# Optimal trading of an asset driven by a hidden Markov process in the presence of fixed transaction costs

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**Abstract**

We consider the problem of the optimal trading of an asset in the presence of fixed transaction costs where the asset price satisfies an SDE of the form $dS_t = dB_t + h(X_t)\,dt$ where $B_t$ is a Brownian motion, $h$ is a known function and $X_t$ is a Markov Chain. We look at two versions of the problem, maximising the long term gain per unit time and maximising a form of discounted gain. It is well known that the optimal trading strategy for such a problem is the solution of a free-boundary problem; we present an intuitive derivation by viewing the optimal trading problem as a pair of simultaneous optimal stopping problems. We also give explicit solutions for a range of examples, and give bounds on the transaction cost above which it is optimal never to buy the asset at all. We show that in the case where Markov Chain $X_t$ is independent of the Brownian motion and has a finite statespace, this critical transaction cost has a simple form.

**Keywords:** Optimal stopping problem, trading problem, hidden Markov model

## 1 Introduction

We consider the problem of optimally trading an asset in the presence of transaction costs. We assume that the asset price process has the form $S_t = B_t + \int_0^t h(X_u)\,du$ where $B$ is a standard Brownian motion and $X$ is a Markov process. The only information available at time $t$ is the past history of the asset; the Markov process $X$ is not directly observable.

At each time $t$, $t \in [0, \infty)$ we assume that we are allowed to hold 0 or 1 units of the asset and that we incur a transaction cost of $\frac{1}{2}c > 0$ whenever we buy or sell the asset. Letting $\xi_t$ denote our holding at time $t$ and $\{T_i\}$, $i \geq 1$ the times when $\xi$ is discontinuous, we look for strategies which maximise $\mathbb{E}(\int_0^\infty e^{-\rho t}\xi_t\,dS_t - \frac{1}{2}c\sum e^{-\rho T_i})$ where $\rho > 0$. For tractability we will frequently just consider the limiting strategy obtained as $\rho \to 0$. This is often also the strategy which maximises $\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}(\int_0^t \xi_u \, dS_u - \frac{1}{2}c \sum I_{\{T_i \le t\}})$. We will refer to the quantity $\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}(\int_0^t \xi_u \, dS_u - \frac{1}{2}c \sum I_{\{T_i \le t\}})$ as the (long run) average gain.

In Morton & Pliska (n.d.) and Pliska & Selby (1994) a similar type of problem is considered. The authors try to maximise the long term log return per unit time in a framework with proportional transaction costs where the asset price follows an $n$-dimensional geometric Brownian motion with constant drift. Here we look at a rather different class of models by introducing an unobserved Markov Chain, and allow ourselves the luxury of a somewhat simpler transaction cost structure and optimality criteria. This approach also introduces the element of filtering the past history of the asset to estimate the current state of $X_t$, a feature also considered in Mandarino (1990), where the Kalman filter is used. More examples of the filtering of hidden Markov models can be found in Elliot, Aggoun & Moore (1995) and in its references.

The outline of this paper is as follows: In the following section, Section 2, we derive a condition for optimality closely related to the Hamilton-Jacobi-Bellman equation, by considering the optimal trading problem as a pair of simultaneous optimal stopping problems.

In Section 3 we consider the case where the instantaneous expected drift is a Markov process of a particular type. We derive the limiting optimal strategy as $\rho \to 0$ and prove that it does maximise the average gain in certain cases. As an example, we derive the optimal trading strategy when the asset follows an Ornstein-Ulhenbeck price process.

In Section 4 we consider the case where $X$ has a finite statespace. We derive conditions on $c$ which determine whether it is ever optimal to hold the asset. We examine both the discounted and average gain cases and as an example consider a simple 2-state model.

Finally, in Section 5 we consider a model where the asset drifts towards a level which follows a Brownian motion; we show that this model is essentially the same as the OU model considered in Section 3.

## 2 Optimality Equations

We assume that the asset price process has the form $S_t = B_t + \int_0^t h_u \, du$, where $B_t$ is a Brownian motion adapted to a filtration $\mathcal{F}$ and $h_u = h(X_u)$ for some Markov process $X_t$ also adapted to $\mathcal{F}$. We denote by $\mathcal{Y}$ the complete filtration generated by $S$ and let $\widehat{h}_t = \mathbb{E}(h_t|\mathcal{Y}_t)$.

We define the innovations process $N_t = S_t - \int_0^t \widehat{h}_u \, du$. Assuming $\mathbb{E}(\int_0^t h_u^2 \, du < \infty)$ for each $t$, $N_t$ is a $\mathcal{Y}$-Brownian motion (see Section VI.8 of Rogers & Williams (1987)). We shall also assume $\mathbb{E}(\int_0^\infty e^{-\rho t}|\widehat{h}_t| \, dt)^2 < \infty$. Note that if $h$ is bounded these conditions are certainly met.

A trading strategy $\xi$ is a $\mathcal{Y}$-previsible process with values in $\{0,1\}$ and with $\xi_0$ given, such that $\mathbb{E}(\sum e^{-\rho T_i})^2 < \infty$ where $\{T_i\}$, $i \geq 1$, are the discontinuities of $\xi$. Our aim is to maximise the expected discounted gain, which we define by

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\xi_t\, dS_t - \tfrac{1}{2}c\sum e^{-\rho T_i}\,\middle|\,\mathcal{Y}_0\right]. \tag{2.1}$$

We assume that the conditional distribution of $X_t$ given $\mathcal{Y}_t$ is is described by a diffusion $p_t \in E \subseteq \mathbb{R}^n$ with generator $\mathcal{G}$, and that $p_t$ satisfies the SDE $dp_t = \sigma(p_t)\, dN_t + \mu(p_t)\, dt$. For example, if $X_t$ has a finite statespace we could let $p_t(i) = \mathbb{P}(X_t = i|\mathcal{Y}_t)$, and if the process $(S_t, X_t)$ is jointly Gaussian we could set $p_t = (\mathbb{E}(X_t|\mathcal{Y}_t), \mathbb{V}\mathrm{ar}(X_t|\mathcal{Y}_t))$. Write $\mathbb{E}^p$ for expectation under the law of $(p_t)$ with $p_0 = p$.

For an arbitrary function $f$ on the statespace of $X$, we define $\widehat{f}(p)$ to be the value $\mathbb{E}f(X)$ under the distribution for $X$ corresponding to $p$.

Note that since $\mathbb{E}(\int_0^T e^{-\rho t}\xi_t\, dN_t)^2 \leq \int_0^T e^{-2\rho t}\, dt < (2\rho)^{-1}$ the collection of random variables $\{\int_0^T e^{-\rho t}\xi_t\, dN_t : T \geq 0\}$ is bounded in $L^2$ and the process $\int_0^t e^{-\rho u}\xi_u\, dN_u$ is a UI martingale. Using the Optional Stopping Theorem and the fact that $dS_t = dN_t + \widehat{h}_t\, dt$, the expected discounted gain, (2.1) equals

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\xi_t \widehat{h}_t\, dt - \tfrac{1}{2}c\sum e^{-\rho T_i}\,\middle|\,\mathcal{Y}_0\right].$$

For simplicity we will restrict attention to Markov strategies which specify $\xi_t$ in terms of $\xi_{t-}$ and $p_t$. Thus a strategy amounts to the specification of two subsets $B$ and $S$ of $E$, the asset, if it is not already held, being bought at the moment $p_t$ enters $B$ and then sold when $p_t$ enters $S$. We look for Markov strategies which are optimal for all initial $\xi_0$ and $p_0$.

**Proposition 2.1** *Suppose that for fixed subsets $B$, $S \subseteq E$ the strategy $\xi$ which buys when $p_t \in B$ and sells when $p_t \in S$ is optimal. Then there is a function $w$ on $E$ such that*

$$(\mathcal{G} - \rho)w = -\widehat{h} \quad \text{on } B^c \cap S^c \tag{2.2}$$

$$w = +\tfrac{1}{2}c \quad \text{on } B \tag{2.3}$$

$$w = -\tfrac{1}{2}c \quad \text{on } S \tag{2.4}$$

$$\sigma \cdot \nabla w = 0 \quad \text{on } \partial B,\ \partial S \tag{2.5}$$

**Proof**

Define the function $g : E \to \mathbb{R}$ as follows: for $p \in S$, set $g(p) = 0$, otherwise set $g(p) = \mathbb{E}^p[\int_0^{H_S} e^{-\rho t}\widehat{h}_t\, dt - \frac{1}{2}c(1 + e^{-\rho H_S})]$, where $H_S$ is the first hitting time of $S$ by $p_t$. First consider the problem: find a stopping time $\tau$ to maximise $\mathbb{E}^p e^{-\rho\tau} g(p_{\tau\wedge H_S})$. Let $B^*$ be the optimal stopping set; we will show that $B = B^*$.

Define the sequence of stopping times $\sigma_n$, $\tau_n$, $n \geq 0$ by

$$\begin{aligned}
\tau_0 &= 0 \\
\sigma_n &= \inf\{t > \tau_n : p_t \in B \cup B^*\} \\
\tau_{n+1} &= \inf\{t > \sigma_n : p_t \in S\}
\end{aligned}$$

Note that $\xi$ buys at most once in $[\sigma_{j-1}, \tau_j)$, $j \geq 1$ and sells at $\tau_j$. Define $T'_j = T$ if $\Delta\xi_T = 1$, $T \in [\sigma_{j-1}, \tau_j)$ and $T'_j = \infty$ otherwise. Note that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\xi_t \widehat{h}_t\, dt - \tfrac{1}{2}c\sum e^{-\rho T_i}\right] = \mathbb{E}\left[\int_0^{\sigma_0} e^{-\rho t}\xi_t \widehat{h}_t\, dt - \tfrac{1}{2}c\sum_{T_i \leq \sigma_0} e^{-\rho T_i}\right]$$
$$+\mathbb{E}\left[\sum_{j=1}^\infty I_{\{T'_j < \infty\}}\left(\int_{T'_j}^{\tau_j} e^{-\rho t}\widehat{h}_t\, dt - \tfrac{1}{2}c(e^{-\rho T'_j} + e^{-\rho\tau_j})\right)\right]$$

Since $\mathbb{E}(\int_0^{\sigma_j} e^{-\rho t}|\widehat{h}_t|\, dt)^2$ and $\mathbb{E}(\sum_{T_i \leq \sigma_j} e^{-\rho T_i})^2$ are both increasing in $j$ and bounded above, we use Dominated Convergence to interchange the order of summation and expectation in the final term of the equation above. Conditioning on $\mathcal{Y}_{\sigma_{j-1}}$ and using the strong Markov property gives

$$\sum_{j=1}^\infty \mathbb{E}[e^{-\rho\sigma_{j-1}}\mathbb{E}^{p_{\sigma_{j-1}}} I_{\{H_B < H_S\}} e^{-\rho H_B} g(p_{H_B})],$$

where the inner expectation is just $\mathbb{E}^{p_{\sigma_j}} e^{-\rho H_B} g(p_{H_B \wedge H_S})$. Thus by the optimality of $\xi$, and that fact that $B^*$ is the optimal stopping set, $\xi$ buys at time $\sigma_{j-1}$ if and only if $p_{\sigma_{j-1}} \in B^*$. Thus $B = B^*$.

We now define $v(p) = \mathbb{E}^p e^{-\rho\tau} g(p_{\tau\wedge H_S})$, so that $v$ satisfies $(\mathcal{G}-\rho)v = 0$ in $S^c \cap B^c$, $v(p) = 0$ on $S$, $v(p) = g(p)$ on $B$ and $\sigma\cdot\nabla v = \sigma\cdot\nabla g$ on $\partial B$. (The final condition is the 'smooth pasting' condition on the decision boundary, see Shiryayev (1978) page 161 or Öksendal (1994) page 202.) Similarly we define $\tilde{g}(p) = 0$ for $p \in B$ and $\tilde{g}(p) = -\mathbb{E}^p[\int_0^{H_B} e^{-\rho t}\widehat{h}_t\, dt + \frac{1}{2}c(1 + e^{-\rho H_B})]$ for $p \notin B$. We let $S^*$ be the stopping set for the problem 'find a stopping time $\tilde{\tau}$ to maximise $\mathbb{E}^p e^{-\rho\tilde{\tau}}\tilde{g}(p_{\tilde{\tau}\wedge H_B})$', and we define the sequence of stopping times $\tilde{\sigma}_n$, $\tilde{\tau}_n$, $n \geq 0$ by

$$\begin{aligned}
\tilde{\tau}_0 &= 0 \\
\tilde{\sigma}_n &= \inf\{t > \tilde{\tau}_n : p_t \in S \cup S^*\} \\
\tilde{\tau}_{n+1} &= \inf\{t > \tilde{\sigma}_n : p_t \in B\}
\end{aligned}$$

We see that $\xi$ sells at most once in $[\tilde{\sigma}_{j-1}, \tilde{\tau}_j)$, $j \geq 1$, and buys at $\tilde{\tau}_j$. Defining $\tilde{T}'_j = T$ if

$\Delta\xi_{\tilde{T}'} = -1$, $T \in [\tilde{\sigma}_{j-1}, \tilde{\tau}_j)$ and $\tilde{T}'_j = \infty$ otherwise, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\xi_t \widehat{h}_t\,dt - \tfrac{1}{2}c\sum e^{-\rho T_i}\right] &= \mathbb{E}\left[\int_0^{\tilde{\sigma}_0} e^{-\rho t}\xi_t \widehat{h}_t\,dt - \tfrac{1}{2}c\sum_{T_i\le\tilde{\sigma}_0} e^{-\rho T_i}\right]\\ &\quad+\mathbb{E}\left[\sum_{j=1}^\infty I_{\{\tilde{T}'_j<\infty\}}\left(\int_{\tilde{\sigma}_{j-1}}^{\tilde{T}'_j} e^{-\rho t}\widehat{h}_t\,dt - \tfrac{1}{2}c(e^{-\rho\tilde{T}'_j} + e^{-\rho\tilde{\tau}_j})\right)\right]\\ &\quad+\mathbb{E}\left[\sum_{j=1}^\infty I_{\{\tilde{T}'_j=\infty\}}\int_{\tilde{\sigma}_{j-1}}^{\tilde{\tau}_j} e^{-\rho t}\widehat{h}_t\,dt\right]\end{aligned}$$

where the final two terms equal

$$\mathbb{E}\left[\sum_{j=1}^\infty\int_{\tilde{\sigma}_{j-1}}^{\tilde{\tau}_j} e^{-\rho t}\widehat{h}_t\,dt\right] + \mathbb{E}\left[\sum_{j=1}^\infty I_{\{\tilde{T}'_j<\infty\}}\left(-\int_{\tilde{T}_j}^{\tilde{\tau}_j} e^{-\rho t}\widehat{h}_t\,dt - \tfrac{1}{2}c(e^{-\rho\tilde{T}'_j} + e^{-\rho\tilde{\tau}_j})\right)\right]$$

Again we can interchange the order of summation and expectation in the final term. Conditioning on $\mathcal{Y}_{\tilde{\sigma}_j}$ and using the strong Markov property, the this becomes

$$\sum_{j=1}^\infty \mathbb{E}[e^{-\rho\tilde{\sigma}_{j-1}}\mathbb{E}^{p_{\tilde{\sigma}_{j-1}}} I_{\{H_S<H_B\}}e^{-\rho H_S}\tilde{g}(p_{H_S})],$$

where the inner expectation is now $\mathbb{E}^{p_{\tilde{\sigma}_{j-1}}}e^{-\rho H_S}\tilde{g}(p_{H_S\wedge H_B})$. Thus $S = S^*$.

Similarly to before, define $\tilde{v}(p) = \mathbb{E}^p e^{-\rho\tilde{\tau}}\tilde{g}(p_{\tilde{\tau}\wedge H_B})$, then $\tilde{v}$ satisfies $(\mathcal{G}-\rho)\tilde{v} = 0$ in $S^c\cap B^c$, $\tilde{v}(p) = 0$ on $B$, $\tilde{v}(p) = \tilde{g}(p)$ on $B$ and $\sigma\cdot\nabla\tilde{v} = \sigma\cdot\nabla\tilde{g}$ on $\partial B$. We now have two functions

$$\begin{aligned}v(p) &= \mathbb{E}^p\left[\int_{H_S\wedge H_B}^{H_S} e^{-\rho t}\widehat{h}_t\,dt - \tfrac{1}{2}cI_{\{H_B<H_S\}}(e^{-\rho H_S} + e^{-\rho H_B})\right]\\ \tilde{v}(p) &= -\mathbb{E}^p\left[\int_{H_S\wedge H_B}^{H_B} e^{-\rho t}\widehat{h}_t\,dt + \tfrac{1}{2}cI_{\{H_S<H_B\}}(e^{-\rho H_S} + e^{-\rho H_B})\right].\end{aligned}$$

Since $0 = \sigma\cdot\nabla(v-g)$ on $\partial B$, noting that $\int_{H_S\wedge H_B}^{H_S} e^{-\rho t}\widehat{h}_t\,dt = \int_0^{H_S} e^{-\rho t}\widehat{h}_t\,dt - \int_0^{H_S\wedge H_B} e^{-\rho t}\widehat{h}_t\,dt$, we see that

$$\begin{aligned}0 &= \sigma\cdot\nabla\mathbb{E}^p\left[-\int_0^{H_S\wedge H_B} e^{-\rho t}\widehat{h}_t\,dt - \tfrac{1}{2}c(I_{\{H_B<H_S\}}(e^{-\rho H_S} + e^{-\rho H_B}) - e^{-\rho H_S})\right]\\ &= \sigma\cdot\nabla\mathbb{E}^p\left[-\int_0^{H_S\wedge H_B} e^{-\rho t}\widehat{h}_t\,dt - \tfrac{1}{2}c(I_{\{H_B<H_S\}}e^{-\rho H_S} - I_{\{H_S<H_B\}}e^{-\rho H_S})\right]\end{aligned}$$

Similarly, since $0 = \sigma\cdot\nabla(\tilde{v}-\tilde{g})$ on $\partial S$

$$\begin{aligned}0 &= \sigma\cdot\nabla\mathbb{E}^p\left[\int_0^{H_S\wedge H_B} e^{-\rho t}\widehat{h}_t\,dt - \tfrac{1}{2}c(I_{\{H_S<H_B\}}(e^{-\rho H_B} + e^{-\rho H_S}) - e^{-\rho H_B})\right]\\ &= \sigma\cdot\nabla\mathbb{E}^p\left[\int_0^{H_S\wedge H_B} e^{-\rho t}\widehat{h}_t\,dt - \tfrac{1}{2}c(I_{\{H_S<H_B\}}e^{-\rho H_B} - I_{\{H_B<H_S\}}e^{-\rho H_S})\right]\end{aligned}$$

Defining the function $w$ on $E$ by

$$w(p) = \mathbb{E}^p \left[ \int_0^{H_S \wedge H_B} e^{-\rho t} \widehat{h}_t \, dt - \tfrac{1}{2} c (I_{\{H_S < H_B\}} e^{-\rho H_B} - I_{\{H_B < H_S\}} e^{-\rho H_S}) \right]$$

we have $(\mathcal{G} - \rho) w = -\widehat{h}$ in $S^c \cap B^c$, $w = +\frac{1}{2}c$ in $B$, $w = -\frac{1}{2}c$ in $S$ and $\sigma \cdot \nabla w = 0$ on $\partial B$, $\partial S$.

$\square$

**Remark** To see the connection between (2.2)–(2.5) and the HJB equation, define

$$V(p, \xi) = \sup \mathbb{E}^p \left[ \int_0^\infty e^{-\rho t} \xi_t \widehat{h}_t \, dN_t - \sum e^{-\rho T_i} \right]$$

to be the maximal expected discounted gain over trading strategies such that $\xi_0 = \xi$, where the conditional distribution of $X_0$ given $\mathcal{Y}_0$ is given by $p$. The HJB equation in this case is

$$\max(\xi \widehat{h} + \mathcal{G} V(p, \xi) - \rho V(p, \xi), -\tfrac{1}{2} c + V(p, 1 - \xi) - V(p, \xi)) = 0 \tag{2.6}$$

where, as before, $\mathcal{G}$ is the generator of $p$ acting on $V(\cdot, \xi)$. We then have

$$\begin{aligned} \xi \widehat{h} + \mathcal{G} V(p, \xi) - \rho V(p, \xi) &= 0 \quad \text{on } B^c \cap S^c, \\ V(p, 1) - V(p, 0) &= \tfrac{1}{2} c \quad \text{on } B, \\ V(p, 0) - V(p, 1) &= \tfrac{1}{2} c \quad \text{on } S, \end{aligned}$$

Setting $w(p) = V(p, 1) - V(p, 0)$, these conditions become

$$\begin{aligned} (\mathcal{G} - \rho) w &= -\widehat{h} \quad \text{on } B^c \cap S^c, \\ w(p) &= +\tfrac{1}{2} c \quad \text{on } B, \\ w(p) &= -\tfrac{1}{2} c \quad \text{on } S, \end{aligned}$$

which are just (2.2), (2.3) and (2.4) again.

# 3 Markovian expected drift

We now consider the special case where $\widehat{h}_t$ is a recurrent diffusion on an interval $I$ with SDE

$$d\widehat{h}_t = \sigma(\widehat{h}_t) \, dN_t - \gamma \widehat{h}_t \, dt,$$

where $\gamma > 0$. We will show that the limiting optimal strategy as $\rho \to 0$ is to buy when $\widehat{h}_t \geq b \geq 0$ and to sell when $\widehat{h}_t \leq s \leq 0$ where $b$ and $s$ satisfy:

$$s'(b) = s'(s), \tag{3.1}$$

$$b - s - c\gamma = [s(b) - s(s)] / s'(s). \tag{3.2}$$

Here the function $s(x)$ is the scale function of $\widehat{h}$, which we define by $s'(x) = e^{2\gamma \int^x u\sigma(u)^{-2}\,du}$, determined up to a positive affine transformation. As an example we will solve the problem in the case when the asset price follows an Ornstein-Ulhenbeck process. Finally, we will show that if $\widehat{h}_t$ is positive recurrent, then, as suggested in the introduction, this strategy also maximises $\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}(\int_0^u \xi_t\, dS_u - \frac{1}{2}c\sum I_{\{T_i\le t\}})$

Our first two lemmas will show that the limiting optimal strategy for the discounted version of the problem has the stated form.

**Lemma 3.1** *Any optimal strategy never buys when $\widehat{h}_t < 0$ and never sells when $\widehat{h}_t > 0$.*

**Proof** Let $\xi$ be an arbitrary strategy and define the sequence of interleaved stopping times $\sigma_n$, $\tau_n$, $n \ge 0$ as follows:

$$\begin{aligned} \sigma_0 &= 0 \\ \tau_n &= \inf(t > \sigma_n : \Delta\xi_t = -1, \widehat{h}_t > 0 \text{ or } \Delta\xi_t = +1, \widehat{h}_t < 0) \\ \sigma_{n+1} &= \inf(t > \tau_n : \widehat{h}_t = 0). \end{aligned}$$

Now define a new strategy $\tilde{\xi}$ by $\tilde{\xi}_0 = \xi_0$ and for $t > 0$,

$$\tilde{\xi} = \begin{cases} \xi & \text{on} \quad \cup_{n\ge0}(\sigma_n, \tau_n) \\ 0 & \text{on} \quad \cup_{n\ge0}[\tau_n, \sigma_{n+1}] \cap \{\Delta\xi_{\tau_n} = +1\} \\ 1 & \text{on} \quad \cup_{n\ge0}[\tau_n, \sigma_{n+1}] \cap \{\Delta\xi_{\tau_n} = -1\} \end{cases}$$

The strategy $\tilde{\xi}$ just mimics $\xi$ until $\xi$ either buys when $\widehat{h}_t < 0$ or sells when $\widehat{h}_t > 0$. It then takes no action until $\widehat{h}_t$ hits 0 when it changes back to $\xi$.

Since $\tilde{\xi}$ is continuous at $\tau_n$, it is clear that $\tilde{\xi}$ is previsible. In addition we have that $\tilde{T}_i \ge T_i$ and so $\mathbb{E}(\sum e^{-\rho\tilde{T}_i})^2 < \infty$. We also have $\tilde{\xi}_u\widehat{h}_u \ge \xi_u\widehat{h}_u$ for all times $u$, and so $\tilde{\xi}$ has a larger expected discounted gain than $\xi$. □

**Lemma 3.2** *Let $b, b' \in I$ with $b' > b$ and suppose it is optimal to buy when $\widehat{h}_t = b \ge 0$. Then there exists $\rho(b') > 0$ such that for $\rho \le \rho(b')$ it is also optimal to buy whenever $\widehat{h}_t = b'$.*

**Proof** Suppose $\widehat{h}_0 = b' > b$ and $\xi_0 = 0$, and let $T$ denote the first time $\widehat{h}$ hits $b$. Let $\xi$ be a Markov strategy which buys when $\widehat{h}_t = b$, but not when $\widehat{h}_t \in U$, where $U$ is an open interval containing $b'$. Let $\tilde{\xi}$ be the strategy identical to $\xi$ except that $\tilde{\xi}$ buys immediately. Thus $\tilde{T}_i = T_i$ for $i \ge 2$ and $\tilde{T}_1, T_1 \le T$. The difference between the expected discounted gains under $\tilde{\xi}$ and $\xi$ is given by

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}(\tilde{\xi}_t - \xi_t)\widehat{h}_t\, dt - \tfrac{1}{2}c\mathbb{E}(e^{-\rho\tilde{T}_1} - e^{-\rho T_1})\,\middle|\, \mathcal{Y}_0\right].$$

We note that $\tilde{\xi}_t \geq \xi_t$ and $\widehat{h}_t > 0$ for $t \in [0, T]$, so that $\widehat{h}_t(\tilde{\xi}_t - \xi_t) > 0$ for $t \in [0, T]$. Thus, by Monotone convergence on each term separately as $\rho \to 0$, and noting that $\tilde{T}_1$ and $T_1$ are a.s. finite since $\widehat{h}_t$ is recurrent, we have

$$\lim_{\rho \to 0} \mathbb{E}\left[\int_0^\infty e^{-\rho t}(\tilde{\xi}_t - \xi_t)\widehat{h}_t\, dt - \tfrac{1}{2}c(e^{-\rho \tilde{T}_1} - e^{-\rho T_1}) \,\middle|\, \mathcal{Y}_0\right] > 0.$$

Thus for $\rho$ sufficiently small it is better to buy immediately. □

Similarly if it is optimal to sell when $\widehat{h}_t = s \leq 0$, $s \in I$, then for any $s' < s$, $s \in I$ and for $\rho$ sufficiently small, it is optimal to sell whenever $\widehat{h}_t = s'$.

Thus the limiting optimal strategy has the form: buy if $\widehat{h}_t \geq b \geq 0$ and sell if $\widehat{h}_t \leq s \leq 0$. It only remains to show $s$ and $b$ satisfy (3.1) and (3.2).

**Proposition 3.3** *The limiting optimal strategy as $\rho \to 0$ is to buy when $\widehat{h}_t \geq b \geq 0$ and to sell when $\widehat{h}_t \leq s \leq 0$, where $b$ and $s$ satisfy:*

$$\begin{aligned} s'(b) &= s'(s), \\ b - s - c\gamma &= (s(b) - s(s))/s'(s), \end{aligned}$$

*and the function $s(x)$ is the scale function of $\widehat{h}$, defined on page 7.*

**Proof** From the optimality condition derived in Section 2, Equations (2.2)–(2.5), the limiting strategy as $\rho \to 0$ gives rise to a function $w$ which satisfies $\frac{1}{2}\sigma(x)^2 w'' - \gamma x w' = -x$ in $(s, b)$; $w(x) = \frac{1}{2}c$ on $[b, \infty)$; $w(x) = -\frac{1}{2}c$ on $(-\infty, s]$, and $w(x)$ is differentiable at $x = s$ and $x = b$.

We first define $f(x) = w(x) - x/\gamma$ so that $f$ satisfies $\frac{1}{2}\sigma(x)^2 f'' - \gamma x f' = 0$ on $(s, b)$. Integrating once and using $s'(x) = e^{2\gamma \int^x u\sigma(u)^{-2}\, du}$ gives $f'/s' = K$, a constant, so using the fact that $w'(b) = w'(s) = 0$ we have $s'(s) = s'(b) = -\frac{1}{K\gamma}$. Integrating again and using $w(b) - w(s) = c$ gives $c - (b - s)/\gamma = -(s(b) - s(s))/(\gamma s'(s))$. Rearranging gives the result. □

**Exmaple** If the price process is an Ornstein-Ulhenbeck process, so $dS_t = \sigma\, dB_t - \gamma S_t\, dt$, then $d\widehat{h}_t = -\gamma\sigma\, dB_t - \gamma \widehat{h}_t\, dt$. The optimal strategy is to buy if $S_t \leq -b/\gamma$ and to sell if $S_t \geq b/\gamma$, where $b$ satisfies

$$2b - \gamma c = 2e^{-b^2/(\gamma\sigma^2)} \int_0^b e^{u^2/(\gamma\sigma^2)}\, du. \tag{3.3}$$

We now show that this equation has a unique solution in $b \geq 0$ and thus determines the optimal trading strategy. Let $\phi(b)$ denote the right hand side of (3.3), so $\phi(b) \geq 0$ and $\phi(b) = 0$ only at $b = 0$. Since $\phi'(b) = 2(1 - \frac{b}{\gamma\sigma^2}\phi)$ and the left hand side of (3.3) has derivative 2 (considered as a function of $b$), there can be at most one solution. To prove existence we note that $\phi(b) \to 0$ as $b \to 0$; we now show $\phi(b) \to 0$ as $b \to \infty$. Writing $u = bv$, we have

$$\phi(b) = 2\int_0^1 be^{b^2(v^2-1)/(\gamma\sigma^2)}\,dv = 2\int_{[0,1)} be^{b^2(v^2-1)/(\gamma\sigma^2)}\,dv.$$

For $x \geq 0$ the maximum of $be^{-b^2x}$ occurs at $b = 1/\sqrt{2x}$, so

$$be^{b^2(v^2-1)/(\gamma\sigma^2)} \leq e^{-\frac{1}{2}}/\sqrt{2(1-v^2)/(\gamma\sigma^2)}, \quad b \geq 0, \quad v \in [0,1)$$

which is integrable on $[0,1)$. Since $be^{b^2(v^2-1)/(\gamma\sigma^2)} \to 0$ as $b \to \infty$ on $[0,1)$, by Dominated Convergence, $\phi(b) \to 0$ as $b \to \infty$.

**Average gain case.** We now consider the problem of maximising

$$\liminf_{t\to\infty} \tfrac{1}{t}\mathbb{E}[\int_0^t \xi_u\,dS_u - \tfrac{1}{2}c\sum I_{\{T_i\leq t\}}|\mathcal{Y}_0],$$

which we shall refer to as the (long run) average gain. Our method is essentially the same as that of the proof of Proposition 1, and we will deduce the same condition (Equations (2.2)–(2.5)) but for $\rho$ equal to 0. Thus if an optimal strategy exists for this second problem, and the equation $\mathcal{G}w = -\widehat{h}$, with the same boundary conditions as before, has a unique solution, then the limiting optimal strategy as $\rho \to 0$ also maximises the average gain. Let $\mathbb{E}^x$ denote expectation under the law of $\widehat{h}_t$ started from $x$. In this Section we will assume that $\widehat{h}_t$ is positive recurrent (so $\mathbb{P}^y(H_x < \infty) = 1$ implies $\mathbb{E}^y(H_x) < \infty$ where $H_x$ denote the first hitting time of $x$ by $\widehat{h}_t$), that $\sigma$ is bounded on $I$ and that $\frac{1}{t}\mathbb{E}(|\widehat{h}_t|) \to 0$ as $t \to \infty$.

First we state and prove a useful lemma:

**Lemma 3.4** *Let $\sigma < \tau$ be two $\mathcal{Y}$-stopping times with $\mathbb{E}(\tau - \sigma) < \infty$. Then $\mathbb{E}(S_\tau - S_\sigma) = -\gamma^{-1}\mathbb{E}(\widehat{h}_\tau - \widehat{h}_\sigma)$.*

**Proof** We have

$$\mathbb{E}(S_\tau - S_\sigma + \gamma^{-1}(\widehat{h}_\tau - \widehat{h}_\sigma)) = \mathbb{E}\int_\sigma^\tau [1 + \gamma^{-1}\sigma(\widehat{h}_u)]\,dN_u.$$

Since $1 + \gamma^{-1}\sigma(\cdot)$ is bounded and $\mathbb{E}(\tau - \sigma) < \infty$, we have that the family of random variables $\{\int_{\sigma\wedge t}^{\tau\wedge t}(1 + \gamma^{-1}\sigma(\widehat{h}_u))\,dN_u : t \geq 0\}$ is $L^2$-bounded and hence UI. Thus $\mathbb{E}\int_\sigma^\tau(1 + \gamma^{-1}\sigma(\widehat{h}_u))\,dN_u = 0$ by a version of the Optimal Stopping Theorem. □

Note that a corollary of this is that the optimal strategy is to buy when $\widehat{h}_t \geq b \geq 0$ and to sell when $\widehat{h}_t \leq s \leq 0$, for some $b$ and $s$ (by very similar arguments to those given earlier).

**Proposition 3.5** *Suppose that, for fixed $s \leq 0 \leq b$, the strategy $\xi$ which buys when $\widehat{h}_t \geq b$ and sells when $\widehat{h}_t \leq s$ is optimal. There is a function $w$ such that $\mathcal{G}w(x) = -x$ on $(s,b)$, $w = +\frac{1}{2}c$ on $[b,\infty)$, $w = -\frac{1}{2}c$ on $(-\infty, s]$ and $w' = 0$ at $b$ and $s$, where $\mathcal{G}$ denotes the generator of $\widehat{h}_t$.*

**Proof** Define $g(x) = 0$ for $x \leq s$, and $g(x) = \mathbb{E}^x \int_0^{H_s} \widehat{h}_t \, dt - c$ for $x > s$, where $H_s$ is the first hitting time of $s$ by $\widehat{h}_t$. Let $B^*$ be the optimal stopping set for the problem: $\max \mathbb{E}^x g(\widehat{h}_{\tau \wedge H_S})$. Let $\bar{b} = b \wedge \inf\{x \in B, x > s\}$; we will show $b = \bar{b}$.

Define the sequence of stopping times $\sigma_n$, $\tau_n$, $n \geq 0$ by

$$
\begin{aligned}
\tau_0 &= 0 \\
\sigma_n &= \inf\{t > \tau_n : \widehat{h}_t \geq \bar{b}\} \\
\tau_{n+1} &= \inf\{t > \sigma_n : \widehat{h}_t \leq s\}
\end{aligned}
$$

and define $G_t = \int_0^t \xi_u \widehat{h}_u \, du - \frac{1}{2} c \sum I_{\{T_i \leq t\}}$. Let $N(t)$ be the greatest index such that $\tau_{N(t)} \leq t$ and write

$$
G_t = G_t - G_{N(t)+1} + G_{\tau_1} + \sum_{n=2}^{N(t)+1} (G_{\tau_n} - G_{\tau_{n-1}}).
$$

We will show $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(G_t - G_{N(t)+1}) = 0$, $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(G_{\tau_1}) = 0$, and then finally that $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\sum_{n=2}^{N(t)} (G_{\tau_{n+1}} - G_{\tau_n}) = K^{-1}\mathbb{E}^{\bar{b}} g(\widehat{h}_{H_b \wedge H_s})$ for some constant $K$. This will imply $b = \bar{b}$.

For the two first statements, note that $\xi$ can buy and sell at most once in each of the intervals $[0, \tau_1)$ and $[t, \tau_{N(t)+1})$. Thus we have

$$
\begin{aligned}
\mathbb{E}(G_{\tau_1}) &\leq \gamma^{-1}(|\widehat{h}_t| + |b| + |s|), \\
\mathbb{E}(G_t - G_{\tau_{N(t)+1}}) &\leq \gamma^{-1}(|\widehat{h}_t| + |b| + |s|).
\end{aligned}
$$

Since $\frac{1}{t}\mathbb{E}(|\widehat{h}_t|) \to 0$ as $t \to \infty$, $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(G_t - G_{N(t)+1}) = 0$ and $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(G_{\tau_1}) = 0$.

For the third statement, $\mathbb{E}(\tau_{n+1} - \tau_n) > 0$ so $\mathbb{E}[N(t)] < \infty$ (see Grimmett & Stirzaker (1992), 10.5.1(b)). As the random variables $G_{\tau_{n+1}} - G_{\tau_n}$, $n \geq 1$, are IID, using Wald's equation (Grimmett & Stirzaker (1992), page 211) and setting $K = \mathbb{E}(\tau_{n+1} - \tau_n)$ gives

$$
\begin{aligned}
\lim_{t\to\infty} \tfrac{1}{t}\mathbb{E}^x[\sum_{n=2}^{N(t)+1} (G_{\tau_{n+1}} - G_{\tau_n})] &= \lim_{t\to\infty} \tfrac{1}{t}\mathbb{E}^x[N(t)]\mathbb{E}^x(G_{\tau_{n+1}} - G_{\tau_n}) \\
&= K^{-1}\mathbb{E}^{\bar{b}}(G_{\tau_{n+1}} - G_{\tau_n}) \\
&= K^{-1}\mathbb{E}^{\bar{b}}[I_{\{H_b < H_s\}} g(\widehat{h}_{H_b})] \\
&= K^{-1}\mathbb{E}^{\bar{b}}[g(\widehat{h}_{H_s \wedge H_b})].
\end{aligned}
$$

Thus $b = \inf\{x \in B^*, x > s\}$. The remainder of the proof is very similar to the proof of Proposition 1: defining $\tilde{g}(x) = 0$ for $x \geq b$ and $\tilde{g}(x) = -\mathbb{E}^x(\int_0^{H_b} \widehat{h}_t \, dt + c)$ for $x < b$, we can show that $s = \sup\{x \in S^*, x < b\}$ where $S^*$ is the optimal stopping set for the stopping problem: $\max \mathbb{E}^x \tilde{g}(\widehat{h}_{H_s \wedge H_b})$. These are just the same stopping problems met in Proposition 1 but with $\rho$ replaced with 0. Thus the points $s$ and $b$ satisfy (2.2)–(2.5) with $\rho = 0$, and the result follows.

□

# 4 Finite statespace models

We consider the model

$$dS_t = dB_t + h(X_t)\,dt$$

where $X_t$ is an irreducible Markov Chain on $\{1, \dots, n\}$ with $Q$-matrix $Q$, and independent of $B$. In this section we will consider the problem of finding a transaction cost above which it is optimal never to buy the asset, in both the discounted and average gain versions of the optimal trading problem. At the end of the section we will consider a simple 2-state model.

Letting $p_t(i) = \mathbb{P}(X_t = i | \mathcal{Y}_t)$, where $\mathcal{Y}$ is as usual the filtration generated by the asset price process, we have (see Section VI.11 of Rogers & Williams (1987))

$$dp_t(i) = p_t(i)(h(i) - \widehat{h}_t)\,dN_t + (Q^\top p_t)(i)\,dt. \tag{4.1}$$

From Section 2 the optimal strategy, which buys when $p_t \in B$ and sells when $p_t \in S$, gives rise to a function $w$ on $\{x_1, \dots, x_n : \sum_i x_i = 1, x_i \geq 0\}$, such that $(\mathcal{G} - \rho)w = -\widehat{h}$. Here $\mathcal{G}$ is the generator of $p_t$ and the function $\widehat{h}$ is defined by $\widehat{h}(x) = \sum_i x_i h(i)$, with boundary conditions $w = +\frac{1}{2}c$ in $B$, $w = -\frac{1}{2}c$ in $S$, and $\sum_i x_i(h(i) - \widehat{h})\partial w/\partial x_i =$ on $\partial B$ and $\partial S$.

Define $\psi(i) = \mathbb{E}(\int_0^\infty e^{-\rho t} h(X_t)\,dt | X_0 = i)$. Note that $(\rho - Q)\psi = h$ so $(\mathcal{G} - \rho)\widehat{\psi} = -\widehat{h}$. Suppose we are about to buy at time 0 and subsequently sell at the random time $\tau$. This will not be an optimal decision if $\mathbb{E}(\int_0^\tau e^{-\rho t} h(X_t)\,dt - \frac{1}{2}c(1 + e^{-\rho\tau})) < 0$. Note that $\mathbb{E}(\int_0^\tau e^{-\rho t} h(X_t)\,dt - \frac{1}{2}c(1 + e^{-\rho\tau})) = \widehat{\psi}_0 - \frac{1}{2}c - \mathbb{E}e^{-\rho\tau}(\widehat{\psi}_\tau + \frac{1}{2}c)$ using the Strong Markov property at $\tau$.

**Proposition 4.1** *Define $\psi^M = \max_i \psi(i)$, $\psi^m = \min_i \psi(i)$. Then, i) if $\psi^M > -\psi^m$ and $p_t$ is irreducible on $\{x_1, \dots, x_n : \sum x_i = 1, x_i > 0\}$ (with respect to Lebesgue measure), it is optimal never to buy the asset if $c > 2\psi^M$, and to buy at some point with probability one if $c < 2\psi^M$; ii) if $\psi^M \leq -\psi^m$, a sufficient condition to ensure that it is optimal never to buy the asset is that $c > \psi^M - \psi^m$.*

**Proof** i) If $\psi^M > -\psi^m$ and $c > 2\psi^M$, then $\widehat{\psi}_\tau + \frac{1}{2}c > 0$ and $\widehat{\psi}_0 - \frac{1}{2}c < 0$, so $\widehat{\psi}_0 - \frac{1}{2}c - \mathbb{E}e^{-\rho\tau}(\widehat{\psi}_\tau + \frac{1}{2}c) < 0$. If $c < 2\psi^M$, choose $k$ such that $\frac{1}{2}c < k < \psi^M$, buy when $\widehat{\psi} \geq k$, which happens with probability one, and sell at some sufficiently large deterministic time $\tau$.

ii) If $\psi^M \le -\psi^m$ then $\psi^M + \psi^m \le 0$. Write $2c = \psi^M - \psi^m + 2c'$, with $c' > 0$. Now

$$\begin{aligned}
\widehat{\psi}_0 - \tfrac{1}{2}c - \mathbb{E}e^{-\rho\tau}(\widehat{\psi}_\tau + \tfrac{1}{2}c) &\le \psi^M - \tfrac{1}{2}c - \mathbb{E}e^{-\rho\tau}(\psi^m + \tfrac{1}{2}c) \\
&= \tfrac{1}{2}(\psi^m + \psi^M - 2c') - \mathbb{E}e^{-\rho\tau}\tfrac{1}{2}(\psi^m + \psi^M + 2c') \\
&= \tfrac{1}{2}(\psi^m + \psi^M)(1 - \mathbb{E}e^{-\rho\tau}) - c'(1 + \mathbb{E}e^{-\rho\tau}) \\
&< 0
\end{aligned}$$

□

**Remark** In Proposition 4.1 we ignore the case where $\psi^M = \psi^m$ since then $h = (\rho - Q)\psi$ is constant and the asset has constant drift.

**Average gain case.** We now suppose that we are considering buying at time 0 and subsequently selling at time $\tau$, which we will assume satisfies $\mathbb{E}(\tau) < \infty$.

Define $\psi^\rho(i) = \mathbb{E}(\int_0^\infty e^{-\rho t} h(X_t)\, dt | X_0 = i)$ (the function $\psi$ of the previous section) and let $\pi$ be the stationary distribution of $X$. Note that as $\rho \to 0$, $\psi^\rho(i) - \pi \cdot h/\rho$ converges, since $\psi^\rho(i) - \pi \cdot h/\rho = \int_0^\infty (e^{-\rho t}(\mathbb{E}(h(X_t)|X_0 = i) - \pi \cdot h)\, dt)$ and $\mathbb{E}(h(X_t)|X_0 = i) - \pi \cdot h$ converges to 0 exponentially fast as $t \to \infty$. Let $\tilde{\psi}^\rho(i) = \psi^\rho(i) - \pi \cdot h/\rho$ and $\tilde{\psi} = \lim_{\rho\to 0} \tilde{\psi}^\rho$, so that $(\rho - Q)\tilde{\psi}^\rho = h - \pi \cdot h$. Now using $\mathbb{E}(\tau) < \infty$ we have

$$\begin{aligned}
\mathbb{E}(S_\tau - S_0) &= \lim_{\rho\to 0} \mathbb{E}\left[\int_0^\infty e^{-\rho t} h(X_t)\, dt - \tfrac{1}{2}c(1 + e^{-\rho\tau})\right] \\
&= \lim_{\rho\to 0} (\widehat{\psi^\rho}_0 - \tfrac{1}{2}c - \mathbb{E}[e^{-\rho\tau}(\widehat{\psi^\rho}_\tau + \tfrac{1}{2}c)]) \\
&= \widehat{\tilde{\psi}}_0 - \mathbb{E}(\widehat{\tilde{\psi}}_\tau) - c + (\pi \cdot h)\mathbb{E}(1 - e^{-\rho\tau})/\rho \\
&= \widehat{\tilde{\psi}}_0 - \mathbb{E}(\widehat{\tilde{\psi}}_\tau) - c + (\pi \cdot h)\mathbb{E}(\tau)
\end{aligned}$$

Thus if $\pi \cdot h > 0$ it is always optimal to buy at some point, and we can ensure a positive expected profit by simply selling at a sufficiently large deterministic future time. If $\pi \cdot h < 0$, a sufficient condition to ensure it is optimal never to buy the asset is $c > \max_i \tilde{\psi}(i) - \min_i \tilde{\psi}(i)$. If $\pi \cdot h = 0$ it is optimal to buy the asset at some point if and only if $c < c^* = \max_i \tilde{\psi}(i) - \min_i \tilde{\psi}(i)$. In the case where $\pi \cdot h = 0$ note that this amounts to the assumption that the asset has no long-term drift; in this case we also have $-Q\tilde{\psi} = h$ and thus $\mathcal{G}\widehat{\tilde{\psi}} = -\widehat{h}$. These both still hold if we add a constant to $\tilde{\psi}$, so without loss of generality assume $\min_i \tilde{\psi}(i) = -\frac{1}{2}c$. We can now write down an explicit solution to (2.2)–(2.5) in the case $c = c^*$ by setting $w(p) = \widehat{\tilde{\psi}}$.

**Exmaple** We consider a model for the asset price with the form

$$dS_t = dB_t + X_t\, dt$$

where $X$ is a Markov chain on $\{-1, +1\}$, independent of the Brownian motion $B$, so $S_t$ alternates between being a Brownian motion with drift $+1$ and a Brownian motion with drift $-1$. We assume that $X$ jumps between $\pm 1$ at a rate $q > \frac{1}{2}\rho$. This model fits into the previous framework through the choice $h(x) = x$. Since $\widehat{h}_t = \mathbb{E}(h(X_t)|\mathcal{Y}_t)$, from (4.1) we have

$$d\widehat{h}_t = (1 - \widehat{h}_t^2)\, dN_t - 2q\widehat{h}_t\, dt.$$

This is of the form considered in Section 2. Here the scale function $s(x)$ satisfies $s'(x) = \exp(4q \int^x u/(1-u^2)^2\, du) = e^{2q/(1-u^2)}$. Thus, using results from Section 2, the optimal strategy is to buy when $\widehat{h}_t \geq b$ and to sell when $\widehat{h}_t \leq -b$, where $b$ satisfies

$$b - qc = e^{-2q/(1-b^2)} \int_0^b e^{2q/(1-u^2)}\, du.$$

This equation has a unique positive solution provided $c < 1/q$. Note that $\psi(1) = \rho/((q+\rho)^2 - \rho^2) = (2q-\rho)^{-1}$ and $\psi(-1) = -\rho/((q+\rho)^2 - \rho^2) = -(2q-\rho)^{-1}$. If $c > 1/q$ then for $\rho$ sufficiently small, $c > \psi^M - \psi^m$ and it is optimal never to buy the asset. If $c = 1/q$, then $\psi^M > \frac{1}{2}c$ and it is optimal to buy when $\widehat{\psi}$ hits a level $b(\rho)$ where $b(\rho) \to 1$ as $\rho \to 0$.

# 5 Reversion to a moving level

In this section we assume the asset price obeys the SDE

$$dS_t = dB_t + h(X_t)\, dt,$$

where $h(x) = -\gamma x$, $\gamma > 0$. The Markov process $X$ is defined by

$$X_t = S_t - \sigma B'_t,$$

where $B'$ is a Brownian motion independent of $B$, $\sigma \geq 0$ is known, and, conditional on $\mathcal{Y}_0$, $X_0$ is Gaussian with mean $\widehat{x}_0$ and variance $v_0$. This model is very similar to that considered in Mandarino (1990).

Since the process $(S_t, X_t)$ is jointly Gaussian, the conditional distribution of $X_t$ given $\mathcal{Y}_t$ is also Gaussian and we need only consider the evolution of the conditional mean, $\widehat{x}_t$ and variance $v_t$ of $X_t$. Introduce the notation $\widehat{f}_t = \mathbb{E}(f(X_t)|\mathcal{Y}_t)$ for an arbitrary function $f$. Since $X$ is a diffusion, the evolution of $\widehat{f}_t$ is given by (see Section VI.8 of Rogers & Williams (1987))

$$\widehat{f}_t = \widehat{f}_0 + \int_0^t (\widehat{fh}_u - \widehat{f}_u\widehat{h}_u + \widehat{\alpha}_u, dN_u) + \int_0^t \widehat{\mathcal{G}f}_u\, du, \tag{5.1}$$

where $\mathcal{G}$ is the generator of $X$ as usual. The process $\alpha_t$ is defined by $d\alpha_t = d[B, M]_t$, where $M_t$ denotes the $\mathcal{F}$-martingale $f(X_t) - \int_0^t \mathcal{G}f_u\, du$.

Applying (5.1) to the function $f(x) = x$, and using that fact that $dX_t = dB_t - \gamma X_t\, dt - \sigma\, dB'_t$, we obtain (supressing some indicies)

$$\begin{aligned} d\widehat{x} &= (-\gamma\widehat{x^2} + \gamma\widehat{x}^2 + 1)\, dN_t - \gamma\widehat{x}\, dt \\ &= (1 - \gamma v_t)\, dN_t - \gamma\widehat{x}_t\, dt. \end{aligned}$$

Similarly, applying (5.1) to the function $f(x) = x^2$, giving

$$d\widehat{x^2} = (-\gamma\widehat{x^3} + \gamma\widehat{x^2}\widehat{x} + 2\widehat{x})\, dN_t + (-2\gamma\widehat{x^2} + 1 + \sigma^2)\, dt.$$

Thus, since $dv_t = d\widehat{x^2} - d\widehat{x}^2$,

$$dv_t = (-\gamma\widehat{x^3} + 3\gamma\widehat{x^2}\widehat{x} - 2\gamma\widehat{x}^2)\, dN_t + (-2\gamma\widehat{x^2} + 1 + \sigma^2 + 2\gamma\widehat{x}^2 - (1 - \gamma v_t)^2)\, dt.$$

The second term simplifies to $(\sigma^2 - \gamma^2 v_t^2)\, dt$, and as $X_t$ given $\mathcal{Y}_t$ is Gaussian, $\widehat{x^3} - 3\widehat{x}\widehat{x^2} + 2\widehat{x}^3 = 0$. Thus the conditional distribution of $X_t$ evolves according to

$$\begin{aligned} d\widehat{x}_t &= (1 - \gamma v_t)\, dN_t - \gamma\widehat{x}_t\, dt \\ dv_t &= (\sigma^2 - \gamma^2 v_t^2)\, dt. \end{aligned}$$

If we are in the steady state, with $v_t \equiv \sigma/\gamma$, we have

$$d\widehat{h}_t = -\gamma(1 - \sigma)\, dN_t - \gamma\widehat{h}_t\, dt.$$

This OU form for $\widehat{h}_t$ has already been considered in Section 2, and the optimal strategy is: buy when $\widehat{h}_t$ hits $b$ and sell when $\widehat{h}_t$ hits $-b$, where $b$ is the unique positive solution to

$$2b - \gamma c = 2e^{b^2/(\gamma|1-\sigma|^2)} \int_0^b e^{u^2/(\gamma|1-\sigma|^2)}\, du.$$

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