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SPREAD OPTION VALUATION AND THE FAST FOURIER TRANSFORM

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WP 26/2000

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July 2000

Abstract

We investigate a method for pricing the generic spread option beyond the classical two-factor Black-Scholes framework by extending the fast Fourier Transform technique introduced by Carr & Madan (1999) to a multi-factor setting. The method is applicable to models in which the joint characteristic function of the underlying assets forming the spread is known analytically. This enables us to incorporate stochasticity in the volatility and correlation structure – a focus of concern for energy option traders – by introducing additional factors within an affine jump-diffusion framework. Furthermore, computational time does not increase significantly as additional random factors are introduced, since the fast Fourier Transform remains two dimensional in terms of the two prices defining the spread. This yields considerable advantage over Monte Carlo and PDE methods and numerical results are presented to this effect.

1 Introduction

Spread Options are derivatives with terminal payoffs of the form: $[(\boldsymbol{S}_1(T) - \boldsymbol{S}_2(T)) - K]_+$, where the two underlying processes $\boldsymbol{S}_1, \boldsymbol{S}_2$ forming the spread could refer to asset or futures prices, equity indices or (defaultable) bond yields. There is a wide variety of such options traded across different sectors of the financial markets; for example, the crack spread and crush spread options in the commodity markets [16, 22], credit spread options in the fixed income markets, index spread options in the equity markets [10] and the spark (electricity/fuel) spread options in the energy markets [9, 18]. They are also applied extensively in the area of real options [23] for both asset valuations and hedging a firm's production exposures. Despite their wide applicability and crucial role in managing the so-called *basis risk*, hedging and pricing of this class of options remain difficult and no consensus on a theoretical framework has emerged.

The main obstacle to a "clean" pricing methodology lies in the lack of knowledge about the distribution of the difference between two non-trivially correlated stochastic processes: the more variety we inject into the correlation structure, the less we know about the stochastic dynamics of the spread. At one extreme, we have the *arithmetic Brownian motion* model in which $\boldsymbol{S}_1, \boldsymbol{S}_2$ are simply two Brownian motions with constant correlation [19]. The spread in this case is also a Brownian motion and an analytic solution for the spread option is thus available. This, however, is clearly an unrealistic model as it, among other things, permits negative values in the two underlying prices/rates. An alternative approach to modelling the spread directly as a *geometric Brownian motion* has also proven inadequate as it ignores the intrinsic multi-factor structure in the correlation between the spread and the underlying prices and can lead to severe misspecification of the option value when markets are volatile [13].

Going one step further we can model the individual prices as geometric Brownian motions in the spirit of Black and Scholes and assume that the two driving Brownian motions have a constant correlation [17, 20, 22]. The resulting spread, distributed as the difference of two lognormal random variables, does not possess an analytical expression for its density, preventing us from deriving a closed form solution to the pricing problem. We can however invoke a conditioning technique which reduces the two dimensional integral for computing the expectation under the martingale measure to a one dimensional integral, thanks to a special property of the normal distribution: conditional on a correlated random variable a normal random variable remains normally distributed.

As we develop a stochastic term structure for volatilities and correlations of the underlying processes, we move out of the Gaussian world and the conditioning technique no longer applies. Furthermore, a realistic model for asset prices often requires more than two factors; for example, in the energy market, random jumps are essential in capturing the true dynamics of electricity or oil prices, and in the equity markets, stochastic volatilities are needed. Interest rate models such as the CIR or affine jump-diffusion models [11] frequently assume more than two factors and non-Gaussian dynamics for the underlying yields. However, the computational times using existing numerical techniques such as Monte Carlo or PDE methods increase dramatically as diffusion models take these issues into account.

In this paper we propose a new method for pricing spread options valid for the class of models which have analytic characteristic functions for the underlying asset prices or market rates. This includes the Variance Gamma (VG) model [15], the inverse Gaussian model [3] and numerous stochastic volatility and stochastic interest rates models in the general affine jump-diffusion family [1, 4, 6, 14, 21]. The method extends the fast Fourier transform approach of Carr & Madan [5] to a multi-factor setting, and is applicable to options with a payoff more complex than a piecewise-linear structure. The main idea is to integrate the option payoff over approximate regions bounding the non-trivial exercise region, analogous to the method of integrating a real function by Riemann sums. As for the Riemann integral, this gives close upper and lower bounds for the spread option price which tend to the true value as we refine the discretisation.

The FFT approach is superior to existing techniques in the sense that changing the underlying diffusion models only amounts to changing the characteristic function and therefore does not alter the computational time significantly. In particular, one can introduce factors such as stochastic volatilities, stochastic interest rates and random jumps, provided the characteristic function is known, to result in a more realistic description of the market dynamics and a more sophisticated framework for managing the volatility and correlation risks involved.

We give a brief review of the FFT pricing method applied to the valuation of a simple European option on two assets in Section 2. In Section 3 our pricing scheme for a generic spread option is set out in detail. Section 4 describes the underlying models implemented for this paper and presents computational results to illustrate the advantage of the approach and the need for a non-trivial volatility and correlation structure. Section 5 concludes and describes current research directions.

2 Review of the FFT Method

To illustrate the application of the fast Fourier Transform technique to the pricing of simple European style options in a multi-factor setting, in this section we derive the value of a correlation option as defined in [2] following the method and notation of [5] in the derivation of a European call on a single asset.

A *correlation option* is a two-factor analog of an European call option, with a payoff of $[\boldsymbol{S}_1(T) - K_1]_+ \cdot [\boldsymbol{S}_2(T) - K_2]_+$ at maturity T, where $\boldsymbol{S}_1, \boldsymbol{S}_2$ are the underlying asset prices. Denoting strikes and asset prices by K_1, K_2, S_1, S_2 and their logarithms by k_1, k_2, s_1, s_2, our aim is to evaluate the following integral for the option price:

$$\begin{aligned} C_T(k_1, k_2) \;&:=\; \mathbb{E}_{\mathbb{Q}}\Big[e^{-rT}\big[\boldsymbol{S}_1(T) - K_1\big]_+ \cdot \big[\boldsymbol{S}_2(T) - K_2\big]_+\Big] \\ &\equiv\; \int_{k_1}^{\infty}\int_{k_2}^{\infty} e^{-rT}\big(e^{s_1} - e^{k_1}\big)\big(e^{s_2} - e^{k_2}\big) q_T(s_1, s_2) ds_2 ds_1 \;\;, \qquad (1) \end{aligned}$$

where $\mathbb{Q}$ is the risk-neutral measure and $q_T(\cdot, \cdot)$ the corresponding joint density of $\boldsymbol{s}_1(T), \boldsymbol{s}_2(T)$.

The *characteristic function* of this density is defined by

$$\begin{aligned}\phi(u_1,u_2) &:= \mathbb{E}_{\mathbb{Q}}\big[\exp(iu_1\boldsymbol{s}_1(T)+iu_2\boldsymbol{s}_2(T))\big]\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(u_1s_1+u_2s_2)}q_T(s_1,s_2)ds_2ds_1.\end{aligned}$$

As in [5, 8], we multiply the option price (1) by an exponentially decaying term so that it is square-integrable in k_1, k_2 over the negative axes:

$$c_T(k_1,k_2) := e^{\alpha_1k_1+\alpha_2k_2}C_T(k_1,k_2) \quad \alpha_1,\alpha_2>0.$$

We now apply a Fourier transform to this modified option price:

$$\begin{aligned}\psi_T(v_1,v_2) &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(v_1k_1+v_2k_2)}c_T(k_1,k_2)dk_2dk_1\\ &= \iint_{\mathbb{R}^2} e^{(\alpha_1+iv_1)k_1+(\alpha_2+iv_2)k_2}\int_{k_2}^{\infty}\int_{k_1}^{\infty} e^{-rT}\big(e^{s_1}-e^{k_1}\big)\big(e^{s_2}-e^{k_2}\big)q_T(s_1,s_2)ds_2ds_1dk_2dk_1\\ &= \iint_{\mathbb{R}^2} e^{-rT}q_T(s_1,s_2)\int_{-\infty}^{s_2}\int_{-\infty}^{s_1} e^{(\alpha_1+iv_1)k_1+(\alpha_2+iv_2)k_2}\big(e^{s_1}-e^{k_1}\big)\big(e^{s_2}-e^{k_2}\big)dk_2dk_1ds_2ds_1\\ &= \iint_{\mathbb{R}^2}\frac{e^{-rT}q_T(s_1,s_2)\; e^{(\alpha_1+1+iv_1)s_1+(\alpha_2+1+iv_2)s_2}}{(\alpha_1+iv_1)(\alpha_1+1+iv_1)(\alpha_2+iv_2)(\alpha_2+1+iv_2)}ds_2ds_1\\ &= \frac{e^{-rT}\phi_T\big(v_1-(\alpha_1+1)i, v_2-(\alpha_2+1)i\big)}{(\alpha_1+iv_1)(\alpha_1+1+iv_1)(\alpha_2+iv_2)(\alpha_2+1+iv_2)}. \qquad (2)\end{aligned}$$

Thus if the characteristic function ϕ_T is known in closed form, the Fourier transform ψ_T of the option price will also be available analytically, yielding the option price itself via an inverse transform:

$$C_T(k_1,k_2) = \frac{e^{-\alpha_1k_1-\alpha_2k_2}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(v_1k_1+v_2k_2)}\psi_T(v_1,v_2)dv_2dv_1.$$

Invoking the trapezoid rule we can approximate this Fourier integral by the following sum:

$$C_T(k_1,k_2) \approx \frac{e^{-\alpha_1k_1-\alpha_2k_2}}{(2\pi)^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-i(v_{1,m}k_1+v_{2,n}k_2)}\psi_T(v_{1,m},v_{2,n})\Delta_2\Delta_1, \qquad (3)$$

where Δ_1, Δ_2 denote the integration steps and

$$v_{1,m} := (m-\frac{N}{2})\Delta_1 \quad v_{2,n} := (n-\frac{N}{2})\Delta_2 \quad m,n=0,\ldots,N-1. \qquad (4)$$

Recall that a two-dimensional *fast Fourier transform* (FFT) computes, for any complex (input) array, $\big\{X[j_1,j_2]\in\mathbb{C}\;\big|\; j_1=0,\ldots,N_1-1, j_2=0,\ldots,N_2-1\big\}$, the following (output) array of identical structure:

$$Y[l_1,l_2] := \sum_{j_1=0}^{N_1-1}\sum_{j_2=0}^{N_2-1} e^{-\frac{2\pi i}{N_1}j_1l_1-\frac{2\pi i}{N_2}j_2l_2}X[j_1,j_2], \qquad (5)$$

for all $l_1 = 0, \dots, N_1 - 1, l_2 = 0, \dots, N_2 - 1$. In order to apply this algorithm to evaluate the sum in (3) above, we define a grid of size $N \times N$, $\Lambda := \{(k_{1,p}, k_{2,q}) : 0 \le p, q \le N-1\}$, where

$$k_{1,p} := (p - \frac{N}{2})\lambda_1, k_{2,q} := (q - \frac{N}{2})\lambda_2$$

and evaluate on it the sum

$$\Gamma(k_1, k_2) := \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-i(v_{1,m}k_1 + v_{2,n}k_2)}\psi_T(v_{1,m}, v_{2,n}).$$

Choosing $\lambda_1\Delta_1 = \lambda_2\Delta_2 = \frac{2\pi}{N}$ gives the following values of $\Gamma(\cdot,\cdot)$ on Λ:

$$\begin{aligned}
\Gamma(k_{1,p}, k_{2,q}) &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-i(v_{1,m}k_{1,p} + v_{2,n}k_{2,q})}\psi_T(v_{1,m}, v_{2,n}) \\
&= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}\left[(m-N/2)(p-N/2)+(n-N/2)(q-N/2)\right]}\psi_T(v_{1,m}, v_{2,n}) \\
&= (-1)^{p+q}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}(mp+nq)}\left[(-1)^{m+n}\psi_T(v_{1,m}, v_{2,n})\right].
\end{aligned}$$

This is computed by the fast Fourier transform of (5) by taking the input array as

$$X[m,n] = (-1)^{m+n}\psi_T(v_{1,m}, v_{2,n}), \quad \forall m, n = 0, \dots, N-1.$$

The result is an approximation for the option price at $N \times N$ different (log) strikes given by

$$C_T(k_{1,p}, k_{2,q}) \approx \frac{e^{-\alpha_1 k_{1,p} - \alpha_2 k_{2,q}}}{(2\pi)^2}\Gamma(k_{1,p}, k_{2,q})\Delta_2\Delta_1 \quad 0 \le p, q \le N.$$

3 FFT Pricing of the Spread Option

3.1 Pricing a Spread Option with Riemann Sums

Let us now consider the *price* of a *spread option*, given by

$$\begin{aligned}
V(K) &:= \mathbb{E}_{\mathbb{Q}}\Big[e^{-rT}\big[\boldsymbol{S}_1(T) - \boldsymbol{S}_2(T) - K\big]_+\Big] \\
&= \int\int_{\Omega} e^{-rT}\big(e^{s_2} - e^{s_1} - K\big)q_T(s_1, s_2)ds_2 ds_1 \\
&= \int_{-\infty}^{\infty}\int_{\log(e^{s_1}+K)}^{\infty} e^{-rT}\big(e^{s_2} - e^{s_1} - K\big)q_T(s_1, s_2)ds_2 ds_1,
\end{aligned}$$

where the *exercise region* is defined as

$$\Omega := \Big\{(s_1, s_2) \in \mathbb{R}^2 \Big| e^{s_2} - e^{s_1} - K \ge 0\Big\}.$$

Transforming the option price with respect to the log of the strike K no longer gives the same kind of simple relationship with the characteristic function as in (2) of the previous section as a consequence of the simple shape of the exercise region Ω of the correlation option. If the boundaries of Ω are made up of straight edges, an appropriate affine change of variables can be introduced to make the method in the previous section applicable. This will not work for the pricing of spread options for which the exercise region is by nature non-linear (see Figure 1).

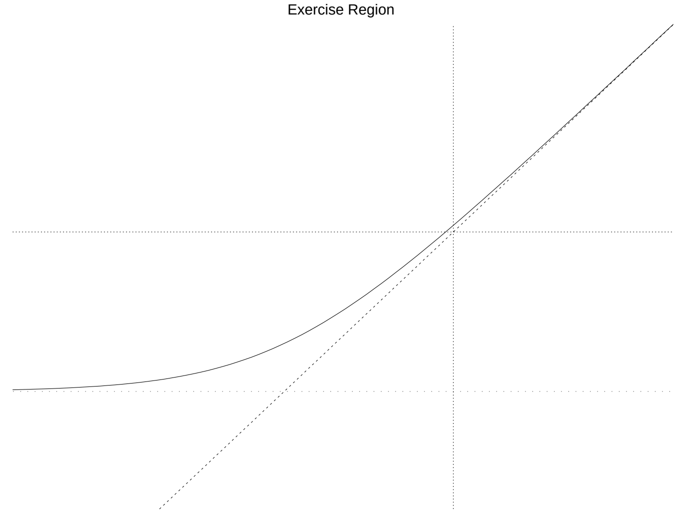

Figure 1: Exercise region of a spread option in logarithmic variables

Notice however from above that the FFT option pricing method gives $N \times N$ prices simultaneously in one transform, that is, integrals of the payoff over $N \times N$ different regions. By subtracting and collecting the correct pieces, we can form tight upper and lower bounds for an integral over a non-polygonal region analogous to integrating by Riemann sums. More specifically, we consider the following modified exercise region:

$$\Omega_\lambda := \left\{ (s_1, s_2) \in \left[-\tfrac{1}{2}N\lambda, \tfrac{1}{2}N\lambda \right) \times \mathbb{R} \;\middle|\; e^{s_2} - e^{s_1} - K \geq 0 \right\}$$

and construct two "sandwiching" regions $\underline{\Omega} \subset \Omega_\lambda \subset \overline{\Omega}$ out of rectangular strips with vertices on the grid of the inverse transform (see Figure 2 and 3).

Take as before an $N \times N$ equally spaced grid $\Lambda_1 \times \Lambda_2$, where

$$\begin{aligned} \Lambda_1 &:= \left\{k_{1,p}\right\} := \left\{ \left(p - \tfrac{1}{2}N\right)\lambda_1 \in \mathbb{R} \middle| 0 \leq p \leq N-1 \right\} \\ \Lambda_2 &:= \left\{k_{2,q}\right\} := \left\{ \left(q - \tfrac{1}{2}N\right)\lambda_2 \in \mathbb{R} \middle| 0 \leq q \leq N-1 \right\} \end{aligned}$$

For each $p = 0, \ldots, N-1$, define

$$\begin{aligned} \underline{k}_2(p) &:= \min_{0 \leq q \leq N-1} \left\{ k_{2,q} \in \Lambda_2 \middle| e^{k_{2,q}} - e^{k_{1,p+1}} \geq K \right\} \\ \overline{k}_2(p) &:= \max_{0 \leq q \leq N-1} \left\{ k_{2,q} \in \Lambda_2 \middle| e^{k_{2,q}} - e^{k_{1,p}} < K \right\}, \end{aligned}$$

the s_2-coordinates of the lower edges of the rectangular strips,

$$\begin{aligned}\underline{\Omega}_p &:= [k_{1,p}, k_{1,p+1}) \times [\underline{k}_2(p), \infty) \\ \overline{\Omega}_p &:= [k_{1,p}, k_{1,p+1}) \times [\overline{k}_2(p), \infty).\end{aligned}$$

Putting these together we obtain two regions bounding Ω_λ:

$$\underline{\Omega} := \bigcup_{p=0}^{N-1} \underline{\Omega}_p \,, \quad \overline{\Omega} := \bigcup_{p=0}^{N-1} \overline{\Omega}_p.$$

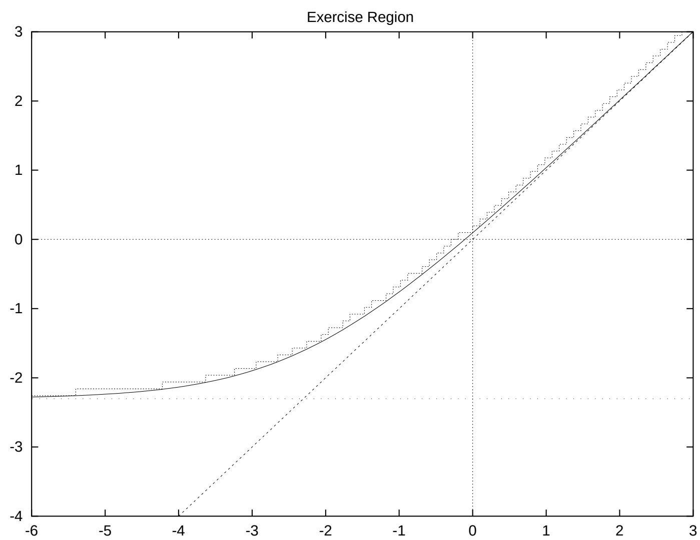

Figure 2: Construction of the boundary of the approximate region $\underline{\Omega}$

Since $\underline{\Omega} \subset \Omega_\lambda$ and the spread option payoff is positive over Ω_λ, we have a lower bound for its integral with the pricing kernel over this region:

$$\begin{aligned}V(K) &:= \int\int_{\Omega_\lambda} e^{-rT}\big(e^{s_2} - e^{s_1} - K\big) q_T(s_1, s_2) ds_2 ds_1 \\ &\gtrsim \int\int_{\underline{\Omega}} e^{-rT}\big(e^{s_2} - e^{s_1} - K\big) q_T(s_1, s_2) ds_2 ds_1. \qquad (6)\end{aligned}$$

Establishing the upper bound is a trickier issue since the integrand is not positive over the entire region $\overline{\Omega}$. In fact, the payoff is strictly negative over $\overline{\Omega} \setminus \Omega_\lambda$ by the definition of Ω_λ. To overcome this, we shall pick some $\epsilon > 0$ such that

$$\overline{\Omega} \subset \Big\{(s_1, s_2) \in \mathbb{R}^2 \Big| e^{s_2} - e^{s_1} - K \geq -\epsilon\Big\}.$$

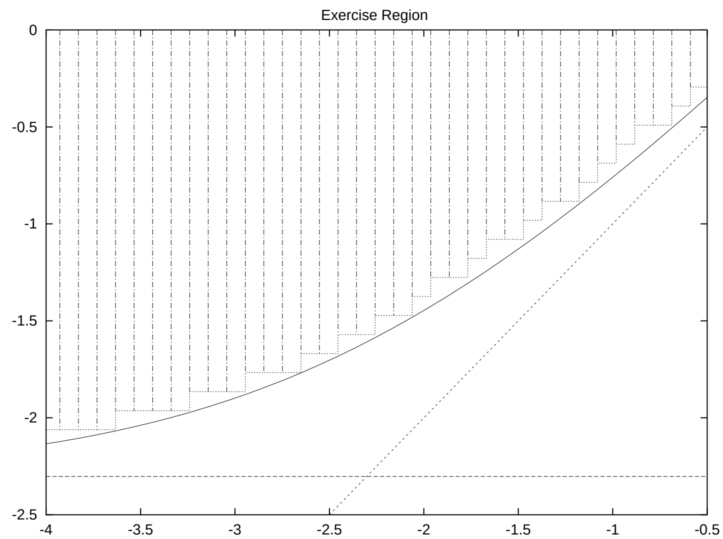

Figure 3: Approximation of the exercise region with rectangular strips

We then have

$$\begin{aligned}
V(K) &= e^{-rT}\Big[\int\int_{\Omega_\lambda}\big(e^{s_2}-e^{s_1}-(K-\epsilon)\big)q_T(s_1,s_2)ds_2ds_1 - \int\int_{\Omega_\lambda}\epsilon\cdot q_T(s_1,s_2)ds_2ds_1\Big] \\
&\lesssim e^{-rT}\Big[\int\int_{\underline{\Omega}}\big(e^{s_2}-e^{s_1}-(K-\epsilon)\big)q_T(s_1,s_2)ds_2ds_1 - \int\int_{\underline{\Omega}}\epsilon\cdot q_T(s_1,s_2)ds_2ds_1\Big] \\
&= e^{-rT}\Big[\int\int_{\overline{\Omega}}\big(e^{s_2}-e^{s_1}\big)q_T(s_1,s_2)ds_2ds_1 - (K-\epsilon)\int\int_{\overline{\Omega}}q_T(s_1,s_2)ds_2ds_1 \\
&\qquad -\epsilon\int\int_{\underline{\Omega}}q_T(s_1,s_2)ds_2ds_1\Big]. \qquad (7)
\end{aligned}$$

By breaking (6) and (7) into two components we can obtain these bounds by integrating $(e^{s_2}-e^{s_1})\cdot q_T(s_1,s_2)$ and the density $q_T(s_1,s_2)$ over $\underline{\Omega}$ and $\overline{\Omega}$, using the fast Fourier Transform method described in the previous section. Set

$$\begin{aligned}
\underline{\Pi}_1 &:= \int\int_{\underline{\Omega}}\big(e^{s_2}-e^{s_1}\big)q_T(s_1,s_2)ds_2ds_1 \qquad & \underline{\Pi}_2 &:= \int\int_{\underline{\Omega}}q_T(s_1,s_2)ds_2ds_1 \\
\overline{\Pi}_1 &:= \int\int_{\overline{\Omega}}\big(e^{s_2}-e^{s_1}\big)q_T(s_1,s_2)ds_2ds_1 \qquad & \overline{\Pi}_2 &:= \int\int_{\overline{\Omega}}q_T(s_1,s_2)ds_2ds_1.
\end{aligned}$$

Equations (6) and (7) can now be written as

$$e^{-rT}\big[\,\underline{\Pi}_1 - K\underline{\Pi}_2\,\big] \lesssim V(K) \lesssim e^{-rT}\big[\,\overline{\Pi}_1 - (K-\epsilon)\overline{\Pi}_2 - \epsilon\underline{\Pi}_2\,\big]. \qquad (8)$$

3.2 Computing the Sums by FFT

We now demonstrate in detail how to compute, by performing two fast Fourier transforms, the four components $\underline{\Pi}_1, \underline{\Pi}_2, \overline{\Pi}_1, \overline{\Pi}_2$ in the approximate pricing equations (8) and hence the spread option prices across different strikes. (In fact, if one only wishes to approximate the option price from below, a single transform is sufficient.) This is set out explicitly for $\underline{\Pi}_1$

below and the other three cases follow similarly.

$$
\begin{aligned}
\underline{\Pi}_1 &:= \int\int_{\underline{\Omega}} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1 = \sum_{p=0}^{N-1} \int\int_{\underline{\Omega}_p} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1 \\
&:= \sum_{p=0}^{N-1} \left[\int_{k_{1,p}}^{\infty} \int_{\underline{k}_2(p)}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1 - \int_{k_{1,p+1}}^{\infty} \int_{\underline{k}_2(p)}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1 \right] \\
&= \sum_{p=0}^{N-1} \Pi_1(k_{1,p}, \underline{k}_2(p)) - \Pi_1(k_{1,p+1}, \underline{k}_2(p)),
\end{aligned}
\tag{9}
$$

where

$$
\Pi_1(k_1, k_2) := \int_{k_1}^{\infty} \int_{k_2}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1.
$$

As before we apply a Fourier transform to the following modified integral:

$$
\underline{\pi}_1(k_1, k_2) := e^{\alpha_1 k_1 + \alpha_2 k_2} \Pi_1(k_1, k_2) \quad \alpha_1, \alpha_2 > 0.
$$

for a simple relationship with the characteristic function:

$$
\begin{aligned}
\chi_1(v_1, v_2) &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(v_1 k_1 + v_2 k_2)} \underline{\pi}_1(k_1, k_2) dk_2 dk_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(\alpha_1 + iv_1)k_1 + (\alpha_2 + iv_2)k_2} \int_{k_2}^{\infty} \int_{k_1}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1 \; dk_2 dk_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) \int_{-\infty}^{s_2} \int_{-\infty}^{s_1} e^{(\alpha_1 + iv_1)k_1 + (\alpha_2 + iv_2)k_2} dk_2 dk_1 \; ds_2 ds_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) \, \frac{e^{(\alpha_1 + iv_1)s_1 + (\alpha_2 + iv_2)s_2}}{(\alpha_1 + iv_1)(\alpha_2 + iv_2)} ds_2 ds_1 \\
&= \frac{\phi_T\big(v_1 - \alpha_1 i, v_2 - (\alpha_2 + 1)i\big) - \phi_T\big(v_1 - (\alpha_1 + 1)i, v_2 - \alpha_2 i\big)}{(\alpha_1 + iv_1)(\alpha_2 + iv_2)} \quad .
\end{aligned}
\tag{10}
$$

Discretising as in the previous section with

$$
\begin{aligned}
\lambda_1 \cdot \Delta_1 &= \lambda_2 \cdot \Delta_2 = \tfrac{2\pi}{N} \\
v_{1,m} := (m - \tfrac{N}{2})\Delta_1 &\qquad v_{2,n} := (n - \tfrac{N}{2})\Delta_2,
\end{aligned}
\tag{11}
$$

we now have via an (inverse) Fast Fourier transform values of $\Pi_1(\cdot, \cdot)$ on all $N \times N$ vertices of the grid $\Lambda_1 \times \Lambda_2$ given by

$$
\begin{aligned}
\Pi_1(k_{1,p}, k_{2,q}) &= \frac{e^{-\alpha_1 k_{1,p} - \alpha_2 k_{2,q}}}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(v_1 k_{1,p} + v_2 k_{2,q})} \chi_1(v_1, v_2) dv_2 dv_1 \\
&\approx \frac{e^{-\alpha_1 k_{1,p} - \alpha_2 k_{2,q}}}{(2\pi)^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{-i(v_{1,m} k_{1,p} + v_{2,n} k_{2,q})} \chi_1(v_{1,m}, v_{2,n}) \Delta_2 \Delta_1 \\
&= \frac{(-1)^{p+q} \cdot e^{-\alpha_1 k_{1,p} - \alpha_2 k_{2,q}}}{(2\pi)^2} \Delta_2 \Delta_1 \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}(mp + nq)} \Big[(-1)^{m+n} \chi_1(v_{1,m}, v_{2,n}) \Big]
\end{aligned}
$$

and hence the values of the $2 \cdot p$ required components in (9). Repeating the same procedure for the other components in (6) and (7) gives the bounds for the spread option value $V(K)$.

4 Numerical Performance

4.1 Underlying Models

Previous works on spread options have concentrated on the two-factor Geometric Brownian motion (GBM) model in which the risk-neutral dynamics of the underlying assets are given by

$$\begin{aligned} d\boldsymbol{S}_1 &= \boldsymbol{S}_1((r-\delta_1)dt + \sigma_1 d\boldsymbol{W}_1) \\ d\boldsymbol{S}_2 &= \boldsymbol{S}_2((r-\delta_2)dt + \sigma_2 d\boldsymbol{W}_2), \end{aligned}$$

where $\mathbb{E}_{\mathbb{Q}}[d\boldsymbol{W}_1 d\boldsymbol{W}_2] = \rho dt$ and r, δ_i, σ_i denote the risk-free rate, dividend yields and volatilities respectively. Working with the log prices, $\boldsymbol{s}_i := \log \boldsymbol{S}_i$, one has the following pair of SDEs:

$$\begin{aligned} d\boldsymbol{s}_1 &= (r-\delta_1-\tfrac{1}{2}\sigma_1^2)dt + \sigma_1 d\boldsymbol{W}_1 \\ d\boldsymbol{s}_2 &= (r-\delta_2-\tfrac{1}{2}\sigma_2^2)dt + \sigma_2 d\boldsymbol{W}_2 \ . \end{aligned}$$

We shall now extend this model to include a third factor, a stochastic volatility for the two underlying processes.

$$\begin{aligned} d\boldsymbol{s}_1 &= (r-\delta_1-\tfrac{1}{2}\sigma_1^2\boldsymbol{\nu})dt + \sigma_1\sqrt{\boldsymbol{\nu}}\ d\boldsymbol{W}_1 \\ d\boldsymbol{s}_2 &= (r-\delta_2-\tfrac{1}{2}\sigma_2^2\boldsymbol{\nu})dt + \sigma_2\sqrt{\boldsymbol{\nu}}\ d\boldsymbol{W}_2 \\ d\boldsymbol{\nu} &= \kappa(\mu-\boldsymbol{\nu})dt + \sigma_\nu\sqrt{\boldsymbol{\nu}}\ d\boldsymbol{W}_\nu \ , \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[d\boldsymbol{W}_1 d\boldsymbol{W}_2] &= \rho\, dt \\ \mathbb{E}_{\mathbb{Q}}[d\boldsymbol{W}_1 d\boldsymbol{W}_\nu] &= \rho_1\, dt \\ \mathbb{E}_{\mathbb{Q}}[d\boldsymbol{W}_2 d\boldsymbol{W}_\nu] &= \rho_2\, dt. \end{aligned}$$

This is a direct generalisation of the single-asset stochastic volatility model [14, 21] and is considered for the case of correlation options in [2]. Applying Ito's lemma and solving the resulting PDE, one obtains an analytical expression for its characteristic function:

$$\begin{aligned} \phi_{sv}(u_1,u_2) &:= \mathbb{E}_{\mathbb{Q}}\Big[\exp\big(iu_1\boldsymbol{s}_1(T) + iu_2\boldsymbol{s}_2(T)\big)\Big] \\ &= \exp\Bigg[iu_1\cdot s_1(0) + iu_2\cdot s_2(0) + \left(\frac{2\zeta(1-e^{-\theta T})}{2\theta-(\theta-\gamma)(1-e^{-\theta T})}\right)\cdot\nu(0) \\ &\quad + \sum_{j=1,2} u_j(r-\delta_j)T - \frac{\kappa\mu}{\sigma_\nu^2}\left[2\cdot\log\left(\frac{2\theta-(\theta-\gamma)(1-e^{-\theta T})}{2\theta}\right) + (\theta-\gamma)T\right]\Bigg] \ , \end{aligned} \tag{12}$$

where

$$\begin{aligned} \zeta &:= -\frac{1}{2}\Big[\big(\sigma_1^2u_1^2 + \sigma_2^2u_2^2 + 2\rho\sigma_1\sigma_2u_1u_2\big) + i\big(\sigma_1^2u_1 + \sigma_2^2u_2\big)\Big] \\ \gamma &:= \kappa - i\big(\rho_1\sigma_1u_1 + \rho_2\sigma_2u_2\big)\sigma_\nu \\ \theta &:= \sqrt{\gamma^2 - 2\sigma_\nu^2\zeta}. \end{aligned}$$

Notice that as we let the parameters of the stochastic volatility process approach the limits

$$\kappa, \mu, \sigma_\nu \to 0, \ \nu(0) \to 1,$$

the three-factor stochastic volatility (SV) model degenerates into the two-factor GBM model and the characteristic function simplifies to that of a bivariate normal distribution:

$$\phi_{gbm}(u_1, u_2) = \exp\Big[\, iu_1 \cdot s_1(0) + iu_2 \cdot s_2(0) + \zeta\, T + \sum_{j=1,2} u_j(r - \delta_j)T\Big].$$

We shall use these two characteristic functions to compute the spread option prices under the GBM and SV model. In the former case the prices computed by the FFT method are compared to the analytic option value obtained by a one dimensional integration based on the conditioning technique. This fails when we introduce a stochastic volatility factor and thus a Monte Carlo pricing method is used as a benchmark for the SV model.

Prices are also compared for the two diffusion models. Given a set of parameter values for the SV model, one can compute from the characteristic function the mean and covariance matrix of $\boldsymbol{s}_1(T), \boldsymbol{s}_2(T)$ under the stochastic volatility assumption. We can then infer for these the parameter values of the two-factor GBM model needed to produce the same moments. Option values may then be computed and compared to the three factor SV prices.

The code is written in C++ and includes the fast Fourier Transform routine FFTW (the *Fastest Fourier Transform in the West*), written by M. Frigo and S.G. Johnson [12]. The experiments were conducted on an Athlon 650 MHz machine running under Linux with 512 MB RAM.

4.2 Computational Results

Table 1 documents the spread option prices across a range of strikes under the two factor Geometric Brownian motion model [22], computed by three different techniques: one-dimensional integration (analytic), the fast Fourier Transform and the Monte Carlo method. The values for the FFT methods shown are the “lower” prices, computed over $\underline{\Omega}$, regions that approach the the true exercise region from below and are therefore all less than the analytic price in the first column. 80000 simulations were used to produce the Monte Carlo prices and the average standard errors are recorded in brackets at the bottom. Note that if one is only interested in computing prices in the two factor world, it is not actually necessary to discretise the time horizon $[0, T]$ as was done here. Since we know the terminal joint distribution of the two asset prices are bivariate normal, they can be simulated directly and one single time step is sufficient. However, the point of this exercise is to acquire an intuition into how the computational time and accuracy varies as one changes the underlying assumptions, since the introduction of extra factors into a model inevitably involves generating the whole paths of these factors.

The average errors of the two methods are computed and recorded in Table 2. First we note that integrating over $\underline{\Omega}$ from below is more accurate than over $\overline{\Omega}$, as one can expect from the less straightforward procedure for constructing the upper bound. For $N = 1024$ the lower bound has an error of roughly one basis point, whereas $N = 2048$ takes us well below this

Table 1: Prices computed by alternative methods under the 2-factor GBM model

	Analytic	Fast Fourier Transform				Monte Carlo	
		No. Discretisation N				Time Steps	
Strikes K		512	1024	2048	4096	1000	2000
0.0	8.513201	8.509989	8.511891	8.512981	8.513079	8.500949	8.516613
0.4	8.312435	8.311424	8.311995	8.312370	8.312385	8.300180	8.315818
0.8	8.114964	8.113877	8.114304	8.114901	8.114916	8.102730	8.118328
1.2	7.920790	7.919520	7.920173	7.920712	7.920741	7.908614	7.924135
1.6	7.729903	7.728471	7.729268	7.729810	7.729852	7.717831	7.733193
2.0	7.542296	7.540686	7.541637	7.542185	7.542242	7.530322	7.545496
2.4	7.357966	7.356278	7.357288	7.357830	7.357901	7.346038	7.361136
2.8	7.176888	7.175080	7.176185	7.176734	7.176818	7.164956	7.180054
3.2	6.999052	6.997200	6.998345	6.998881	6.998979	6.987070	7.002243
3.6	6.824451	6.822477	6.823721	6.824259	6.824371	6.812353	6.827700
4.0	6.653060	6.651047	6.652306	6.652852	6.652976	6.640874	6.656364
						(0.018076)	(0.018184)

$S_1(0) = 96$ $\delta_1 = 0.05$ $\sigma_1 = 0.1$ $S_2(0) = 100$ $\delta_2 = 0.05$ $\sigma_2 = 0.2$
$r = 0.1$ $T = 1.0$ $K = 4.0$ $\rho = 0.5$
Note: 80000 simulations have been used in the Monte Carlo method

error level. From Table 3 they take 4.28 and 18.46 seconds respectively, clearly outperforming the Monte-Carlo method. For the same level of accuracy, one would require simulations far more than 80000, which already take 304.95 seconds (606.40 seconds for the case of 2000 time steps) to generate. Although the Monte Carlo code employed uses no variance reduction technique other than antithetic variates and its speed could be significantly improved, the method is still unlikely to beat the FFT method in performance.

Table 2: Accuracy of alternative methods for the 2-factor GBM model: Error in b.p.

Fast Fourier Transform			Monte Carlo				
Number of Discretisation	Lower	Upper	Number of Simulations	Time Steps 1000		2000	
512	4.44	25.60	10000	129.15	(0.051839)	70.81	(0.050949)
1024	1.13	13.90	20000	22.34	(0.036225)	40.67	(0.035899)
2048	0.32	7.20	40000	7.44	(0.025737)	7.63	(0.025733)
4096	0.10	3.65	80000	18.34	(0.018076)	4.94	(0.018184)

$S_1(0) = 96$ $\delta_1 = 0.05$ $\sigma_1 = 0.1$ $S_2(0) = 100$ $\delta_2 = 0.05$ $\sigma_2 = 0.2$
$r = 0.1$ $T = 1.0$ $K = 4.0$ $\rho = 0.5$

A close examination of Table 3 reveals the real strength of the FFT method. As we introduce a stochastic volatility factor, the Monte Carlo technique needs to generate this value at each time step, which is then multiplied with the increments $d\boldsymbol{W}_1, d\boldsymbol{W}_2$ of the Brownian

Table 3: Computing Time of Alternative Methods

Fast Fourier Transform				
Number of	10 Strikes		100 Strikes	
Discretisation	GBM	SV	GBM	SV
512	1.04	1.11	1.10	1.20
1024	4.28	4.64	4.48	4.83
2048	18.46	19.54	18.42	19.74
4096	74.45	81.82	76.47	81.27

Monte Carlo: 1000 Time Steps				
Number of	10 Strikes		100 Strikes	
Simulation	GBM	SV	GBM	SV
10000	38.2	144.87	41.95	151.75
20000	76.22	288.09	83.81	303.31
40000	152.5	576.25	168.48	606.53
80000	304.95	1152.9	335.20	1212.76

Monte Carlo: 2000 Time Steps				
Number of	10 Strikes		100 Strikes	
Simulation	GBM	SV	GBM	SV
10000	75.57	287.41	79.83	295.21
20000	157.28	574.18	159.08	590.23
40000	303.37	1149.25	317.49	1184.32
80000	606.40	2298.37	636.33	2359.05

motions to give the asset price in the next period. As indicated across the columns this increases the computational time by almost a factor of 4. Recalling the FFT method described in the previous section, we notice that only a different characteristic function is substituted when more factors are included, and the transform remain two dimensional. Comparing the times for the GBM and SV models, we observe only a 5 to 9 percent increase and falling as we increase the discretisation number. The extra computing time is due to the more complex expression of the characteristic function with a larger set of parameters. For both methods however, increasing the number of strikes does not result in dramatic increases in the computational times.

Table 4 shows the spread option prices for different strikes under the three factor SV model. The Monte Carlo prices with a discretisation of 2000 time steps oscillate around those computed by the FFT method. Since we observe that in the two factor case the errors of the Monte Carlo method remain high even for 80000 simulations, more experiments need to be conducted for a conclusive judgment on this point.

Table 4: Prices computed by alternative methods under the 3-factor SV model

	Fast Fourier Transform		Monte Carlo			
	No. of Discretisations		No. of Simulations			
Strikes K	512	2048	10000	20000	40000	80000
2.0	7.546895	7.543618	7.514375	7.567536	7.572211	7.523968
2.2	7.451878	7.452998	7.421861	7.475093	7.479742	7.431489
2.4	7.357703	7.363377	7.330142	7.383470	7.388080	7.339813
2.6	7.264298	7.274876	7.239209	7.292616	7.297218	7.248961
2.8	7.171701	7.186990	7.149234	7.202571	7.207191	7.158919
3.0	7.079987	7.099819	7.060043	7.113303	7.117954	7.069687
3.2	6.989008	7.013731	6.971625	7.024808	7.029515	6.981272
3.4	6.898826	6.928373	6.884026	6.937119	6.941875	6.893664
3.6	6.809471	6.843671	6.797246	6.850283	6.854984	6.806859
3.8	6.720957	6.759903	6.711328	6.764275	6.768886	6.720859
4.0	6.633232	6.676768	6.626221	6.679076	6.683587	6.635661
			(0.052702)	(0.036984)	(0.025739)	(0.018206)

$r = 0.1$	$T = 1.0$	$\rho = 0.5$	
$S_1(0) = 96$	$\delta_1 = 0.05$	$\sigma_1 = 0.5$	$\rho_1 = 0.25$
$S_2(0) = 100$	$\delta_2 = 0.05$	$\sigma_2 = 1.0$	$\rho_1 = -0.5$
$\nu(0) = 0.04$	$\kappa = 1.0$	$\mu = 0.04$	$\sigma_\nu = 0.05$

Note: 2000 time steps have been used for the Monte Carlo simulation.

Finally, Figure 5 plots the difference in the spread option values under the 3-factor stochastic volatility model and the 2-factor geometric Brownian motion model. Under the SV model, knowing the characteristic function of $\boldsymbol{s}_1, \boldsymbol{s}_2$, we can calculate their means and covariance matrix, which can then be used as the implied parameters $r - \delta_i$ and σ_i, $i = 1, 2$, and ρ for the GBM model. We repeat the procedure for different values of ρ_1, ρ_2, the correlation parameters between the Brownian motions $\boldsymbol{W}_i, i = 1, 2$ driving the asset prices and $\boldsymbol{W}_\nu$ driving the stochastic volatility factor $\boldsymbol{\nu}$. When ρ_1, ρ_2 are high, a large increment $\boldsymbol{W}_\nu$ in (12) is more likely to induce simultaneously large values of $\boldsymbol{W}_i, i = 1, 2$, and $d\boldsymbol{\nu}$. This increases the volatilities of both $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ and hence the spread and the spread option value. Compared with the two factor GBM model, the SV model of (12) obviously exhibits a richer structure for the spread option value which can be used by traders with forward views on the term structures of volatilities and correlations of the components of the spread [16].

5 Conclusions and future directions

We have described and implemented an efficient method of computing, via a construction of suitable approximate exercise regions, the value of a generic spread option under models for which the characteristic function of the two underlying asset prices is known in closed form. This takes us well beyond the two factor constant correlation Gaussian framework found in the

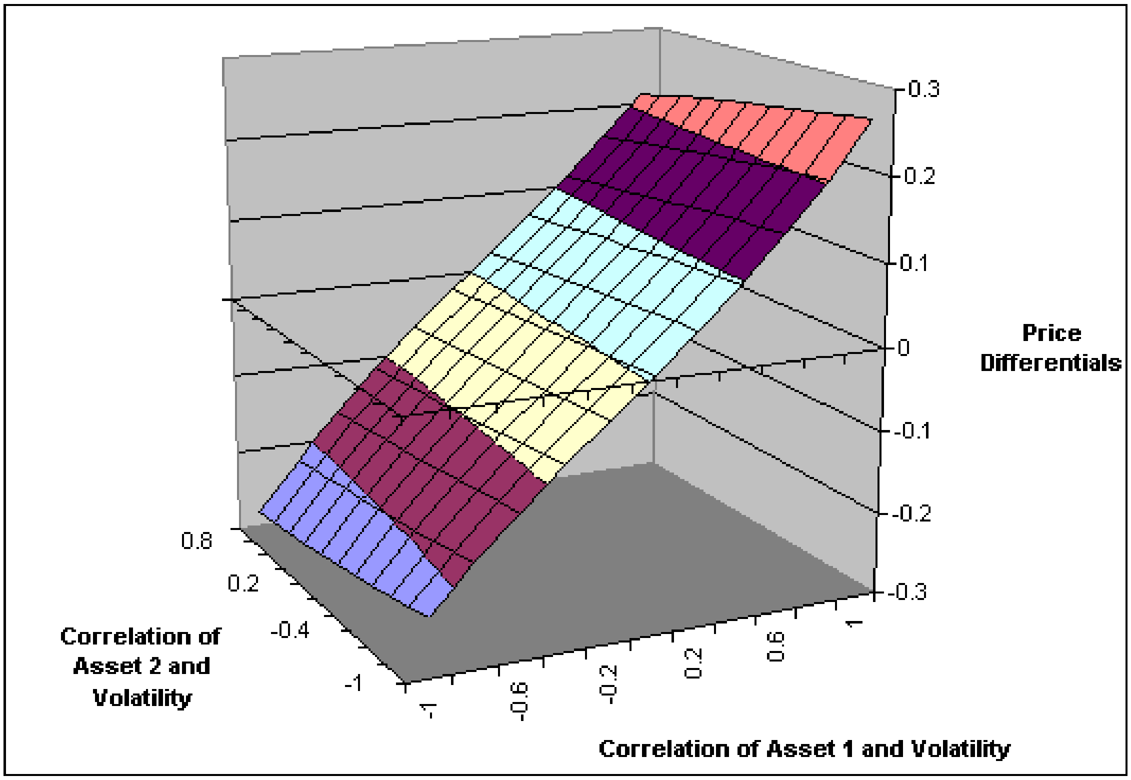

Figure 4: Price Difference between SV Model and the GBM Model with Implied Parameters

existing literature, which is commonly assumed only for its tractability. In particular, one can now price spread options under many multi-factor models in the affine jump-diffusion family. For example, an index spread option in the equity markets can be priced under stochastic volatility models. Spark and crack spread options in the energy market can now be valued with asset price spikes and random volatility jumps, with major implications for trading, as well as for asset and real option valuation.

Furthermore, switching between alternative diffusion models only amounts to substituting a different characteristic function for the underlying prices/rates, leaving the dimension of the transform and the summation procedure unchanged. As more factors are introduced more time is devoted to the inexpensive evaluation of the more complex characteristic function, but *not* to the fast Fourier Transform algorithm. This significantly cuts down the increase of computational times expected when one applies the generic PDE or Monte Carlo approaches to such a high dimensional option pricing problem.

The computational advantage of the approach is demonstrated with numerical experiments for both the two factor geometric Brownian motion and the three factor stochastic volatility models. Price differentials between the models as one varies the parameters of the volatility process confirm the significance of a non-trivial correlation structure in the model dynamics.

One possible direction to enrich the volatility and correlation structure further is to assume a four factor model with two correlated stochastic volatility processes [7]. The calibration issue also remains to be resolved in detail, where the focus of concern will be an efficient procedure for backing out a implied correlation surface from observed option prices.

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