Optimization for Business and Economics

This set of notes will introduce the mathematical methods useful in analytics. As it is assumed the reader has at least some exposure to calculus, the nuts and bolts are kept brief. Some of the material here is based on Nicholson and Snyder's *Microeconomic Theory: Basic Principles and Extensions*. This is a great reference book for those wishing to delve into the mathematics of economics.

1 Objective Functions

We typically approach problems in economics as *optimization problems*. Essentially, an economic agent, be it a consumer, firm, or governmental entity, wishes to achieve the best outcome possible, that is, the optimal outcome. A consumer's objective is usually to maximize utility. In this course, we will be focusing on firms, and for the most part, a firm's goal is to maximize profit. Some variables within the problem are within the problem are within the agent's control. We'll call them *choice variables*. The agent's problem is to choose values for the choice variables such that the optimal outcome is achieved.

We define the relationship between the choice variables and outcomes using *objective functions*. In economics, models often give the firm the ability to choose either price or quantity and let the market determine the other. More complex models may have more than one choice variable, a concept we will explore in section 5. For now, we'll stick with an abstract objective function $f(x)$. Here, x is the choice variable.

2 Derivatives

Before we go any further, we need to know what exactly a derivative is. The derivative of a function is itself a function, one that describes how the function's output changes as the input changes. For a function $f(x)$, we denote its derivative as either $\frac{df}{dx}$ or $f'(x)$. Sometimes we will also use capital letters to denote the original function and lower-case for the derivative ($F(x)$ and its derivative $f(x)$).

2.1 Basic Derivative Rules

You should be aware of the basic rules of derivatives, which I will outline here. Let a be an arbitrary constant and $f(x)$, $g(x)$ be arbitrary functions.

1. $\frac{da}{dx} = 0$
2. $\frac{d[af(x)]}{dx} = a\frac{df(x)}{dx}$
3. $\frac{d(x^a)}{dx} = ax^{(a-1)}$
4. $\frac{d[log(x)]}{dx} = \frac{1}{x}$, where $log(x)$ is the natural logarithm of x
5. $\frac{d[f(x)+g(x)]}{dx} = f'(x) + g'(x)$
6. $\frac{d[f(x)*g(x)]}{dx} = f(x) * g'(x) + f'(x) * g(x)$ (Product rule)
7. $\frac{d[f(x)/g(x)]}{dx} = \frac{f'(x)*g(x)-f(x)*g'(x)}{[g(x)]^2}$ (Quotient rule)

3 First-Order Conditions

We approach these types of problems using calculus. We find maxima and minima of the objective function by taking the first derivative and setting this equal to zero. We call this equation the *first-order condition* (FOC). For example, suppose we have an objective function

$$f(x) = -x^2 + 10x + 5.$$

Taking the first derivative, we have

$$f'(x) = -2x + 10.$$

Hence, the first-order condition is given by

$$-2x + 10 = 0.$$

With a little algebra, we can see that $x = 5$ is the value we're after.

4 Second-Order Conditions

First-order conditions don't tell the whole story. A point satisfying the first-order condition need not be a maximum. It can also be a minimum or an inflection point. To check this, we need to verify concavity; this is done with the second-order condition. If the second derivative is negative, we have a concave function, and if the function is concave, we have a maximum. Continuing with our example from the last section, the second derivative of $f(x)$ is

$$f''(x) = -2$$

Note that this is a constant and that it is negative. From this, we know that $f(x)$ is concave for all values of x. If a function is concave, we know that the point really is a maximum. Let's look at a few more examples, where the first-order condition doesn't necessarily give us a maximum. Consider $f(x) = x^4$. We have the FOC

$$f'(x) = 4x^3 = 0.$$

The only x that satisfies this condition is $x = 0$. But is this actually a maximum? Let's take the second derivative and find out:

$$f''(x) = 12x^2.$$

Since x^2 is nonnegative for all x, the function is not concave, but *convex*. This means that $x = 0$ is a minimum.

Now consider $f(x) = x^3$. This gives us the FOC

$$f'(x) = 3x^2 = 0.$$

This tells us our point of interest lies at $x = 0$. Now, let us examine the second derivative.

$$f''(x) = 6x.$$

Note that the sign of $f''(x)$ depends on x. Specifically, $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$. Lastly, we also notice that $f''(0) = 0$. The function is concave to the left of 0 and convex to the right of 0. In this case, $x = 0$ is an *inflection point* – neither a minimum or a maximum.

5 Partial Derivatives

In game theoretic models, agents are aware of the impact that the action of others can have on their own outcomes. When these agents (firms, generally) make their decisions, they have to take into account what others might do. Let's look at an example objective function. Suppose we have two agents who we'll creatively call 1 and 2. Each makes a choice of their own x but can't affect what the other does. Suppose agent 1's objective function is

$$f_1(x_1, x_2) = 10x_1 + x_1x_2 - {x_1}^2$$

Although he might like to, agent 1 change x_2, only x_1. Thus, x_1 is his only choice variable and x_2 can be treated as a constant in this decision. We now take the derivative of the objective function *with respect to* x_1. We call this a *partial derivative.* Partial derivatives are denoted as $\frac{\partial f}{\partial x_1}$. Note the "curly" ∂ that is used here.

Continuing with our example, let's find the optimal x_1. The FOC here is

$$10 + x_2 - 2x_1 = 0,$$

again noting that we treat x_2 as we would any other constant, such as the 10. Solving for x_1, we have

$$\frac{10 + x_2}{2} = x_1.$$

Note also that the second derivative (again, with respect to x_1) is -2, so this is a maximum. The above function gives us the optimal x_1 for any possible value of x_2. As such, we will call this a *best-response function.* Now, suppose the second agent's objective function is

$$f_2(x_1, x_2) = 10x_2 + x_1x_2 - {x_2}^2.$$

Now, the first order condition is

$$\frac{10 + x_1}{2} = x_2.$$

Solving these as a system of equations, we have

$$x_1 = 10$$

$$x_2 = 10.$$

5.1 Single Agent, Multiple Choices

We also need to use partial derivatives to solve decision problems with only one agent, but more than one choice variable. There are many economic situations where this is relevant, including consumer choice problems (how much of good x and how much of good y to consume?), labor-leisure choice problems, and cost minimization problems (how much labor and capital is efficient for a target output?). In a microeconomic theory course, this kind of problem is typically *constrained*, meaning there are certain off-limits values for the choice variables. Typically, this includes nonnegativity constraints or budget constraints. We will not be worrying about constraints in this course.

Consider the following objective function:

$$f(x_1, x_2) = x_1 x_2 + x_1 + x_2 - x_1^2 - x_2^2.$$

Suppose we want to maximize the value of the objective function. We need to find the optimal values for x_1 and x_2. To do this, take both first order conditions. First, with respect to x_1:

$$x_2 - 2x_1 + 1 = 0$$

and then with respect to x_2:

$$x_1 - 2x_2 + 1 = 0.$$

To solve the problem, solve these two equations simultaneously. We get

$$x_1 = 1$$

$$x_2 = 1.$$

5.2 Application to OLS

When doing ordinary least squares (OLS), the objective function is the residual sum of squares, which we seek to minimize.

$$RSS = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2$$

The first order conditions will involve taking first derivatives with respect to all $\hat{\beta}$ and setting these equal to zero. Solving this system will yield the OLS estimators.

6 References

Nicholson, Walter and Snyder, Christopher. *Microeconomic Theory: Basic Principles and Extensions*, 10th Edition. 2008.