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Exponential Growth of Fixed-Mix Strategies in Stationary Asset Markets[1]

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Abstract. The paper analyzes the long-run performance of dynamic investment strategies based on fixed-mix portfolio rules. Such rules prescribe rebalancing the portfolio by transferring funds between its positions according to fixed (time-independent) proportions. The focus is on asset markets where prices fluctuate as stationary stochastic processes. Under very general assumptions, it is shown that any fixed-mix strategy in a stationary market yields an exponential growth of the portfolio with probability one.

Key words: Asset allocation, Fixed-mix strategies, Stationary markets, Exponential growth, Products of random matrices, Stochastic version of the Perron–Frobenius theorem

JEL Classification: G11, F31

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1 Introduction

The paper examines the performance of fixed-mix dynamic investment strategies in an asset market where prices fluctuate as stationary stochastic processes. A fixed-mix strategy prescribes transferring, at each step, a fixed share $\alpha_{kj} > 0$ of the jth position of the portfolio to the kth position $(k, j = 1, ..., K)$. We analyze the asymptotic behavior of the portfolio of a trader who employs such an investment rule systematically during a sufficiently long time period. In the deterministic case, we always have convergence to a stable state. But in case of any, even slightest, random price fluctuations, the value of the portfolio will grow exponentially with probability one. This result might seem counterintuitive, since fixed-mix strategies are self-financing and a stationary price process has zero trend. The purpose of the paper is to derive and discuss the result in the framework of a fairly general stochastic model of an asset market.

Consider a financial market with K assets whose prices change in time and depend on random factors. Randomness is described as follows. There is a stochastic process $..., s_{-1}, s_0, s_1, ...$ with values in a space S. The value of s_t characterizes the "state of the economy" at time $t \in \{0, \pm 1, \pm 2, ...\}$. The vector of asset prices

$$p_t(s^t) = (p_t^1(s^t), ..., p_t^K(s^t)), \quad [p_t^k(s^t) > 0]$$

at time $t = 0, 1, ...$ depends on the history

$$s^t = (..., s_{t-1}, s_t)$$

of the process (s_t). (The space S and all functions of $s^t \in ... \times S \times S$ are supposed to be measurable.)

An *investment strategy* (*trading strategy*) is a sequence of non-negative vector functions

$$h_t(s^t) = (h_t^1(s^t), ..., h_t^K(s^t)), \; t = 0, 1, 2, ... ,$$

where $h_t = h_t(s^t)$ is the *portfolio* of assets at time t. The component $h_t^k(s^t)$ of the vector $h_t(s^t)$ represents the number of units of asset k in the portfolio h_t. The choice of the portfolio may depend on time and on information about the process (s_t): therefore h_t depends on t and s^t. We assume that all the coordinates of $h_t(s^t)$ are non-negative: short sales are ruled out.

A trading strategy h_t is said to be *self-financing* if

$$p_t h_t = p_t h_{t-1}.$$

(Here $p_t h_t = \sum_{k=1}^K p_t^k h_t^k$ and $p_t h_{t-1} = \sum_{k=1}^K p_t^k h_{t-1}^k$ are scalar products.) A trader using a self-financing strategy h_t rebalances, at every time period, the portfolio within the budget constraint determined by the cost $p_t h_{t-1}$ of the previous portfolio h_{t-1} in the current prices p_t. To begin with, we will assume that there are no transaction costs. The case of transaction costs will be considered later in the paper.

Let α_{kj}, $k, j \in \{1, ..., K\}$, be a (non-random) matrix such that

$$\alpha_{kj} > 0, \ \sum_{k=1}^{K} \alpha_{kj} = 1. \tag{1}$$

A strategy h_t, $t \geq 0$, is called a *fixed-mix strategy associated with the matrix* $\alpha = (\alpha_{kj})$, or, for short, an *α-strategy*, if

$$p_t^k h_t^k = \sum_{j=1}^{K} \alpha_{kj} p_t^j h_{t-1}^j \tag{2}$$

for all k, t. The number α_{kj} indicates what share of position j should be transferred to position k. Such strategies express in a simple form the idea of diversification of a portfolio. Every asset is converted at each step into a "fixed mix" of all the assets. Clearly any α-strategy is self-financing (to show this add up equations (2) over k and use (1)).

In an important special case, α_{kj} does not depend on j:

$$\alpha_{kj} = \alpha_k. \tag{3}$$

Then $\alpha_k > 0$ and $\alpha_1 + ... + \alpha_K = 1$. In this case, (2) reduces to

$$p_t^k h_t^k = \alpha_k \sum_{j=1}^{K} p_t^j h_{t-1}^j = \alpha_k p_t h_{t-1}, \quad k \in \{1, ..., K\}. \tag{4}$$

An investor using a strategy of this kind divides the available wealth $p_t h_{t-1}$ according to the proportions $\alpha_1, ..., \alpha_K$ and spends the amount $\alpha_k p_t h_{t-1}$ for purchasing $\alpha_k p_t h_{t-1}/p_t^k$ units of asset k.

Fixed-mix strategies—especially of the form (4)—are quite common in financial theory and practice; see, e.g., Perold and Sharpe (1988), Mulvey and Ziemba (1998), Browne (1998) and Dempster (2001, 2002).

Strategies of the general form (2) can be considered, in particular, in the context of the modelling of currency markets. This aspect of our analysis is inspired, in particular, by recent work of Kabanov (1999) and Kabanov and Stricker (2001). Consider a frictionless market where K currencies are traded. The exchange rates $\pi_t^{kj} = \pi_t^{kj}(s^t) > 0$ fluctuate randomly in time, depending on the stochastic factors (s_t). Here, π_t^{kj} denotes the amount of currency k which can be purchased by selling one unit of currency j at time t. Assume the trader divides the amount $h_{t-1}^j \geq 0$ of currency j available at the beginning of a time period $(t-1, t]$ according to the proportions $\alpha_{kj} > 0$ ($\sum_k \alpha_{kj} = 1$) and exchanges $\alpha_{kj} h_{t-1}^j$ into currency k. Then the amount of currency k obtained at time t will be equal to

$$h_t^k = \sum_{j=1}^{K} \alpha_{kj} \pi_t^{kj} h_{t-1}^j. \tag{5}$$

By virtue of no-arbitrage considerations, exchange rates in a frictionless market satisfy

$$\pi_t^{kj} = \pi_t^{km}\pi_t^{mj} \tag{6}$$

for all k, m and j. This implies

$$\pi_t^{kj} = 1/\pi_t^{jk},\ \pi_t^{jj} = 1. \tag{7}$$

Let us regard currency 1 as a "numeraire" and define $p_t^k = \pi_t^{1k}$. It follows from (6) and (7) that $\pi_t^{kj} = p_t^j/p_t^k$, and so (5) can be written in the form (2).

In this paper we focus on *stationary* markets. We say that the market under consideration is stationary if the stochastic process (s_t), $t = 0, \pm 1, \pm 2, ...$, is stationary, and the price vectors p_t do not explicitly depend on t:

$$p_t = p(s^t). \tag{8}$$

The last assumption means that the random asset prices might depend on the current random situation and foregoing random events, but the structure of this dependence does not change in time. In the above model of currency exchange, the counterpart of condition (8) is

$$\pi_t^{kj} = \pi^{kj}(s^t), \tag{9}$$

which implies (8), when p_t is defined by $p_t^k = \pi_t^{1k}$.

A comment on the notion of stationarity of a stochastic process is in order. Recall that this notion formalizes the idea of time invariance of all probabilistic characteristics of the process. Stationarity requires that, for each $k = 1, 2, ...$, the expected value of any integrable function $\phi(s_t, ..., s_{t+k})$ is independent of t. This is equivalent to the assumption that $E\phi(s^t)$ is independent of t for any integrable function $\phi(s^t)$ on $... \times S \times S$. An example of a process of this kind is a sequence of independent identically distributed random variables. Stationary processes should not be confused with (temporally) homogeneous ones, such as Markov chains with time-invariant ("stationary"—in another terminology) transition probabilities.

A starting point for this study was the following question. Consider the foreign exchange model outlined above. Suppose a trader has selected some fixed-mix strategy defined by a non-zero initial portfolio h_0 and a strictly positive matrix (α_{kj}) satisfying (1). Assume that the trader systematically applies the rule of currency exchange specified by the matrix (α_{kj}). How will the portfolio h_t behave in the long run? Will it stabilize in one sense or another, will it grow or will it generally decrease?

The intuition here might be based on the following typical argumentation. First of all, the property of self-financing, combined with the assumption of stationarity, might seem to rule out possibilities of unbounded growth. Further, if we consider the deterministic case, i.e., assume that S is a singleton, then a stationary process will reduce to a constant and the vector function $p(s^t)$ will become a constant vector $p = (p^1, ..., p^k) > 0$. By setting $x_t^k = p^k h_t^k$, we can write equation (2) in the form

$$x_t^k = \sum_{j=1}^{K} \alpha_{kj} x_{t-1}^j.$$

Therefore $x_t = A^t x_0$, where $A = (\alpha_{kj}) > 0$ and $\sum_k \alpha_{kj} = 1$. As is well-known, the sequence x_t—and hence h_t—will *converge* to some strictly positive vector; see, e.g., Kemeny and Snell (1960). These considerations might lead to the (wrong) conjecture of the convergence of h_t to a stationary distribution in the stochastic case.

The correct answer seems to be rather unexpected: the portfolio process h_t will grow almost surely in every coordinate at an exponential rate! This conclusion turns out to be valid for *any* strictly positive matrix α_{kj} defining the fixed-mix strategy. Furthermore, the conclusion obtains for *any* ergodic stationary sequence (s_t) with price process $p(s^t)$ satisfying a mild assumption of non-degeneracy. The very general nature of this fact, as well as its counterintuitive (at first glance) character, motivated us to write this article.

In the paper, we give a rigorous proof of the above statement regarding the exponential growth. We begin with the analysis of a frictionless market. Then we consider a version of the model involving transaction costs. We show that our results can be extended to the case of a market with friction, provided the transaction costs are small enough. In the course of the study some properties of the model are established that are of independent interest—in particular, the existence of so-called balanced strategies. Finally, we demonstrate how our results can be modified—quite easily—to include into consideration asset markets where relative proportions of prices, rather than the prices themselves, change in time as stationary stochastic processes.

The article is organized as follows. In Section 2 we formulate and discuss the main results. In Section 3 we provide their proofs. The Appendix contains a statement of a general fact regarding random dynamical systems (a stochastic version of the Perron-Frobenius theorem) used in this work.

2 The main results

Throughout the paper we will assume that the prices $p^k(s^t) > 0$ satisfy

$$E|\ln p^k(s^t)| < \infty,\ k \in \{1, ..., K\}, \tag{10}$$

and the process (s_t) is stationary and ergodic. The letter E denotes the expectation with respect to the underlying probability measure P. Additionally, we impose the following requirement of non-degeneracy of the price process $p(s^t)$:

(A) The vector $\bar{p}(s^t) = (\bar{p}^1(s^t), ..., \bar{p}^K(s^t))$ of normalized prices

$$\bar{p}^j(s^t) := \frac{p^j(s^t)}{\sum_m p^m(s^t)},\ j \in \{1, ..., K\},$$

is not constant a.s. with respect to s^t.

According to (A), one cannot find a constant vector c for which $\bar{p}(s^t) = c$ almost surely (a.s.) with respect to the given probability P (we will often omit "a.s." when this does not lead to ambiguity). Since (s_t) is stationary, if condition (A) holds for some t, it holds for all t. As long as the process (s_t) is ergodic, hypothesis (A) is equivalent to hypothesis (B) below.

(B) With positive probability, the ratios

$$\frac{p^j(s^t)}{p^j(s^{t-1})} \tag{11}$$

are not constant with respect to j.

For a proof of the equivalence of (A) and (B) see the next section.

Let $\alpha = (\alpha_{kj})$ be a matrix satisfying (1). Consider the α-strategy (h_t), $h_t = (h_t^1, ..., h_t^K)$, defined by (2). A central result is as follows.

Theorem 1. *For each $k \in \{1, 2, ..., K\}$, the limit*

$$\lim_{t \to \infty} \frac{1}{t} \ln h_t^k \tag{12}$$

exists and is strictly positive almost surely. Furthermore, this limit does not depend on k, and we have

$$\lim_{t \to \infty} \frac{1}{t} \ln h_t^k = \lim_{t \to \infty} \frac{1}{t} \ln p_t h_t \quad \text{(a.s.)}. \tag{13}$$

The fact that the limit (12) is positive implies that h_t^k tends to infinity at an exponential rate. According to (13), the wealth $p_t h_t$ of the investor grows with the same positive exponential rate. In the special case described in (3) and (4), Theorem 1 was established in the previous work of Evstigneev and Schenk–Hoppé (2001). The random dynamical system governed by equations (4) is closely related to those considered by Hakansson and Ziemba (1995) and Algoet and Cover (1988).

We outline the main stages of the proof of Theorem 1. The following notion plays an important role in our analysis. A trading strategy $\{h_t\}$ is said to be *balanced* if

$$h_t(s^t) = \gamma(s^1) \dots \gamma(s^t) \, \tilde{h}(s^t) \quad \text{(a.s.)}, \; t = 1, 2, \dots, \tag{14}$$

where $\gamma(\cdot) > 0$ is a scalar-valued function and $\tilde{h}(\cdot) > 0$ is a vector function such that

$$E|\ln \gamma(s^t)| < \infty \; \text{ and } \; |\tilde{h}(s^t)| = 1.$$

The norm $|h|$ of a vector $h = (h^i)$ is defined as $\sum_i |h^i|$. For $t = 0$, we assume in (14) that $h_0(s^0) = \tilde{h}(s^0)$.

For a balanced path, all the ratios $h_t^i(s^t)/h_t^j(s^t) = \tilde{h}^i(s^t)/\tilde{h}^j(s^t)$, $i \neq j$, describing the proportions between the amounts of different assets in the portfolio, form stationary stochastic processes. (We assume for the moment that $\tilde{h}^j(s^t) > 0$.) Furthermore, the random growth rate of the amount of each asset $i = 1, ..., K$ in the portfolio, $h_t^i(s^t)/h_{t-1}^i(s^{t-1}) = \gamma(s^t)\tilde{h}^i(s^t)/\tilde{h}^i(s^{t-1})$, is a stationary process. In the deterministic case—when S consists of a single point—formula (14) reduces to $h_t = \gamma^t \tilde{h}$, where $\gamma > 0$ is a constant and $\tilde{h}$ is a nonnegative vector normalized by the condition $|\tilde{h}| = 1$. Such strategies exhibit growth with *constant* proportions and at a *constant* rate. Our notion of a balanced strategy is closely related to the notion of a balanced path introduced by Radner (1971) in the context of stochastic models of economic growth.

An α-strategy is determined by a matrix α and an initial portfolio. By virtue of the following theorem, the initial portfolio can be chosen in such a way that the resulting α-strategy is balanced.

Theorem 2. *For each $\alpha = (\alpha_{kj}) > 0$, there exists a unique balanced α-strategy*

$$\bar{x}_t(s^t) = \lambda(s^t)\lambda(s^{t-1})...\lambda(s^1)\tilde{x}(s^t) \tag{15}$$

($E|\ln \lambda(s^t)| < \infty$, $|\tilde{x}(s^t)| = 1$, $\bar{x}_0(s^0) = \tilde{x}(s^0)$). We have

$$E|\ln \tilde{x}^k(s^t)| < \infty,\ k = 1, 2, ..., K, \tag{16}$$

for each coordinate $\tilde{x}^k(s^t)$ of the vector $\tilde{x}(s^t)$.

Let us sketch the idea of the proof of Theorem 2. Denote by $A_t = A(s^t) = (a^{kj}(s^t))$ the positive random $K \times K$ matrix defined by

$$a^{kj}(s^t) = \alpha_{kj}\frac{p^j(s^t)}{p^k(s^t)}. \tag{17}$$

As a consequence of (10), we have

$$E|\ln a^{kj}(s^t)| < \infty,\ k, j \in \{1, ..., K\}. \tag{18}$$

In view of (2) and (17), an α-strategy (h_t) can be represented as

$$h_t(s^t) = A(s^t)A(s^{t-1})...A(s^1)h_0(s^0). \tag{19}$$

By virtue of the stationarity of (s_t), functions $\lambda(\cdot)$ and $\tilde{x}(\cdot)$ satisfying $E|\ln \lambda(s^t)| < \infty$ and $|\tilde{x}(s^t)| = 1$ generate a balanced α-strategy if and only if

$$\lambda(s^t)\tilde{x}(s^t) = A(s^t)\tilde{x}(s^{t-1}) \text{ (a.s.)}. \tag{20}$$

The existence of a solution to this equation follows from a stochastic version of the Perron–Frobenius theorem presented in the Appendix. Note that we cannot solve (20) by using the conventional Perron–Frobenius theorem (for each fixed s^t) because the vector $\tilde{x}(s^t)$ on the left-hand side of equation (20) does not coincide with the vector $\tilde{x}(s^{t-1})$ on the right-hand side. The function $\tilde{x}(s^t)$ is the result of the application of the "time shift" operator $s^{t-1} \to s^t$ to $\tilde{x}(s^{t-1})$.

Theorem 3 below shows that the growth rate of the portfolio of an investor employing any (not necessarily balanced) α-strategy is completely determined by the expected value of $\ln \lambda$.

Theorem 3. *Let (h_t) be an α-strategy with initial portfolio h_0 satisfying*

$$|h_0(s^0)| > 0 \tag{21}$$

for all s^0. Then, for each $k = 1, 2, ..., K$, we have

$$\lim_{t\to\infty} \frac{1}{t} \ln h_t^k = \lim_{t\to\infty} \frac{1}{t} \ln p_t h_t = \lim_{t\to\infty} \frac{1}{t} \ln |h_t| = E \ln \lambda(s^t) \ \ (a.s.). \tag{22}$$

Recall that, by virtue of the stationarity of (s_t), the expectation $E \ln \lambda(s^t)$ does not depend on t.

Theorem 1 is an immediate consequence of Theorem 3 and the following result.

Theorem 4. *We have*

$$E \ln \lambda(s^t) > 0.$$

Finally, we provide a version of Theorem 1 pertaining to a model with transaction costs. A convenient framework for this aspect of our study is the foreign exchange model outlined in the previous section. We modify it by introducing transaction costs.

Assume that the exchange rates $\pi_t^{kj} = \pi^{kj}(s^t)$ satisfying (6) are now replaced by $\rho_t^{kj} = (1 - \delta_t^{kj})\pi_t^{kj}$, where $\delta_t^{kj} = \delta^{kj}(s^t)$ are random variables with values in $[0, d]$, $0 < d < 1$. One unit of currency j can be exchanged to $\rho_t^{kj} = \rho^{kj}(s^t)$ units of currency k. The numbers δ_t^{kj} represent the rates of (proportional) *transaction costs.* As before, we set $p^k(s^t) = \pi_t^{1k}(s^t)$. In this setting, an α-strategy

$$g_t(s^t) = (g_t^1(s^t), ..., g_t^K(s^t)), \ t = 0, 1, 2, ... ,$$

is defined by

$$g_t^k = \sum_{j=1}^{K} \alpha_{kj}(1 - \delta_t^{kj})\pi_t^{kj} g_{t-1}^j = \sum_{j=1}^{K} \alpha_{kj}(1 - \delta_t^{kj})\frac{p_t^j}{p_t^k} g_{t-1}^j \tag{23}$$

$(t \geq 1,\ k \in \{1, ..., K\})$. We set

$$\delta(s^t) := \max_{kj} \delta^{kj}(s^t).$$

We have $0 < 1 - d \leq 1 - \delta^{kj}(s^t) \leq 1$, and so

$$\ln(1 - d) \leq \ln(1 - \delta^{kj}(s^t)) \leq 0,\ \ \ln(1 - d) \leq \ln(1 - \delta(s^t)) \leq 0. \tag{24}$$

The result below is an extension of Theorem 1 to the case of small transaction costs.

Theorem 5. *Let $\alpha = (\alpha_{kj})$ be a matrix satisfying (1). If the absolute value of the expectation $E \ln(1 - \delta(s^t))$ is small enough, then all the assertions of Theorem 1 remain valid for the strategy (23) starting from any initial portfolio $g_0(s^0)$ with $|g_0(s^0)| > 0$.*

To analyze the dynamics of strategies of the form (23) consider the random matrix

$$B_t = B(s^t) = (b^{kj}(s^t)),\ b^{kj}(s^t) = \alpha_{kj}(1 - \delta^{kj}(s^t))\frac{p^j(s^t)}{p^k(s^t)}.$$

We have

$$g_t(s^t) = B(s^t)B(s^{t-1})...B(s^1)g_0(s^0),\ t = 1, 2, ... ,$$

and

$$E|\ln b^{kj}(s^t)| < \infty,\ k, j \in \{1, ..., K\}, \tag{25}$$

by virtue of (24) and (10). We will prove Theorem 5 by using assertions (i)–(iv) contained in the following theorem.

Theorem 6. *(i) For any matrix (1), there exists a unique pair $(\tilde{y}(\cdot), \mu(\cdot))$ such that $|\tilde{y}(s^t)| = 1$, $E|\ln \mu(s^t)| < \infty$, $E|\ln \tilde{y}^k(s^t)| < \infty$, $k = 1, 2, ..., K$, and $\mu(s^t)\tilde{y}(s^t) = B(s^t)\tilde{y}(s^{t-1})$. (ii) The sequence $\bar{y}_t(s^t) := B_t B_{t-1}...B_0\bar{y}_0$, $\bar{y}_0 = \tilde{y}(s^0)$, satisfies $\bar{y}_t(s^t) = \mu(s^t)...\mu(s^1)\tilde{y}(s^t)$, $t = 1, 2, ...$ (iii) The assertions of Theorem 3 hold with g_t and μ in place of h_t and λ. (iv) We have*

$$E \ln \mu \geq E \ln \lambda + E \ln(1 - \delta(s^t)), \tag{26}$$

and so $E \ln \mu > 0$ when $|E \ln(1 - \delta(s^t))|$ is small enough.

We conclude this section with a remark regarding a further generalization of the above results. Of course the concept of a stationary market, where asset prices p_t change as stationary stochastic processes, is an idealization. A more realistic assumption on $p_t = (p_t^1, ..., p_t^K)$ is that only the relative proportions p_t^j/p_t^k, rather than p_t^k themselves, are stationary. So, assume now that p_t is of the form

$$p_t = \xi_t \hat{p}_t,$$

where $\hat{p}_t = \hat{p}(s^t)$ is a process satisfying the assumptions we previously imposed on p_t, and $\xi_t = \xi_t(s^t) > 0$ is *any* sequence of strictly positive random variables. The multipliers ξ_t might represent the dynamics of a price index capturing a trend of price changes in the market. The normalized prices $\hat{p}_t$ are free of this trend. It is easily seen from formulas (2) and (23) that the equations for the portfolio h_t governed by a fixed-mix strategy—with or without transaction costs—do not depend on ξ_t, and so all our conclusions regarding the asymptotic behavior of h_t remain valid. As regards the value $p_t h_t$ of the portfolio h_t, we can see that

$$\frac{1}{t}\ln p_t h_t = \frac{1}{t}\ln \xi_t + \frac{1}{t}\ln \hat{p}_t h_t,$$

and so the properties of growth of $p_t h_t$ are determined by those of ξ_t and $\hat{p}_t h_t$. In our context, the process $\hat{p}_t h_t$, exhibits an exponential growth with probability one. Consequently, if the price index ξ_t grows at an exogenous exponential rate r, then the value $p_t h_t$ of the portfolio h_t will grow almost surely at a rate r' *strictly greater* than r. Stationary random fluctuations of the relative price processes p_t^j/p_t^k may be regarded as the "driving force" sustaining this growth enhancement.

3 Proofs

Equivalence of (A) and (B). If condition (B) is not satisfied, then there exists a function $\zeta(s^t) > 0$ such that $p^j(s^t) = \zeta(s^t)p^j(s^{t-1})$ (a.s.) for all j. This implies $\bar{p}^j(s^t) = \bar{p}^j(s^{t-1})$ (a.s.). In view of the ergodicity of (s_t), this can be true only if there exists a non-random vector c satisfying $\bar{p}^j(s^t) = c$ (a.s.), which is ruled out by (A). Conversely, if (A) does not hold, then $\bar{p}^j(s^t) = c^j > 0$ (a.s.), and so $p^j(s^t) = c^j\nu(s^t)$, where $\nu(s^t) = \sum_m p^m(s^t) > 0$. Consequently, the ratio (11) is equal to the function $\nu(s^t)/\nu(s^{t-1})$ independent of j, which contradicts (B). □

We will prove Theorems 1-6 in this order: $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$, $6 \Rightarrow 5$.

Proof of Theorem 2. To prove the existence and uniqueness of $(\lambda(\cdot), \tilde{x}(\cdot))$ we will employ Theorem A.1 presented in the Appendix. Let Ω denote the space $... \times S \times S \times ...$ of sequences $\omega = (..., s_{-1}, s_0, s_1, ...)$, $s_t \in S$. Consider the σ-algebra $\mathcal{F} = ... \times \mathcal{S} \times \mathcal{S} \times ...$ on Ω, where $\mathcal{S}$ is the σ-algebra defining the measurable structure on S. Let T be the left shift, i.e. the mapping $\Omega \to \Omega$ transforming (s_t) into (s'_t), where $s'_t = s_{t+1}$. The measure P on $\mathcal{F}$ induced by the stochastic process (s_t), $t = 0, \pm 1, \pm 2, ...$ is invariant under T by virtue of the stationarity of (s_t). Consequently, T is an automorphism of $(\Omega, \mathcal{F}, P)$. For each $\omega = (..., s_{-1}, s_0, s_1, ...)$, we set $s^t(\omega) = (..., s_{t-1}, s_t)$ and consider the matrix

$$D(\omega) = A(s^1(\omega)) > 0,\ D(\omega) = (d^{kj}(\omega)) = (a^{kj}(s^1(\omega))),\ k, j \in \{1, ..., K\}.$$

In view of (18), $E|\ln d^{kj}(\omega)| < \infty$. Therefore, by virtue of Theorem A.1 (with $\tau = 1$) and Remark A.1, there exists a solution $(\phi(\cdot), z(\cdot))$ to the equation

$$\phi(\omega)z(T\omega) = D(\omega)z(\omega)\ \text{(a.s.)}\quad [\phi(\omega) > 0,\ |z(\omega)| = 1,\ z(\omega) > 0] \tag{27}$$

such that $z(\omega)$ is measurable with respect to the σ-algebra generated by $s^0(\omega)$ and $\phi(\omega)$ is measurable with respect to the σ-algebra generated by $s^1(\omega)$. Thus we can represent $z(\omega)$ and $\phi(\omega)$ in the form

$$z(\omega) = \tilde{x}(s^0(\omega)),\ \phi(\omega) = \lambda(s^1(\omega)),$$

where $\tilde{x}(\cdot)$ and $\lambda(\cdot)$ are measurable functions, and so (27) can be written as

$$\lambda(s^1)\tilde{x}(s^1) = A(s^1)\tilde{x}(s^0) \text{ (a.s.)} \quad [\lambda(s^1) > 0,\ |\tilde{x}(s^0)| = 1,\ \tilde{x}(s^0) > 0]. \tag{28}$$

The uniqueness of a solution to (28)—up to the equivalence of functions coinciding a.s.—follows from the uniqueness part of Theorem A.1.

Further, we have

$$\lambda(s^t) = |A(s^t)\tilde{x}(s^{t-1})| = \sum_{kj} a^{kj}(s^t)\tilde{x}^j(s^{t-1}) \in [Ka_*(s^t), Ka^*(s^t)], \tag{29}$$

where $a_*(s^t)$ and $a^*(s^t)$ are the smallest and the greatest elements of $A(s^t)$. From (18), we obtain $E|\ln\lambda(s^t)| < \infty$. Finally,

$$1 \geq \tilde{x}^k(s^t) = \frac{\sum_j a^{kj}(s^t)\tilde{x}^j(s^{t-1})}{\lambda(s^t)} \geq \frac{a_*(s^t)}{\lambda(s^t)},$$

which yields (16). □

Proof of Theorem 3. We can represent h_t and $\bar{x}_t$ in the form

$$h_t = A_t ... A_2 h_1,\ \bar{x}_t = A_t ... A_2 \bar{x}_1, \tag{30}$$

(see (17) and (19)), where

$$h_1^i(s^1) = \sum_{j=1}^{K} a^{ij}(s^1)h_0^j(s^0),\ \bar{x}_1^i(s^1) = \sum_{j=1}^{K} a^{ij}(s^1)\bar{x}_0^j(s^0).$$

Since $a^{ij}(s^1) > 0$, $|h_0(s^0)| > 0$ and $|\bar{x}_0(s^0)| = 1$, there exist strictly positive functions $c(s^1)$ and $C(s^1)$ satisfying

$$c(s^1)\bar{x}_1(s^1) \leq h_1(s^1) \leq C(s^1)\bar{x}_1(s^1) \tag{31}$$

(coordinatewise). We have $\bar{x}_t^k(s^t) = \lambda(s^t)...\lambda(s^1)\tilde{x}^k(s^t)$, and so

$$\frac{1}{t}\ln\bar{x}_t^k(s^t) = \frac{1}{t}\sum_{l=1}^{t}\ln\lambda(s^l) + \frac{1}{t}\ln\tilde{x}^k(s^t) \to E\ln\lambda(s^t) \text{ (a.s.)} \tag{32}$$

by virtue of the Birkhoff ergodic theorem. In the last formula, $t^{-1}\ln\tilde{x}^k(s^t) \to 0$ in view of (16). Analogously, we obtain

$$\frac{1}{t}\ln|\bar{x}_t(s^t)| \to E\ln\lambda(s^t) \text{ (a.s.)}. \tag{33}$$

This, combined with the relations

$$|\ln(p_t\bar{x}_t) - \ln|\bar{x}_t|| \leq \sum_{k=1}^{K}|\ln p^k(s^t)|,\ E|\ln p^k(s^t)| < \infty,$$

yields

$$\frac{1}{t}\ln|p_t\bar{x}_t| \to E\ln\lambda(s^t) \ \ \text{(a.s.)}. \tag{34}$$

Now, observe that relations (31) and (30) imply

$$c(s^1)\bar{x}_t(s^t) \le h_t(s^t) \le C(s^1)\bar{x}_t(s^t) \tag{35}$$

because the non-negative matrix $A_t ... A_2$ preserves the coordinatewise partial order. Inequalities (35) with $c(s^1) > 0$ and $C(s^1) > 0$ make it possible to replace in (32), (33) and (34) $\bar{x}_t$ by h_t, which yields (22). □

Proof of Theorem 4. Consider the pair $(\lambda(\cdot), \tilde{x}(\cdot))$ generating the balanced strategy (15). By virtue of (20),

$$\lambda(s^t)\tilde{x}^k(s^t) = \sum_{j=1}^{K} \alpha_{kj}\frac{p^j(s^t)}{p^k(s^t)}\tilde{x}^j(s^{t-1}),\ k \in \{1, ..., K\}. \tag{36}$$

By applying the Perron-Frobenius theorem, we find a vector $r = (r_1, ..., r_K) > 0$ such that

$$r_k = \sum_j \alpha_{kj} r_j,\ k \in \{1, ..., K\}.$$

We define

$$w^k(s^t) = \frac{p^k(s^t)\tilde{x}^k(s^t)}{r_k}.$$

Then we have $p^k(s^t)\tilde{x}^k(s^t) = r_k w^k(s^t)$, and so

$$\lambda(s^t) r_k w^k(s^t) = \sum_{j=1}^{K} \alpha_{kj} r_j \frac{p^j(s^t)}{p^j(s^{t-1})} w^j(s^{t-1}). \tag{37}$$

Put $\beta_{kj} = r_k^{-1}\alpha_{kj} r_j$. The numbers β_{kj} satisfy

$$\beta_{kj} > 0,\ \sum_{j=1}^{K} \beta_{kj} = 1. \tag{38}$$

In view of (37), we can write

$$\lambda(s^t) = \sum_{j=1}^{K} \beta_{kj} \frac{p^j(s^t)}{p^j(s^{t-1})}\frac{w^j(s^{t-1})}{w^k(s^t)},\ k \in \{1, ..., K\}. \tag{39}$$

It follows from (38) and Jensen's inequality that, for each k,

$$E\ln\left[\sum_{j=1}^{K} \beta_{kj}\frac{p^j(s^t)}{p^j(s^{t-1})}\frac{w^j(s^{t-1})}{w^k(s^t)}\right] \ge \sum_{j=1}^{K} \beta_{kj} E\ln\left[\frac{p^j(s^t)}{p^j(s^{t-1})}\frac{w^j(s^{t-1})}{w^k(s^t)}\right]. \tag{40}$$

All the expectations we consider here (in particular, those in (40)) are finite by virtue of (10) and (16).

Define $\theta_k = E \ln w^k(s^t)$ and denote by k^* that value of k for which θ_k is a minimum. The expression on the right-hand side of (40) (denote it by J_k) equals

$$J_k = \sum_{j=1}^{K} \beta_{kj} E \ln \frac{w^j(s^{t-1})}{w^k(s^t)} = \sum_{j=1}^{K} \beta_{kj}(\theta_j - \theta_k), \tag{41}$$

because

$$E \ln \frac{p^j(s^t)}{p^j(s^{t-1})} = 0 \text{ and } E \ln w^j(s^{t-1}) = E \ln w^j(s^t) = \theta_j.$$

Consequently,

$$J_{k^*} = \sum_{j=1}^{K} \beta_{k^*j}(\theta_j - \theta_{k^*}) \geq 0, \tag{42}$$

since $\theta_j \geq \theta_{k^*}$. By combining (39), (40) and (42), we find that $E \ln \lambda(s^t) \geq 0$.

To show that the last inequality is strict, it suffices to verify that, for any k, inequality (40) is strict. To this end, in turn, it is sufficient to show, that the expression

$$\frac{p^j(s^t)}{p^j(s^{t-1})} \frac{w^j(s^{t-1})}{w^k(s^t)} \tag{43}$$

is not constant with respect to j with positive probability. Suppose this is not true for some k (and hence for each k). Then there exists a function $\psi(s^t)$ such that

$$\frac{p^j(s^t)}{p^j(s^{t-1})} w^j(s^{t-1}) = \psi(s^t), \ j \in \{1, ..., K\} \text{ (a.s.)}. \tag{44}$$

¿From (39) we find

$$\lambda(s^t) = \sum_{j=1}^{K} \beta_{kj} \frac{\psi(s^t)}{w^k(s^t)} = \frac{\psi(s^t)}{w^k(s^t)} \text{ (a.s.)}$$

for each k (since $\sum_j \beta_{kj} = 1$). Thus $w^k(s^t) = \psi(s^t)/\lambda(s^t)$ (a.s.), $k = 1, ..., K$, and formula (44) leads to the equality

$$\frac{p^j(s^t)}{p^j(s^{t-1})} = \psi(s^t)\lambda(s^{t-1})/\psi(s^{t-1}) \text{ (a.s.)},$$

which contradicts assumption (B). □

Proof of Theorem 1: immediate from Theorems 3 and 4. □

Proof of Theorem 6. Proofs of assertions (i)–(iii) of Theorem 6 can be obtained by repeating with minor changes the proofs of Theorems 2 and 3 (with B, b, y and μ in place of A, a, x and λ, respectively). To derive (iv) observe that

$$B_t \geq (1 - \delta_t) A_t \geq 0,\ t = 1, 2, ...,$$

and so

$$\mu_t ... \mu_1 \tilde{y}_t = B_t ... B_1 \bar{y}_0 \geq (1 - \delta_t) ... (1 - \delta_1) A_t ... A_1 \bar{y}_0,$$

where $\mu_t = \mu(s^t)$ and $\tilde{y}_t = \tilde{y}(s^t)$. Since $|\tilde{y}_t| = 1$, we have

$$\frac{\mu_t ... \mu_1}{(1 - \delta_t) ... (1 - \delta_1)} \geq |A_t ... A_1 \bar{y}_0|.$$

By using the ergodic theorem and the last equality in (22) (applied to the sequence $A_t ... A_1 \bar{y}_0$), we find

$$E \ln \frac{\mu(s^t)}{(1 - \delta(s^t))} = \lim \frac{1}{t} \ln \frac{\mu_t ... \mu_1}{(1 - \delta_t) ... (1 - \delta_1)} \geq \lim \frac{1}{t} \ln |A_t ... A_1 \bar{y}_0| = E \ln \lambda(s^t),$$

which yields (26). □

Proof of Theorem 5: follows from Theorem 6. □

A A stochastic Perron-Frobenius theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $T : \Omega \to \Omega$ its automorphism, i.e., a one-to-one mapping such that T and T^{-1} are measurable and preserve the measure P. Let $D(\omega)$ be a measurable function taking values in the set of non-negative $K \times K$ matrices. Define

$$H(t, \omega) = D(T^{t-1}\omega) D(T^{t-2}\omega) ... D(\omega),\ t = 1, 2, \ldots, \tag{45}$$

and $H(0, \omega) = I$ (the identity matrix). Then we have

$$H(t, T^s\omega) H(s, \omega) = H(t + s, \omega),\ t, s \geq 0, \tag{46}$$

i.e., the matrix function $H(t, \omega)$ is a *cocycle* over the dynamical system $(\Omega, \mathcal{F}, P, T)$ (see, e.g., Arnold 1998).

For a matrix $D > 0$, denote by $\kappa(D)$ the ratio of the smallest element of the matrix to its greatest element. Let the following condition hold.

(*) There is a (non-random) integer $\tau > 0$ for which $H(\tau, \omega) > 0$ and

$$\int |\ln \kappa(H(\tau, \omega))| \, P(d\omega) < \infty.$$

Theorem A.1. *There exists a measurable vector function $z(\omega) > 0$ and a measurable scalar function $\phi(\omega) > 0$ such that*

$$\phi(\omega) z(T\omega) = D(\omega) z(\omega),\ |z(\omega)| = 1\ (a.s.). \tag{47}$$

The pair of functions $(\phi(\cdot), z(\cdot)) > 0$ *satisfying (47) is determined uniquely up to the equivalence with respect to the measure* P. *If* $s \to \infty$, *then*

$$\frac{H(s, T^{-s}\omega)a}{|H(s, T^{-s}\omega)a|} \to z(\omega) \ \ (a.s.), \tag{48}$$

where convergence is uniform in $a \geq 0$, $a \neq 0$.

The above result may be regarded as a generalization of the Perron–Frobenius theorem on eigenvalues and eigenvectors of positive matrices: $z(\cdot)$ and $\phi(\cdot)$ play the roles of an "eigenvector" and an "eigenvalue" of the cocycle $H(t, \omega)$. Theorem A.1 is a special case of Theorem 1 in Evstigneev (1974); see also Arnold, Demetrius and Gundlach (1994), Theorem 3.1.

Remark A.1. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be sub-σ-algebras $\mathcal{F}$ such that the random matrices $D(T^{-1}\omega), D(T^{-2}\omega), ...$ are $\mathcal{F}_0$-measurable and the random matrices $D(T\omega), D(T^{-1}\omega), ...$ are $\mathcal{F}_1$-measurable. By virtue of (48) and (45), the functions $z(\cdot)$ and $\phi(\cdot)$ are measurable with respect to the σ-algebras $\mathcal{F}_0$ and $\mathcal{F}_1$ completed by all sets of measure zero. From this it follows that we can select versions of $z(\cdot)$ and $\phi(\cdot)$, satisfying (47), which are $\mathcal{F}_0$- and $\mathcal{F}_1$-measurable, respectively.

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