \tag{21}$$

with

$$\varphi = \frac{K+2}{(\overline{Y_t} - \tilde{\mu} 1)' S_{nt}^{-1} (\overline{Y_t} - \tilde{\mu} 1)}. \tag{22}$$

Although they have similar forms, the concept behind the Bayes-Stein estimator is quite different from the truncation estimator. The truncation estimator adjusts the maximum likelihood estimate $\bar{Y}_t$ based on the correlation between securities prices, or equivalently the scores on the latent market factors. The prior distribution is multivariate normal and the conditional covariance (specific variance) is diagonal. The Bayes-Stein estimator shrinks all the $\bar{Y}_{it}$ toward the grand mean, based on variance reduction. In this case, the prior is univariate normal, and the conditional covariance is not diagonal.

3 PARAMETER ESTIMATION ERROR

A number of possible estimators for the expected rate of return on securities have been presented. The truncation estimator is based on the Brownian motion model for dynamics, and incorporates structural information about the covariance between rates of return on securities. The estimator has logical appeal, but the standard assessment of an estimator is based on estimation error. This error is now considered both theoretically and numerically. Since it is understood that estimation is based on data at time$t$, the time subscript will be dropped in all expressions.

Each of the estimators can be written as

$$\hat{\lambda} = \hat{\mu}1 + (I - \hat{B})(\overline{Y} - \hat{\mu}1). \tag{23}$$

So $\hat{B} = 0$ gives $\hat{\lambda} = \overline{Y}$, $\hat{B} = DS^{*^{-1}}$ gives $\hat{\lambda} = \hat{\lambda}_{Tr}$, and $\hat{B} = (1 - \frac{n}{\varphi + I})I$ gives $\hat{\lambda} = \hat{\lambda}_{BS}$. In each case $\hat{B}$ can be viewed as an estimate of $B = \Delta\Sigma^{-1}$, the Bayes value. Of course, the assumptions underlying the prior would differ for each estimator.

3.1 Theoretical Risk

The standard criterion for comparing estimators is based on the mean squared error matrix:

$$E(\hat{\lambda} - \lambda)(\hat{\lambda} - \lambda)'. \tag{24}$$

(In the analysis of this section, the expectation in the MSE is with respect to the conditional and prior distributions, so the dynamic Bayes model is assumed to be correct.) The risk of an estimator is defined as the trace of the MSE matrix:

$$R(\lambda, , \hat{\lambda}) = trE(\hat{\lambda} - \lambda)(\hat{\lambda} - \lambda)'. \tag{25}$$

It is well known that the Bayes estimate minimizes the risk. The work here will focus on the additional risk that is incurred by using an empirical Bayes estimate. Consider the difference

$$\hat{\lambda}_{\hat{B}} - \hat{\lambda}_B = B(\hat{\mu}1 - \mu 1) + (B - \hat{B})(\overline{Y} - \hat{\mu}1).$$

The additional risk for an empirical Bayes estimator is

$$R^{+}(\hat{\lambda}_{\hat{B}}) = trEB(\hat{\mu}1 - \mu 1)(\hat{\mu}1 - \mu 1)'B' + trE(B - \hat{B})(\overline{Y} - \hat{\mu}1)(\overline{Y} - \hat{\mu}1)'(B - \hat{B})'.$$

This expression can be simplified to

$$R^{+}(\hat{\lambda}_{\hat{B}}) = \frac{1}{K^2 n}(1'\Sigma 1) trB11'B' + \frac{1}{n} trE(B - \hat{B})S(B - \hat{B}). \tag{26}$$

The relevant information for the truncation estimator is the correlation between asset prices, and in particular the underlying factors generating the correlation. To simplify the model, it is assumed that Σ_n has eigenvalues

$$\gamma_1 \geq ... \geq \gamma_m > \gamma_{m+1} = ... = \gamma_K = \delta^2.$$

So $\Sigma_n = \Lambda_n \Gamma_n \Lambda_n'$, with $\Gamma_n = diag(\gamma_1, ..., \gamma_m, \delta^2, ..., \delta^2)$, and then $\Sigma_n = \Lambda_n^* \Lambda_n^{*\prime} + \delta^2 I$ for $\Lambda_{nj}^* = (\gamma_j - \delta^2)^{\frac{1}{2}} \Lambda_{nj}, j = 1, ..., m$. (With the time subscript dropped, n denotes the number of equally spaced time points.)

The above structure specifies the rank of Γ_n, that is, the number of factors, and assumes a common error variance. To focus on the covariance, it will be assumed that the error variance is known. (Note that an estimate of the error variance is given in (16).) With the assumed structure, the additional risk for the truncation estimator and the mean rate of return can be compared. A convenient measure for additional risk is the *relative savings loss* (RSL) (Efron and Morris, 1972).

Definition 1 - Relative Savings Loss

Consider the Bayes estimator $\hat{\lambda}_B$, the maximum likelihood estimator $\overline{Y_t}$ and an alternative estimator $\hat{\lambda}$. Then the relative savings loss for the estimator $\hat{\lambda}$ is

$$RSL(\hat{\lambda}) = \frac{R^+(\hat{\lambda})}{R^+(\overline{Y})}. \tag{27}$$

So the RSL of the optimal Bayes estimator is 0, and the RSL for the sample mean is 1 . Intuitively, the risk for the truncation estimator is between the risk for the Bayes estimator and that for the mean, i.e. the RSL is less than 1 for the truncation estimator. There are two important components in the risk and RSL expressions for the truncation estimator: (i) sampling error; and (ii) truncation error. These components are considered separately-sampling error when there is no truncation error, and truncation error when there is no sampling error.

3.2 Sampling error when m is known

If the number of factors in the random rates of return is known, then the empirical Bayes estimator $\hat{\lambda}_{Tr}$ is an improvement on the mean rate of return $\overline{Y}$, provided there is sufficient data to identify model parameters.

Proposition *1*

Suppose that the number of factors $m < K$ *and the volatility* $\Delta = \delta^2 I$ *in the random rate of return model are known. Then there is a sample size* n^* *such that*

$$RSL(\hat{\lambda}_{Tr}) \leq 1,$$

if $n \geq n^*$.

Proof:

Consider the additional risk for the truncation estimator,

$$R^{+}(\hat{\lambda}_{Tr}) = \frac{1}{nK^2}(1'\Sigma 1)trB11'B' + \frac{1}{n}trE(B-\hat{B})S_n(B-\hat{B})',$$

where $B = \Delta\Sigma^{-1}$and $\hat{B} = \Delta S^{*^{-1}}$. Also $S_n = LGL'$ for $G = diag(g_1, ..., g_K)$. With m and $\Delta = \delta^2 I$ known, then $B = \Delta\Sigma^{-1} = \delta^2\Lambda\Gamma^{-1}\Lambda'$ and $\hat{B} = \Delta S^{*^{-1}} = \delta^2 LG^{*^{-1}}L'$, where $\Gamma = diag(\gamma_1, ..., \gamma_m, \delta^2, ..., \delta^2)$ and $G^* = (g_1, ..., g_m, \delta^2, ..., \delta^2)$. Consider

$$trE(B-\hat{B})S_n(B-\hat{B})' = trEBS_nB' - 2trEBS_n\hat{B}' + trE\hat{B}S_n\hat{B}'.$$

$>$From the structure it follows that

$$trE\hat{B}S_n\hat{B}' = E(\sum_{j=i}^{m}\frac{\delta^4}{g_j} + \sum_{j=m+1}^{K} g_j)$$

and

$$trEBS_nB' = \sum_{j=1}^{m}\frac{\delta^4}{\gamma_j} + \delta^2(K-m).$$

From the asymptotic moments of eigenvalues (Seber, 1984) and after some simplification

$$trEBS_n\hat{B}' = \sum_{j=1}^{m}\frac{\delta^4}{\gamma_j} + \sum\nolimits_{i=1}^{m} Eg_j + O(n^{-2}).$$

As well

$$E\sum_{j=1}^{m}\frac{1}{g_j}=\sum_{j=1}^{m}\frac{1}{\gamma_j}+\frac{1}{n}\{(K-m)\sum_{j=1}^{m}\frac{1}{\delta^2-\gamma_j}\}+O(n^{-2})$$

$$E\sum_{j=m+1}^{I}g_j=(K-m)+\frac{1}{n}\{(K-m)\sum_{j=1}^{m}\frac{\gamma_j}{\delta^2-\gamma_j}\}+O(n^{-2}).$$

Substituting expressions into the additional risk for $\hat{\lambda}_{Tr}$ yields

$$R^{+}(\hat{\lambda}_{Tr})=\frac{1}{nK^2}(1'\Sigma 1)trB11'B'+\frac{1}{n}\{\frac{K-m}{n}\sum_{j=1}^{m}\frac{\delta^4-\gamma_j}{\delta^2-\gamma_j}\}+O(n^{-2}).$$

For $\overline{Y}$, it follows that

$$R^{+}(\overline{Y})=\frac{1}{nK^2}(1'\Sigma 1)trB11'B'+\frac{1}{n}(\sum_{j=1}^{m}\frac{\delta^4}{\gamma_j}+\delta^2(K-m)).$$

Consider that

$$\left[\{\frac{K-m}{n}\sum_{i=1}^{m}\frac{\delta^4-\gamma_j}{\delta^2-\gamma_j}\}-\{\sum_{j=1}^{m}\frac{\delta^4}{\gamma_j}+\delta^2(K-m)\}\right]\leq 0$$

if

$$n\geq\frac{1}{\delta^2}\sum\nolimits_{i=1}^{m}\frac{\gamma_j-\delta^4}{\gamma_j-\delta^2}=n^*.$$

In that case,

$$RSL(\hat{\lambda}_{Tr}) = \frac{R^+(\hat{\lambda}_{Tr})}{R^+(\overline{Y}_t)} \leq 1$$

as required.

Usually the relative savings from using $\hat{\lambda}_{Tr}$ in place of $\overline{Y}$ would be substantial. This will be demonstrated with an example.

Example 1

Suppose there are $K = 5$ risky assets with prices defined by equations (2) and (3). Assume there is $m = 1$ factor in the random rates of return and the covariance matrix for log prices is

$$\Sigma = \begin{bmatrix} .10 & .0395 & .0395 & .0395 & .0395 \\ .0395 & .10 & .0395 & .0395 & .0395 \\ .0395 & .0395 & .10 & .0395 & .0395 \\ .0395 & .0395 & .0395 & .10 & .0395 \\ .0395 & .0395 & .0395 & .0395 & .10 \end{bmatrix}.$$

The eigenvalues are $\gamma = .250$ and $\delta^2 = .0625$ with multiplicity 4. From the expressions for additional risk

$$R^+(\hat{\lambda}_{Tr}) = \frac{.078125}{n} + \frac{5.25}{n^2}$$

and

$$R^+(\overline{Y}) = \frac{.343725}{n}.$$

Then the relative savings loss for the truncation estimator is $RSL(\hat{\lambda}_{Tr}) = 0.227 + \frac{15.27}{n}$. So $RSL(\hat{\lambda}_{Tr}) \leq 1$ if $n \geq 20$.

3.3 Asymptotic Truncation Error

In the situation where the number of factors is unknown, then an estimate m^* for m is required before estimating Λ. Understating the number of factors, presumably because the contribution of some factors is insignificant, is the rationale for the label truncation estimator. If the factors are equally important, so that the eigenvalues of Γ are equal, then the impact of truncation will be greater. This worst case will be considered since it provides a bound on the extra error from truncation. To simplify analysis, it will be assumed that the sample size n is large and therefore

$$\Sigma = \Lambda\Lambda' + \delta^2 I$$

will be used in the truncation estimator. Also, $\sigma^2 = \frac{1}{K} tr\Sigma$.

Proposition 2

Suppose Σ has eigenvalues γ with multiplicity m and δ^2 with multiplicity $K - m$, where $\gamma > \delta^2$. Consider $\hat{\lambda}_{Tr}$, the truncation estimator with number *of factors* $m^* < m$.

(i) If $m < \frac{K}{2}$, *then*$RSL(\hat{\lambda}_{Tr}) \leq 1$.

(ii), If $m > \frac{K}{2}$, *then there exists a value* $\delta_1^2, 0 \leq \delta_1^2 \leq \sigma^2$, *such that* $RSL(\hat{\lambda}_{Tr}) \leq 1$ *for*$\delta^2 \geq \delta_1^2$.

Proof:

With the covariance Σ having eigenvalues γ with multiplicity m and δ^2 with multiplicity $K-m$, then the m^* truncation estimates for δ^2 and $\Lambda_{\cdot j}$ are

$$d^2 = \delta^2 + \frac{(m-m^*)}{(K-m^*)}\phi$$

and

$$l_{\cdot j} = (1 + \frac{\delta^2 - d^2}{\phi})^{\frac{1}{2}}\Lambda_{\cdot j}, j = 1, ..., m^*,$$

where $\phi = \gamma - \delta^2$. With $\Sigma^* = L'L + D$, then $tr\Sigma = tr\Sigma^*$. The truncation estimator, based on Σ, has

$$\hat{B} = \left[\delta^2 + (\frac{m-m^*}{K-m^*})\phi\right]\left(1 + \frac{\delta^2 - d^2}{\phi}\right)\Lambda_1\Lambda_1',$$

where $\Lambda = (\Lambda_1, \Lambda_2)$ and Λ_1 is $K \times m^*$ and Λ_2 is $K \times (m - m^*)$. Then the additional risk for the truncation estimator is

$$R^+(\hat{\lambda}_{Tr}) = \frac{1}{K^2 n}(1'\Sigma 1) tr B11'B + \frac{1}{n} tr(B-\hat{B})\Sigma(B-\hat{B})' =$$
$$\frac{1}{K^2 n}(1'\Sigma 1) tr B11'B + \frac{1}{n}\frac{\phi(m-m^*)(K^2-2Km^*+mm^*)}{(\phi+\delta^2)(K-m^*)^2}.$$

For the mean $\overline{Y}$, the additional risk is

$$R^+(\overline{Y_t}) = \frac{1}{K^2 n}(1'\Sigma 1) tr B11'B + \frac{1}{n}\frac{K\delta^2(\phi+\delta^2)-m\phi\delta^2}{\phi+\delta^2}.$$

Without loss of generality let $\sigma^2 = 1$. Then

$$RSL(\hat{\lambda}_{Tr}) \leq 1$$

is equivalent to

$$\phi(m-m^*)(K^2-2Km^*+mm^*) \leq (K-m^*)^2[K\delta^2(\phi+\delta^2)-m\phi\delta^2].$$

This becomes $a\delta^4 + b\delta^2 + c \geq 0$, where $a = (K-m^*)(2m-K)$, $b = (K-m^*)(K-m) + (K^2-2m^*K+mm^*)$, and $c = -(K^2-2m^*K+mm^*)$. For the quadratic in δ^2 there are two cases.

(i) If $m < \frac{K}{2}$, the quadratic is concave with one root less than zero and the other greater than one. So $a\delta^4 + b\delta^2 + c \geq 0$ for $\delta^2 \in (0, \sigma^2)$ and $RSL(\hat{\lambda}_{Tr}) \leq 1$.

(ii) If $m \geq \frac{K}{2}$, the quadratic is convex with one root less than zero and the other between zero and one, say $0 < \delta_1^2 < 1$. Then $a\delta^4 + b\delta^2 + c \geq 0$ and $RSL(\hat{\lambda}_{Tr}) \leq 1$ for $\delta^2 \in (\delta_1^2, \sigma^2)$.

If the number of market factors is small, the effect of truncation cannot make the additional risk greater than that for the mean rate of return. If the number of factors is large, then the truncation estimator dominates when the variance specific to each asset is close to the variance of the common factors.

Example 2

To illustrate the effects of truncation, consider an example with $K = 8$ assets. The covariance structure is determined by the matrix $\Lambda_m = m^{-\frac{1}{2}} E_m$, where E_m is defined by the first m columns of the matrix

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

The error variance is $\delta^2 = 1$. The variance of each log-price is 2 and

$tr\Sigma = 16$. The eigenvalues of Σ are $\gamma = \frac{8}{m} + 1$, with multiplicity m, and $\delta^2 = 1$, with multiplicity $K - m$. This is the situation of the proposition, where truncation error is greatest. In Table 2, the relative savings loss for the truncation estimator, with the range of values of$m$ and m^*, is presented.

		m							
		1	2	3	4	5	6	7	8
	1	0.9903	0.9575	0.9219	0.9007	0.8995	0.9249	0.9252	1.0
	2		0.9513	0.9804	0.8809	0.8769	0.9061	0.9391	1.0
	3			0.8944	0.8581	0.8490	0.8817	0.9214	1.0
m^*	4				0.8351	0.8154	0.8419	0.8960	1.0
	5					0.7812	0.8055	0.8572	1.0
	6						0.7586	0.7921	1.0
	7							0.6975	1.0
	8								1.0

Table 1: Relative Savings loss with Truncation: Worst Case Scenario

As expected, including more factors (less truncation) improves the relative performance of the empirical Bayes estimator.

3.4 Simulation Results

The analytic results have established the relative advantage of the truncation estimator. To calibrate the size of the advantage, two data sets of actual asset returns are now considered: (i) end of month prices for 24 leading stocks from the Toronto Stock Exchange (TSE); (ii) end of month prices for 24 leading stocks from the New York Stock Exchange (NYSE). For both markets, the data covers the years 1990 -2002. The correlation structure for prices is different for the exchanges, so a performance comparison of estimators will show the significance of structure. The percent of variance accounted for by the top 5 eigenvalues for each correlation matrix is shown in Table 2.

Table 2: Leading Eigenvalues (% of variance)

	γ_1	γ_2	γ_3	γ_4	γ_5	Total
NYSE	21.9	12.4	7.9	6.6	5.5	54.3
TSE	28.3	18.7	14.8	8.9	6.7	77.4

>From the data on monthly closing prices for the set of 24 stocks on the Toronto Stock Exchange and the separate set of stocks on the New York Stock Exchange, the price increments (natural log of gross monthly rates of return) were computed. The mean vector and covariance matrix of the increments for each exchange was computed, and these values were used as parameters in the dynamic model. Trajectories of prices for 50 months were simulated and the expected rates of return were estimated by the various methods: average, truncation estimator, and Bayes-Stein estimator. For the truncation estimator the number of factors was preset at $m^* = 5$. Since the TSE data has a more compact market structure, it is expected that the truncation estimator with $m^* = 5$ will perform better in that case. This expiriment was repeated 1000 times and the root mean squared error for each estimator was computed. The results are shown in Table 2.

NYSE Stock	Estimator			TSE Stock	Estimator		
	Tr	Avg	BS		Tr	Avg	BS
$S_{NYSE,1}$	1.9725	2.8432	2.1462	$S_{TSE,1}$	1.7668	4.5063	3.1027
$S_{NYSE,2}$	4.1581	4.8305	4.2945	$S_{TSE,2}$	2.7083	4.4457	3.2007
$S_{NYSE,3}$	1.8415	2.9378	2.2138	$S_{TSE,3}$	1.9782	4.3989	3.0300
$S_{NYSE,4}$	2.2514	3.0818	2.3202	$S_{TSE,4}$	1.7466	4.3747	3.0131
$S_{NYSE,5}$	2.5917	2.8565	2.2977	$S_{TSE,5}$	1.6950	4.5301	3.1046
$S_{NYSE,6}$	2.8090	2.9188	2.2270	$S_{TSE,6}$	1.8265	4.6527	3.2517
$S_{NYSE,7}$	4.9898	5.3249	5.8516	$S_{TSE,7}$	1.6682	4.2899	2.9744
$S_{NYSE,8}$	2.1711	2.9338	2.2597	$S_{TSE,8}$	1.8485	4.3125	2.9897
$S_{NYSE,9}$	2.3748	3.1987	2.5019	$S_{TSE,9}$	1.8717	4.8132	3.3046
$S_{NYSE,10}$	2.0109	3.0979	2.4649	$S_{TSE,10}$	2.0050	4.4784	3.1089
$S_{NYSE,11}$	2.5031	3.0759	2.4323	$S_{TSE,11}$	2.3007	4.5624	3.2394
$S_{NYSE,12}$	1.9821	2.9752	2.2563	$S_{TSE,12}$	2.2495	5.0386	3.4564
$S_{NYSE,13}$	2.0238	2.9266	2.2242	$S_{TSE,13}$	2.6598	4.3629	3.1579
$S_{NYSE,14}$	1.9424	3.1798	2.5272	$S_{TSE,14}$	1.9208	4.5340	3.1757
$S_{NYSE,15}$	1.9684	3.0972	2.3393	$S_{TSE,15}$	1.7682	4.4389	3.0406
$S_{NYSE,16}$	2.2664	2.6545	2.0528	$S_{TSE,16}$	1.9065	4.3880	3.0011
$S_{NYSE,17}$	2.2813	2.7824	2.1058	$S_{TSE,17}$	2.1732	4.8737	3.2946
$S_{NYSE,18}$	4.5069	5.5240	5.7066	$S_{TSE,18}$	2.1746	4.7862	3.3036
$S_{NYSE,19}$	1.8315	2.8630	2.2293	$S_{TSE,19}$	2.2255	4.4604	3.1621
$S_{NYSE,20}$	5.3000	5.6623	6.3081	$S_{TSE,20}$	1.8554	4.3601	3.0257
$S_{NYSE,21}$	2.3198	2.7973	2.1474	$S_{TSE,21}$	1.6668	4.2875	2.9623
$S_{NYSE,22}$	2.2613	3.2026	2.3916	$S_{TSE,22}$	2.0788	4.5036	3.1099
$S_{NYSE,23}$	2.7386	2.8809	2.3177	$S_{TSE,23}$	2.0384	4.4373	3.1094
$S_{NYSE,24}$	1.8536	2.9950	2.2968	$S_{TSE,24}$	1.9518	4.1811	2.8882
AVG	2.6229	3.3600	2.8297	AVG	2.0026	4.5007	3.1253

Table 3: %Root Mean Squared Error

The truncation estimator has smaller mean squared error for most stocks on the NYSE and for all stocks on the TSE. As predicted, the performance is better for the stocks on the TSE.

4 WEALTH EFFECTS

The signifigance of quality estimates for model parameters gets highlighted when those values becomes inputs to portfolio decisions and the accumulation of wealth over time. The dynamic process for determining a portfoliois illustrated in Figure 2, where at discrete points in time the model parameters are re-estimated to include new information acquired for prices. The revised estimates are fed into the portfolio choice model, and a new strategy is calculated. So the investment strategy depends on the estimated parameters.

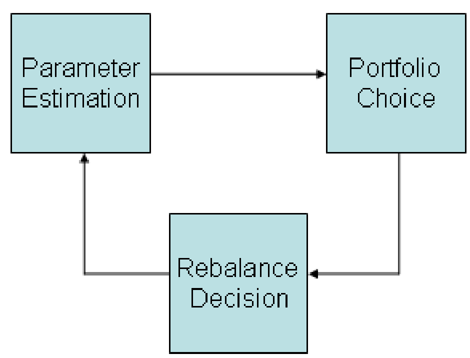

Figure 1: Dynamic Investment Process

This process will be illustrated for the portfolio selection problem with constant relative risk aversion utility.

4.1 Wealth Loss

Consider an investor with wealth w_t at time t and investment strategy, given estimator B,

$$X_B(t) = (x_{B1}(t), ..., x_{BK}(t))',$$

where $x_{Bi}(t)$ is the fraction invested in risky security $i, i = 1, ..., K$, and $x_{B0}(t) = 1 - \sum_{i=1}^{K} x_{Bi}(t)$ is the fraction in the risk-less security. Let $\hat{\alpha}_{Bi}(t) = \hat{\lambda}_{Bi}(t) + \frac{1}{2}\delta_i^2, i = 1, ..., K$. The wealth in period $t+1$ with the strategy $X_B(t)$ is

$$W_B(t+1) = w_t \exp\left\{ X_B(t)'(\hat{\alpha}_B(t) - r) + r - \frac{1}{2} X_B(t)'\Delta X_B(t) + X_B(t)'\Delta Z \right\}, \tag{28}$$

where $Z' = (Z_1, ..., Z_K), Z_i \propto N(0,1)$.

The objective is to maximize the expected utility of wealth at time$t+1$, with

$$Eu(W(t+1)) = E\left[\frac{W(t+1)^\beta - 1}{\beta}\right], \tag{29}$$

where $\beta < 1$ and $\beta \neq 0$. This is the constant relative risk aversion power utility function. When $\beta = 0$,$u(w) = ln(w)$. For the dynamic pricing model with random rates of return, the optimal strategy is similar to the solution given by Merton (1992):

$$X_B(t) = \frac{1}{1-\beta}\Delta^{-1}(\hat{\alpha}_B(t) - re). \tag{30}$$

It is assumed with (30) that the Bayes pricing model from (1)–(3) is correct.

A derivation of the optimal solution is given in MacLean, Ziemba and Li (2003). It is significant that the conditional covariance of log-prices, Δ, is a key component of the solution. So the investment strategy depends on the conditional mean and the conditional covariance, and values for both are required to implement the strategy. The empirical Bayes (truncation) estimation provides estimates for both parameters.

When estimates for the parameters are used in calculating the strategy, so that

$$X_{\hat{B}}(t) = \frac{1}{1-\beta}\hat{\Delta}^{-1}(\hat{\alpha}_{\hat{B}}(t) - re), \tag{31}$$

then $t+1$ period wealth is

$$W_{\hat{B}}(t+1) = w_t \exp\left\{X_{\hat{B}}(t)'(\hat{\alpha}_B(t) - r) + r - \frac{1}{2}X_{\hat{B}}(t)'\hat{\Delta}X(t) + X(t)'\hat{\Delta}Z\right\}. \tag{32}$$

The ratio

$$WL(X_{\hat{B}}) = E\log\frac{W_B(t+1)}{W_{\hat{B}}(t+1)} \tag{33}$$

indicates the wealth loss from estimating the parameters in the Bayes model. A comparison of the empirical Bayes (truncation) strategy with the mean rate strategy (with an independent estimate for covariance) is defined by the relative wealth loss.

Definition 2- Relative Wealth Loss

Let $X_{\overline{Y}}(t), X_{\hat{B}}(t)$ be the investment strategies from using the maximum liklihood, and empirical Bayes (truncation) estimates, respectively, for the random rates of return. The one period ahead relative wealth loss is defined as

$$RWL(X_{\hat{B}}) = \frac{WL(X_{\hat{B}})}{WL(X_{\overline{Y}})}.$$

Following from the results on mean squared error, the wealth loss should be less for the strategy based on the truncation estimator than for the strategy based on the mean rate of return. In developing the comparison assume that $\Delta = \delta^2 I$ is given.

Proposition 3

Assume that $\Delta = \delta^2 I$ *and* m *are known. Then*

$RWL(X_{\hat{B}}) < RSL(\hat{\lambda}_{\hat{B}})$ *if* $\beta < 0$

$RWL(X_{\hat{B}}) = RSL(X_{\hat{B}})$ *if* $\beta = 0$

$RWL(X_{\hat{B}}) > RSL(\hat{\lambda}_{\hat{B}})$ *if* $0 < \beta < 1.$

Proof:

Consider $WL(X_{\hat{B}}) = ElnW(t+1) - ElnW(t+1)$

$$= \frac{-\beta}{(1-\beta)^2}\delta^{-2}E(\hat{\lambda}_B - \hat{\lambda}_{\hat{B}})'(\hat{\lambda}_B + (\frac{1}{2}\delta^2 - r)e) + \frac{1}{2(1-\beta)^2}\delta^{-2}E(\hat{\lambda}_{\hat{B}} - \hat{\lambda}_B)'(\hat{\lambda}_{\hat{B}} - \hat{\lambda}_B)$$
$$= \frac{-\beta}{(1-\beta)^2}\delta^{-2}E(\hat{\lambda}_B - \hat{\lambda}_{\hat{B}})'(\hat{\lambda}_B + (\frac{1}{2}\delta^2 - r)e) + \frac{1}{2(1-\beta)^2}\delta^{-2}R^+(\hat{\lambda}_{\hat{B}}, \hat{\lambda}_B).$$

Also

$$E(\hat{\lambda}_B - \hat{\lambda}_{Tr})'\hat{\lambda}_B = E(\hat{\lambda}_B - \hat{\lambda}_{\overline{Y}})\hat{\lambda}_B + E(\hat{\lambda}_{\overline{Y}} - \hat{\lambda}_{Tr})\hat{\lambda}_B.$$

Furthermore

$$E(\hat{\lambda}_{\overline{Y}} - \hat{\lambda}_{Tr})'\hat{\lambda}_B = \left[\hat{B}(\overline{Y} - \hat{\mu}1)\right]' \left[\overline{Y} - B(\overline{Y} - \mu 1)\right] = -EtrB(\overline{Y} - \hat{\mu}1)(\overline{Y} - \mu 1)'\hat{B}' = -EtrBS_n\hat{B}'$$

and

$$E(\hat{\lambda}_B - \hat{\lambda}_{\overline{Y}})'\hat{\lambda}_B = E\left[[-B(\overline{Y} - \mu 1)]'(\mu 1 + (I - B)(\overline{Y} - \mu 1)\right] =$$
$$EB'(\overline{Y} - \mu 1)'(\overline{Y} - \mu 1)B - E(\overline{Y} - \mu 1)'B(\overline{Y} - \mu 1) = EtrBS_nB' - EtrBS_n = EtrBS_nB - tr\Delta.$$

So

$$WL(X_{Tr}) = \frac{1}{2(1-\beta)^2}\delta^{-2}R^+(\hat{\lambda}_{Tr}, \hat{\lambda}_B) + \frac{\beta}{(1-\beta)^2}\delta^{-2}tr\Delta,$$

and

$$WL(X_{\overline{Y}}) = \frac{1}{2(1-\beta)^2}\delta^{-2}R^{+}(\hat{\lambda}_{\overline{Y}}, \hat{\lambda}_B) + \frac{\beta}{(1-\beta)^2}\delta^{-2}tr\Delta - \frac{\beta}{(1-\beta)^2}\delta^{-2}trEBS_nB.$$

With $RWL(X_{Tr}) = \frac{WL(X_{Tr})}{WL(X_{\overline{Y}})}$ and $RSL(\hat{\lambda}_{Tr}) = \frac{R^{+}(\hat{\lambda}_{Tr})}{R^{+}(\hat{\lambda}_{\overline{Y}})}$, the statement in the theorem follows.

The wealth loss depends upon the risk aversion at the time of decision. In the decision rule, the risk aversion parameter β defines a fraction of capital invested in the optimal growth portfolio: $\frac{1}{1-\beta}$. When $\beta < 0$, the control of decision risk also reduces the impact of estimation error. Correspondingly, when $\beta > 0$, the overinvestment increases the effect of estimation error. When comparing the decisions based on the empirical Bayes and mle estimates, the improvement in parameter estimation translates into better wealth performance.

Corollary 1

Suppose that the number of factors $m < K$ *and the volatility* $\Delta = \delta^2 I$ *are known. Then there exists a value* n^* *such that* $RWL(X_{Tr}) < 1$ *when* $n \geq n^*$ *and* $\beta \leq 0$.

If the number of factors is unknown, then additional error from truncation will be included in the estimator and the investment decision. Consider the bias introduced by truncation

$$\Theta(\hat{\lambda}_{Tr}) = E(\hat{\lambda}_B - \hat{\lambda}_{Tr})'1. \tag{34}$$

If this bias is sufficiently small, then the relationship between wealth loss and estimation error are retained.

Proposition 4

Consider assets with price dynamics defined by (1)-(3), and investment strategy defined by (29). Let $\hat{\lambda}_{Tr}$ *be the truncation estimate with number of factors* $m^* < m$. *If the bias for the truncation estimate satisfies*

$$(\frac{1}{2}\delta^2 - r)\Theta(\hat{\lambda}_{\hat{B}}) < trEBS_nB',$$

then $RWL(X_{\hat{B}}) < RSL(\hat{\lambda}_{\hat{B}})$ *if* $\beta < 0$

$RWL(X_{\hat{B}}) = RSL(X_{\hat{B}})$*if* $\beta = 0$

$RWL(X_{\hat{B}}) > RSL(\hat{\lambda}_{\hat{B}})$ *if* $0 < \beta < 1$.

Proof:

$>$From the first statement in Proposition 3, it follows that

$$RWL(X_{\hat{B}}) = \frac{R^{+}(\hat{\lambda}_{\hat{B}}) + 2\beta tr\Delta - 2\beta(\frac{1}{2}\delta^2 - r)\Theta(\hat{\lambda}_{\hat{B}})}{R^{+}(\hat{\lambda}_{\overline{Y}}) + 2\beta tr\Delta - 2\beta trEBS_nB'}.$$

If the bias inequality is satisfied, then the statements relatingRWL and RSL hold.

The ordering on estimation loss generates an ordering on wealth loss for alternative investment strategies when the number of factors in the price dynamics is truncated. For the worst case, where the truncated factors are as important as those retained, the ordering follows from Proposition 2.

Corollary 2

Suppose Σ *has eigenvalues* γ *with multiplicity* m *and* δ^2 *with multiplicity* $K - m$, *where* $\gamma > \delta$ *and* $\sigma^2 = \frac{1}{K}tr\Sigma$. *Let the number of factors in* $\hat{\lambda}_{\hat{B}}$ *be* $m^* < m$, *with* $\beta \leq 0$, *and assume* $(\frac{1}{2}\delta^2 - r)\Theta(\hat{\lambda}_{\hat{B}}) < trEBS_nB'$. *Then*

(i)$RWL(\hat{\lambda}_{\hat{B}}) < 1$ *when* $m \leq \frac{K}{2}$

(ii)$RWL(\hat{\lambda}_{\hat{B}}) < 1$ *when* $m < \frac{K}{2}$ *and* $\delta^2 \geq \delta_1^2$, *for appropriate choice of*

$\delta_1^2, 0 \leq \delta_1^2 \leq \sigma^2$.

The empirical Bayes or truncation estimator in general has smaller mean squared error than the mle, and that saving translates directly into improved decisions and wealth accumulation in the case of log utility ($\beta = 0$). The log or optimal growth strategy is aggressive, even in the Bayes case where parameter values are known. A fractional log strategy, based on a negative power utility function ($\frac{1}{1-\beta} < 1$), controls the inherent risk, and also lowers the loss from estimation error. In contrast, the levered strategies, from positive power utility functions ($\frac{1}{1-\beta} > 1$), exacerbate the losses.

4.2 Application to NYSE

The dynamic investment process is now implemented with the data from the New York Stock Exchange. The approach is to forecast the prices for the next month using the Truncation or Bayes-Stein estimator, then calculating the investment strategy for the expected utility maximization criteria. The log utility is assumed, so that $\beta = 0$. The segment of the data series from January, 1995 to December 1996 is used. The forecast and strategy are developed from past prices, and the return is calculated from the actual prices. This backcast is worked forward for 24 months, with the resulting accumulated capital shown in Table 4. The returns are very large, as is characteristic of that period, and capital is borrowed to invest in high return securities. The relevant statistic is the relative return for the comparative estimators.

The superiority of the truncation estimator, which was demonstrated in the simulation, translates into improved strategies and greater wealth.

Month	Bayes-Stein	Truncation	Ratio
1	1000.00	1000.00	1.00
2	2753.84	3114.25	1.13
3	4734.16	5755.60	1.22
4	6155.03	7495.27	1.22
5	7072.46	8635.16	1.22
6	8800.46	10852.75	1.23
7	8057.76	9693.90	1.20
8	9839.54	12188.45	1.24
9	9915.73	12495.84	1.26
10	12373.73	16287.52	1.32
11	12945.05	17198.59	1.33
12	14734.91	19844.46	1.35
13	15114.73	20335.51	1.35
14	15905.18	21378.58	1.34
15	15753.94	21355.97	1.36
16	16988.85	23065.39	1.36
17	17817.41	24258.14	1.36
18	16744.71	22670.64	1.35
19	17483.44	23516.54	1.35
20	19935.02	26538.49	1.33
21	21354.56	28376.71	1.33
22	25299.94	33494.38	1.32
23	24760.45	32790.56	1.32
24	28880.96	37855.50	1.31

Table 4: Actual Wealth Trajectory

The 24 month test interval is appropriate for comparing estimators since it doesn't include the market collapse following an overvaluation of securities. A model including shocks with depending on the size of the overvaluation would supplement the model to cover a market bubble.

5 CONCLUSION

The estimation of the rates of return on assets is a critical ingredient to a successful investment strategy. An effective combination of modeling and data can result in significant improvement in capital accumulation. In this paper a Bayes dynamic pricing model is the basis for a truncation estimator of the instantnaeous rate of return on assets. The key to the truncation estimator is the correlation between asset prices. The common information in price movements contributes to improved estimation of individual estimates.

The truncation estimator is compared to well known estimators - mean and Bayes-Stein. If asset prices follow the geometric Brownian motion model, then analytic results establish the superiority of the truncation estimator. From simulation results, the truncation estimator outperforms the alternatives in general.

The savings in estimation error with the proposed estimator translate into better decisions and wealth. A back test on data from the New York Stock Exchange emphasize the gains. The analytic formulation provides an assessment of risk aversion in the face of uncertain returns. In particular, the risk aversion index can be used to offset the loss from estimation error with the truncation estimator.

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