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Global Fund Management Using Stochastic Optimization

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Abstract

We apply a stochastic optimization framework to find the optimal investment policy for a global fund in a discrete time and space setting. This multiperiod framework incorporates realistic constraints and market frictions such as portfolio restrictions and transaction costs and models the preferences of the fund using utility functions which only penalize downside performance. Two vector autoregressive models of asset returns, exchange rates and macroeconomic variables are presented for the purpose of generating scenarios for the optimization problem. Scenario tree generation is discussed and optimization based methods of generating moment matching and arbitrage free scenario trees are given. Comparisons to alternative methods of portfolio management are given and a historical backtest is presented to judge how the methodology would have done in practice.

Keywords

Stochastic Optimization; Global Fund Management; Multiperiod Framework; Vector Autoregressive Models; Moment Matching Arbitrage Free Scenario Tree; Historical Backtest

1 Introduction

The management of the portfolio of a fund with global investments is a complex problem. A practical method should have the flexibility to accommodate different attitudes to risk and different preferences and should take into account multiple periods, market frictions such as transaction costs and portfolio constraints such as position limits. In addition, it should be based on a realistic representation of the stochastics of the relevant factors such as the asset prices and exchange rates.

Many of the existing methods of solving global fund management problems such as *mean-variance analysis* have serious short comings in this regard. In particular, these methods are usually one period (static), have inflexible attitudes to risk and preferences and ignore market frictions.

In this paper we apply *stochastic optimization* to solve the global fund management problem and show that it meets all the requirements of a practical method described above. Like many numerical methods, this approach has benefited from the advance of computing technology and the development of effective algorithms (see e.g. Gassmann [10], Dempster and Thompson [7] and Scott [21]). This is testified to by the growing body of literature concerning the application of stochastic optimization to fund management problems. This body includes Consigli and Dempster [4], Geyer et al. [11] and Dempster et al. [6] who develop models for pension fund, Carino et al. [3] who develop a model for an insurance company, Zenios [25] who develops a model for fixed income securities, and Berger and Mulvey [1] who develop a model for individual investors. The present paper complements this body by focusing on a global fund, emphasizing practical aspects such as scenario tree generation and making comparisons to alternative methods of portfolio management.

The rest of this paper is organized as follows. In Section 2 we describe the stochastic optimization framework which includes the problem set up, possible objective functions and constraints, solution methods and a method of implementing the approach in practice. Section 3 presents two vector autoregressive models of assets, exchange rates and macroeconomic variables used to generate scenarios for the fund management problems. In Section 4 we derive and test methods of scenario tree generation which match moments of the variables in the sampled scenarios to those of a given distribution and derive a method for generating arbitrage free scenario trees. In Section 5 we discuss alternative methods of portfolio management and compare them to the stochastic optimization based methodology. Section 6 presents a historical backtest to show how the framework would have done had it been implemented in practice, and Section 7 concludes. Throughout this paper boldface is used to denote random entities.

2 Stochastic Optimization Framework

In this section we describe the framework for solving global fund management problems using stochastic optimization. We focus on what is normally called strategic asset allocation, which is concerned with allocation across broad asset classes such as stocks and bonds of a given country.

2.1 Set Up

We consider a discrete time and space setting. It is assumed that the fund operates from the view point of one currency which we call the *home currency.* Unless otherwise mentioned, all quantities are assumed to be in local currencies. There are $T+1$ times indexed by $t = 1, \ldots, T+1$ where $T+1$ corresponds to the *planning horizon.* Uncertainty is represented by a finite set of *states* or *scenarios* of the world denoted by Ω. Let $pr(\omega)$ denote the probability of state ω for $\omega \in \Omega$. Let I denote the set of assets which take the form of stock, bond and cash indices. The fund begins with an *initial endowment* of assets given by $\{x_i : i \in I\}$. The fund trades at $t = 1, \ldots, T$, i.e. at all times except for the planning horizon. A *trading strategy* is given by $\{(x_{it}(\omega), x_{it}^+(\omega), x_{it}^-(\omega)) : i \in I, t = 1, \ldots, T, \omega \in \Omega\}$ where:

- $x_{it}(\omega)$ denotes the amount held of asset i between time t and time $t+1$ in state ω
- $x_{it}^+(\omega)/x_{it}^-(\omega)$ denotes the amount bought/sold of asset i at time t in state ω. The introduction of the buy/sell variables is used to account for proportional transaction costs. Denote by f and g the *proportional transaction cost* of buying and selling assets respectively.

The *asset returns* are given by $\{v_{it}(\omega) : i \in I, t = 2, \ldots, T+1, \omega \in \Omega\}$ where:

- $v_{it}(\omega)$ denotes the net return on asset i between time $t-1$ and time t in state ω.

The *exchange rates* expressed as $\dfrac{\text{home currency}}{\text{local currency}}$ are given by $\{p_{it}(\omega) : i \in I, t = 1, \ldots, T+1, \omega \in \Omega\}$ where:

- $p_{it}(\omega)$ denotes the exchange rate of asset i at time t in state ω.

A trading strategy results in a *wealth before rebalancing* of $w_t(\omega)$ for $t = 2, \ldots, T+1$ and $\omega \in \Omega$, and a *wealth after rebalancing* of $W_t(\omega)$ for $t = 1, \ldots, T$ and $\omega \in \Omega$. The difference in these two quantities reflects the transaction costs.

2.2 Objective Functions

Subject to the constraints, the fund acts by choosing the trading strategy which maximizes the *expected utility of the wealth process* which is assumed to take the form:

$$E[U(\boldsymbol{w_2},\ldots,\boldsymbol{w_{T+1}})] = \sum_{\omega\in\Omega} pr(\omega) \sum_{t=2}^{T+1} u_t(w_t(\omega)), \tag{1}$$

where $u_t : \mathbb{R} \to \mathbb{R}$ is the *period utility function*. u_t defines the preferences of the fund over the wealth at time t and should thus have the standard properties of being strictly increasing (more wealth is preferred to less) and concave (risk averse). The period utility functions considered in this paper are:

- *downside-quadratic*: $u_t(w_t) = \gamma_{1t}w_t - \gamma_{2t}(w_t - \tilde{w}_t)^2_-; \gamma_{1t}, \gamma_{2t}, \tilde{w}_t \geq 0$
- *downside-linear*: $u_t(w_t) = \gamma_{1t}w_t - \gamma_{2t}(w_t - \tilde{w}_t)_-; \gamma_{1t}, \gamma_{2t}, \tilde{w}_t \geq 0$,

where $(w_t - \tilde{w}_t)_- = max(0, \tilde{w}_t - w_t)$. Note that both downside utility functions reduce to the *linear* utility function ($u_t(w_t) = w_t$) for $\gamma_{1t} = 1$ and $\gamma_{2t} = 0$.

The $\tilde{w}_t$ parameter denotes a *target wealth* so that both downside utility functions aim to maximize wealth and at the same time penalize downside deviations of the wealth from the target.

2.3 Constraints

The constraints in this framework can be classified into two types. The first type of constraints are called *mandatory constraints* and appear in every problem. These are:

- *Cash balance constraints (financing constraints)*. These ensure that the net flow of cash at each time and in each state is zero

$$\sum_{i\in I} p_{it}(\omega)(gx^-_{it}(\omega) - fx^+_{it}(\omega)) = 0; t = 1,\ldots,T, \omega\in\Omega. \tag{2}$$

- *Inventory balance constraints.* These give the position in each asset at each time and in each state

$$
\begin{aligned}
x_{i1}(\omega) &= x_i + x_{i1}^{+}(\omega) - x_{i1}^{-}(\omega); i \in I, \omega \in \Omega \qquad (3)\\
x_{it}(\omega) &= x_{it-1}(\omega)(1 + v_{it}(\omega)) + x_{it}^{+}(\omega) - x_{it}^{-}(\omega); i \in I, t = 2, \ldots, T, \omega \in \Omega. \qquad (4)
\end{aligned}
$$

- *Nonnegativity constraints.* These constrain the buy and sell variables to be nonnegative

$$x_{it}^{+}(\omega), x_{it}^{-}(\omega) \geq 0; i \in I, t = 1, \ldots, T, \omega \in \Omega. \qquad (5)$$

The second type of constraints are called *regulatory and performance constraints* and are problem specific. The regulatory and performance constraints considered in this paper are:

- *Solvency constraints.* These constrain the wealth of the fund generated by the trading strategy to be nonnegative at each time

$$w_t(\omega) \geq 0; t = 2, \ldots, T + 1, \omega \in \Omega. \qquad (6)$$

- *Short sale limits.* These limit the amount the trading strategy can short assets

$$p_{it}(\omega)x_{it}(\omega) \geq \bar{x}_i; i \in I, t = 1, \ldots, T, \omega \in \Omega. \qquad (7)$$

- *Position limits.* These limit the amount invested in assets to be less than some proportion of the fund wealth

$$p_{it}(\omega)x_{it}(\omega) \leq \beta_i W_t(\omega); i \in I, t = 1, \ldots, T, \omega \in \Omega. \qquad (8)$$

- *Turnover constraints.* These limit the change in the amount invested in assets from one time to the next to be less than some proportion of the fund wealth

$$|p_{it}(\omega)x_{it}(\omega) - p_{it-1}(\omega)x_{it-1}(\omega)| \leq \alpha_i W_t(\omega); i \in I, t = 1, \ldots, T, \omega \in \Omega. \tag{9}$$

All the above constraints are piecewise linear.

2.4 Mathematical Formulation

The objective (1) with one of the period utility functions given in Section 2.2 along with the mandatory constraints (2) - (5) and some subset of the regulatory and performance constraints (6) - (9) constitute the mathematical formulation of the global fund management problem in *deterministic equivalent form.* It is a convex but possibly nonlinear *dynamic stochastic program* (DSP) and is closely related to the CALM model of Dempster [5].

2.5 Scenario Tree Representation of Ω

The uncertainty (Ω) can be represented in the form of a *scenario tree* in which each path through the tree corresponds to a scenario $\omega \in \Omega$ and each *node* in the tree corresponds to a time along one or more scenarios. An example scenario tree is given in Figure 1 for $T + 1 = 3$ and $|\Omega| = 4$.

We call the number of branches emanating from each node in the scenario tree the *branching factor* of the tree. Thus the branching factor of the scenario tree in Figure 1 is 2.

The stochastic optimization approach has the advantage of being able to accommodate a wide range of stochastic models of the underlying factors. This is because the scenario tree is usually generated using Monte Carlo methods so all that is requried for a stochastic

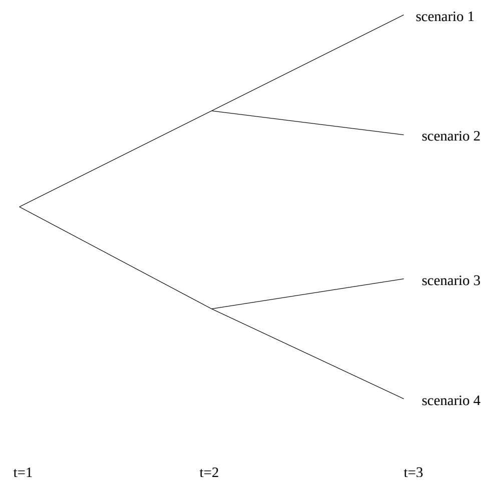

Figure 1: Example scenario tree

model to be used in this framework is that it can be simulated. In this paper we consider vector autoregressive models of the underlying factors.

2.6 Problem Generation and Solution Methods

The fund management DSP is formulated in the deterministic equivalent form of Section 2.4 using a modeling language and generated in a standard mathematical programming format. The method employed in solving the generated DSP will depend on the choice of period utility function. The downside-quadratic period utility problems can be solved with *interior point methods* (see e.g. Wright [24]), and the downside-linear period utility problems can be solved with either interior point or the *simplex method* (see e.g. Luenberger [17]).

2.7 Implementation

In practice a separate DSP is solved at each trading time and only the first stage solution is implemented. This approach is depicted in Figure 2 for a problem with two trading times.

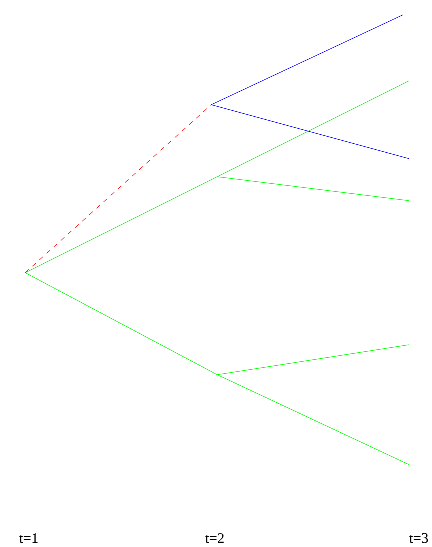

Figure 2: Implementation of framework

Here the dotted red line corresponds to the realized path of the variables and the green and blue scenario trees correspond to the $t = 1$ and $t = 2$ scenario trees respectively. There are several reasons for implementing the approach in this manner. The first is that, as depicted schematically in Figure 2, the actual values of the variables at $t = 2$ are unlikely to coincide with the values of the variables in any of the simulated scenarios of the $t = 1$ scenario tree. If this were to be the case then the optimal investment policy would be undefined. A second reason is that the asset return and exchange rate model's parameters can be updated at each trading time. Since this model is only an approximation to the real stochastics, using the information contained in the most recent history can improve the scenario simulation.

3 Vector Autoregressive Models of Assets and Exchange Rates

In this section we present two *vector autoregressive* (VAR) models of assets and exchange rates which we use to generate scenario trees for the global fund management problem. The first, VARSIM 2.1, is a model of global assets and exchange rates. The second, USMACRO, is a model of the US macroeconomy. In developing the models emphasis was placed on (covariance or weak) stationarity and reasonable first and second moments. Conditions for stationarity and formulas for the unconditional mean and conditional variance of a VAR are given in Appendix A. A more detailed description of the development of these models can be found in Villaverde [22]. Other models of asset returns and exchange rates can be found in Wilkie [23], Mulvey and Thorlacius [19], Duval et al. [9] and Dempster et al. [6].

Both VARSIM 2.1 and USMACRO are VAR(3)s and can be expressed as:

$$\boldsymbol{y_t} = \mu + \sum_{i=1}^{3} \phi_i y_{t-i} + \boldsymbol{\eta_t}, \tag{10}$$

where $\boldsymbol{y_t}$ are the variable net returns at time t, μ is a vector of constants, ϕ_i is the lag i coefficient matrix and the $\boldsymbol{\eta_t}$ are distributed $N(0, \Sigma)$ and are uncorrelated across time.

The models are estimated using monthly data, and each variable in the VAR is estimated separately using ordinary least squares with backwards variable selection and a 95% confidence level. The $\boldsymbol{\epsilon_t}$ terms below refer to standard normal random variables that are correlated across variables but uncorrelated across time.

3.1 VARSIM 2.1

VARSIM 2.1 is a monthly model of EU stock, EU cash, EU bond, US stock, US/EU foreign exchange (fx), JP stock and JP/EU fx. Variables in this model are only allowed to depend on other variables in its country (for EU variables this includes the US/EU exchange rate). Let:

- $rseu$ - return on EU stock
- $rceu$ - return on EU cash
- $rbeu$ - return on EU bond
- $rsus$ - return on US stock
- $rxus$ - return on US fx
- $rsjp$ - return on JP stock
- $rxjp$ - return on JP fx.

The model estimated from February 1988 to February 2001 is given by:

$$\boldsymbol{rseu_t} = .0130 + .0461\boldsymbol{\epsilon_t^{rseu}} \tag{11}$$

$$\boldsymbol{rceu_t} = .0003 + .2190 rceu_{t-1} + .3741 rceu_{t-2} + .3749 rceu_{t-3} \tag{12}$$

$$-.0243 rbeu_{t-2} + .0045 rxus_{t-1} + .0007\boldsymbol{\epsilon_t^{rceu}} \tag{13}$$

$$\boldsymbol{rbeu_t} = .9413 rceu_{t-1} + .1607 rbeu_{t-1} + .0100\boldsymbol{\epsilon_t^{rbeu}} \tag{14}$$

$$\boldsymbol{rsus_t} = .0106 + .0397\boldsymbol{\epsilon_t^{rsus}} \tag{15}$$

$$\boldsymbol{rxus_t} = .0310\boldsymbol{\epsilon_t^{rceu}} \tag{16}$$

$$\boldsymbol{rsjp_t} = .0596\boldsymbol{\epsilon_t^{rsjp}} \tag{17}$$

$$\boldsymbol{rxjp_t} = .0348\boldsymbol{\epsilon_t^{rxjp}}. \tag{18}$$

This model is stationary with unconditional means and conditional standard deviations and correlations given in Tables 1 and 2.

Variable	**Mean**	**St. Dev.**
rseu	0.0130	0.0461
rceu	0.0051	0.0007
rbeu	0.0057	0.0100
rsus	0.0106	0.0397
rxus	0.0000	0.0310
rsjp	0.0000	0.0596
rxjp	0.0000	0.0348

Table 1: VARSIM 2.1 means and standard deviations

3.2 USMACRO

USMACRO is a monthly model for US stock, US cash, US bond, US/EU exchange rate, US short term interest rate, US money supply, US CPI and US GDP. Let:

- $rcus$ - return on US cash
- $rbus$ - return on US bond
- $rrus$ - return on US interest rate
- $rmus$ - return on US money supply
- $rpus$ - return on US CPI
- $ryus$ - return on US GDP.

Correlations							
	rseu	rceu	rbeu	rsus	rxus	rsjp	rxjp
rseu	1.00	0.08	0.30	0.68	-0.43	0.47	-0.29
rceu		1.00	0.26	0.04	-0.01	-0.07	-0.05
rbeu			1.00	0.23	-0.08	0.05	-0.11
rsus				1.00	-0.11	0.36	-0.15
rxus					1.00	-0.06	0.45
rsjp						1.00	-0.14
rxjp							1.00

Table 2: VARSIM 2.1 correlations

The model estimated from February 1988 to February 2001 is given by:

$$
\begin{aligned}
\boldsymbol{rsus_t} &= 2.3484 ryus_{t-2} + .0398 \boldsymbol{\epsilon_t^{rsus}} && (19)\\
\boldsymbol{rcus_t} &= .0006 rsus_{t-3} + .9841 rcus_{t-1} - .0017 rbus_{t-1} - .0021 rbus_{t-2} && (20)\\
&\quad +.0021 rrus_{t-1} + .0014 rrus_{t-2} + .0138 rpus_{t-3} + .0108 ryus_{t-2} && (21)\\
&\quad +.00009 \boldsymbol{\epsilon_t^{rcus}} && (22)\\
\boldsymbol{rbus_t} &= -.1130 rsus_{t-1} + 1.8775 rcus_{t-1} + .2844 rbus_{t-1} - .9658 rpus_{t-1} && (23)\\
&\quad +.0106 \boldsymbol{\epsilon_t^{rbus}} && (24)\\
\boldsymbol{rxus_t} &= -.2487 rsus_{t-1} + .8102 rbus_{t-1} - .5631 rmus_{t-3} + .0290 \boldsymbol{\epsilon_t^{rxus}} && (25)\\
\boldsymbol{rrus_t} &= .2522 rsus_{t-1} - 1.2235 rbus_{t-1} + .0816 rrus_{t-1} + .2392 rrus_{t-3} && (26)\\
&\quad +.8467 rmus_{t-3} + .0407 \boldsymbol{\epsilon_t^{rrus}} && (27)\\
\boldsymbol{rmus_t} &= .0089 - 1.1602 rcus_{t-1} + 1.0926 rpus_{t-2} + .0059 \boldsymbol{\epsilon_t^{rmus}} && (28)\\
\boldsymbol{rpus_t} &= .4629 rcus_{t-1} - .0301 rbus_{t-1} - .0373 rbus_{t-2} - .0126 rrus_{t-1} && (29)\\
&\quad +.0836 rmus_{t-3} + .1517 rpus_{t-1} + .0017 \boldsymbol{\epsilon_t^{rpus}} && (30)\\
\boldsymbol{ryus_t} &= .0016 + .8140 ryus_{t-1} - .1859 ryus_{t-3} + .0011 \boldsymbol{\epsilon_t^{ryus}}. && (31)
\end{aligned}
$$

This model is stationary with unconditional means and conditional standard deviations and correlations given in the Tables 3 and 4.

4 Scenario Tree Generation

The generation of the scenario tree for the fund management problem is a crucial step in the stochastic optimization approach. Since the resulting solution is based on the representation of uncertainty as given by the scenario tree, the usefulness of the entire framework critically depends on how well the scenario tree is able to approximate reality. While there seems to be no optimal method of scenario tree generation, the goal of any proce-

Variable	**Mean**	**St. Dev.**
rsus	0.0105	0.0398
rcus	0.0043	0.0009
rbus	0.0062	0.0106
rxus	-0.0012	0.0290
rrus	0.0008	0.0407
rmus	0.0066	0.0059
rpus	0.0025	0.0017
ryus	0.0044	0.0011

Table 3: USMACRO means and standard deviations

Correlations								
	rsus	rcus	rbus	rxus	rrus	rmus	rpus	ryus
rsus	1.00	-0.01	0.25	-0.17	-0.08	0.07	-0.21	0.03
rcus		1.00	-0.09	0.08	0.06	-0.08	0.11	0.16
rbus			1.00	0.00	-0.39	0.00	-0.06	-0.24
rxus				1.00	-0.15	0.14	0.00	-0.10
rrus					1.00	0.04	-0.05	0.19
rmus						1.00	-0.40	-0.13
rpus							1.00	0.08
ryus								1.00

Table 4: USMACRO correlations

dure should be the best representation of the underlying uncertainty. Methods of scenario tree generation existing in the literature include random sampling (Bradley and Crane [2]), binary lattices (Zenios [25]), adjusted random sampling that matches the means and variances of the underlying processes (Carino et al. [3]) and a general method for satisfying specified statistical properties of the underlying processes (Hoyland and Wallace [15]).

In this section we first address the problem of suppressing sampling error in scenario trees using moment matching algorithms. We then present experiments which show that as compared to the benchmark method of random sampling, these methods lead to greater stability and accuracy. Here the *stability* of a problem refers to how the expected utility and first stage solution change with respect to the seed used to generate the scenario tree. The *accuracy* of a problem refers to how well the optimal expected utility and first stage solution approximate the solution obtained by solving the same model with a very large scenario tree. We then derive a general method for generating arbitrage free scenario trees and explain how it can be combined with the moment matching algorithms to generate arbitrage free moment matching scenario trees.

4.1 Moment Matching and Sampling Error

If the scenarios in the scenario tree are generated by randomly sampling from a given distribution using Monte Carlo methods there will always be *sampling error*, i.e. differences in the given distribution and the distribution of the sampled scenarios. These errors can lead to a representation of the uncertainty in the scenario tree that is very different than that implied by the given distribution, which in turn can result in errors in the solution of the resulting stochastic program.

Clearly the larger the branching factor the smaller the sampling error. However, as the computational burden grows with the branching factor, the problem becomes how to generate the smallest sampling error for a given branching factor.

The sampling error can be reduced by generating the scenario tree such that the moments of the variables in the sampled scenarios are matched to those of the given distribution. We refer to this process as *moment matching* and derive a general optimization based method of achieving this below.

Suppose under the given distribution a random vector $\mathbf{x}$ has a k^{th} marginal moment of μ^k and a covariance matrix of σ. Suppose the random vector $\mathbf{y}$ is generated by randomly sampling from this given distribution and is thus defined on the scenario tree. Let $E[\boldsymbol{y^k}]$ and $Cov[\boldsymbol{y}]$ denote the k^{th} marignal sample moment and sample covariance of $\boldsymbol{y}$ or the k^{th} marignal moment and covariance of $\boldsymbol{y}$ on the scenario tree. Because of sampling error, it will usually be the case that $\sigma \neq Cov[\boldsymbol{y}]$ and $\mu^k \neq E]\boldsymbol{y^k}]$ for any k. The idea of the proposed moment matching method is to find a new random vector on the scenario tree $\mathbf{z}$ that is close to $\mathbf{y}$ subject to moment matching conditions on $\mathbf{z}$. To match the first n marginal moments and the covariance we solve the following optimization problem:

$$min_{\mathbf{z}} E[||\mathbf{z} - \mathbf{y}||_p] \tag{32}$$

$$\text{s.t. } E[\boldsymbol{z^k}] = \mu^k; k = 1, \ldots, n \tag{33}$$

$$Cov[\mathbf{z}] = \sigma, \tag{34}$$

where $|| \cdot ||_p$ denotes the l_p norm. This problem finds a random vector $\mathbf{z}$ that is close to the original randomly sampled vector $\mathbf{y}$ with the first n marginal moments and covariance matched to the given distribution. It has nonlinear constraints but can be solved with sequential quadratic programming (see e.g. Gill [12]).

4.2 Stability

This section presents an experiment which compares the stability of problems using random sampling, mean matching and mean-covariance matching methods of scenario tree generation with VARSIM 2.1 [1]. For simplicity we consider a one stage problem with a planning horizon of one month. For each tree generation method and for branching factors ranging from 10 to 100 we generate 100 one stage scenario trees using different seeds. For each scenario tree we then solve a problem with an initial portfolio of 1 in cash in the home currency and a downside-linear ($\gamma_1 = 5, \gamma_2 = 1000, \tilde{w} = 1.01$) period utility function. We use .5% transaction costs and the realistic *T3 constraint structure* defined in Appendix B [2].

For each method and each branching factor the mean and standard deviation (with respect to seed) of the portfolio weights and expected utility are recorded. This allows us to determine the minimum branching factor needed for the problem to be stable where we consider a problem to be stable if the standard deviation of each asset weight is less than 0.10 and the standard deviation of the expected utility is less than 10% of its mean.

Table 5 gives that the minimum branching factors needed for stability for each scenario tree generation method. The smaller minimum branching factors for the moment matching methods given evidence that they lead to more stable problems.

4.3 Accuracy

This section addresses the accuracy of the problem of the previous section using the same three scenario tree generation methods. To address this issue we repeated the stability

[1] This instance of VARSIM 2.1 was estimated from February 1988 to February 2001. The initial conditions of the problem are those in February 2001.

[2] This constraint structure was suggested by Pioneer Investments.

Method	Minimum Branching Factor
Random Sampling	>100
Mean Matching	30
Mean-Covariance Matching	10

Table 5: Minimum branching factor needed for stability

experiment of the previous section with scenario trees generated using random sampling with larger and larger branching factors until the problem appeared to converge. We took convergence to occur at a branching factor of 10000 [3].

Table 6 gives the mean allocation for each of the three methods from the stability experiment of the previous section with a branching factor of 100 and the mean allocation using random sampling from a stability experiment with a branching factor of 10000. The fact that the moment matching allocations are closer to the allocation produced with 10000 branches gives evidence that they lead to more accurate problems.

	EUstock	EUcash	EUbond	USstock	JPstock
Random Sampling	.1124	.2500	.5866	.0471	.0039
Mean Matching	.1446	.2500	.5766	.0288	.0000
Mean-Covariance Matching	.1427	.2500	.5832	.0242	.0000
10000 Branch Problem	.1526	.2500	.5825	.0149	.0000

Table 6:

[3] At this branching factor all standard deviations were less than 0.10.

4.4 Arbitrage

An *arbitrage* is a trading strategy which is guaranteed not to lose money and is expected to make money. As such it is generally assumed that arbitrage opportunities do not exist. In this section we derive a method for generating one stage scenario trees which are arbitrage free. Multistage arbitrage free trees can then be generated by applying this algorithm at each node.

Consider a one stage problem with I assets and J possible states. Let S denote the vector of initial asset prices (in the home currency) and let D denote the matrix of terminal payoffs (in the home currency) where D_{ij} denotes the payoff of asset i in state j. A *state price vector* is a vector $\pi \in \mathbb{R}^J$ such that:

$$S = D\pi \tag{35}$$

$$\pi \gg 0 \tag{36}$$

where $\pi \gg 0$ denotes that each element of the vector π is strictly greater than 0. It is well known (see e.g. Duffie [8], Section 1A) that there is no arbitrage if and only if there is a state price vector. This suggests the following algorithm for generating an arbitrage free scenario tree. First generate an initial value of D, denoted by D^k, using random sampling. Then obtain a new value of D, denoted by D^f, by solving the following optimization problem:

$$\min_{\pi, D^f_{ij}: i\in I, j\in J} \sum_{i\in I} \sum_{j\in J} (D^k_{ij} - D^f_{ij})^2 \tag{37}$$

$$\text{s.t. } S = D^f \pi \tag{38}$$

$$\pi \geq \delta, \tag{39}$$

where δ is chosen to be a small strictly positive vector. This problem determines new terminal payoffs which are close to the original randomly generated payoffs (in the least squares sense) for which there is a state price vector. This problem has nonconvex constraints but can be solved to at least a local constrained optimum with sequential quadratic programming.

5 Comparison to Alternative Methods of Portfolio Management

In this section we first discuss the solution of one stage portfolio management problems using mean-variance analysis and compare this method to a one stage stochastic optimization approach. We then discuss the solution of multistage portfolio management problems using a sequence of one stage problems and empirically compare this method to a multistage stochastic optimization approach.

5.1 Comparison of One Stage Methods

The most common method used to solve one stage portfolio management problems is the mean-variance analysis pioneered by Markowitz [18]. The approach is characterized by maximizing a weighted combination of the two goals of having a high expected portfolio return and a low portfolio return variance, or equivalently by maximizing a utility function which is quadratic in the portfolio return. While mean-variance analysis is easy to implement, it has serious drawbacks.

A major deficiency of this approach is that it assumes that only the mean and variance of the portfolio return are important in determining optimality. If the portfolio return is normally distributed then it is completely characterized by its first two moments and the mean-variance analysis is sensible. However, if the portfolio return is not normal then other aspects of the return distribution such as the skewness and kurtosis may need to be considered to determine optimality. There is a large body of evidence which points to the fact that stock and bond returns are not normally distributed. This is particularly true for daily and high frequency data (see e.g. Kon [16]), but may also be true data at a monthly frequency (see e.g. Villaverde [22]). In addition, the inclusion of derivatives such as options would also tend to make the portfolio return nonnormal.

A second serious short coming of mean-variance analysis is that the canonical implementation ignores market frictions such as transaction costs and portfolio restrictions such as position limits. It has recently been extended to incorporate these frictions (see e.g. Horniman et al. [14]), but at the cost of increased complexity.

Lastly we note that mean-variance analysis is subsumed by a one stage stochastic optimization approach. Setting the period utility function equal to:

$$u_t(w_t) = \gamma_{1t} w_t - \gamma_{2t} w_t^2; \gamma_{1t}, \gamma_{2t} \geq 0 \tag{40}$$

and ignoring the constraints in Section 2.3 results in the mean-variance analysis.

5.2 Comparison of Multistage Methods

The most common method used to solve multistage portfolio management problems is to solve a *sequence of one stage problems.* This approach is depicted in Figure 3 for a problem with five rebalancing times.

Regardless of how each single stage problem is formulated there can be serious drawbacks to solving a multistage problem with a sequence of one stage problems. In general a series

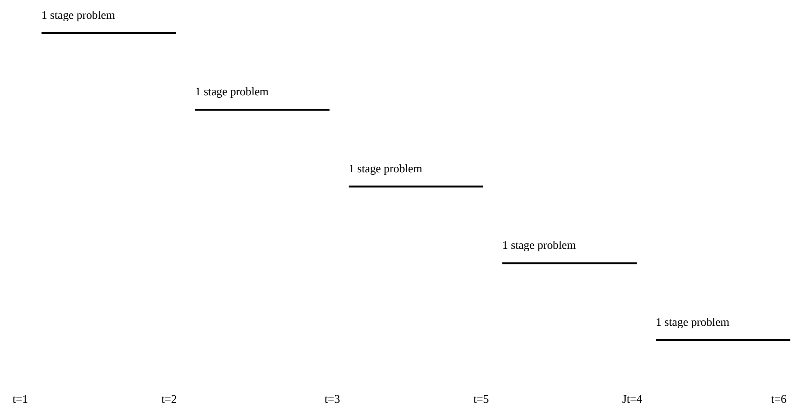

Figure 3: One stage formulation

of one stage problems may not be multistage optimal if there are aspects of the problem that link two or more stages together. Scherer [20] notes the following three conditions:

- *C1* - there are transaction costs
- *C2* - the utility function does not have constant relative risk aversion
- *C3* - asset returns are time dependent or serially correlated.

Below we make two comparisons of the series of one stage formulation (or rolloing one stage formulation) and a multistage formulation. In both cases a stochastic optimization approach is used. In the first comparison the two formulations are assessed based on the expected utility over the same scenarios used to generate the trading strategies. In other words, the scenario tree used to construct the problems is then used as a set of *test scenarios*. This test is meant to compare the two strategies for a given representation of uncertainty and implicitly assumes that the scenario tree sufficiently reflects the underlying uncertainty. In this comparison we first consider violations of conditions C1 - C3 individually and then a realistic case where all three conditions are violated at once.

In the second comparison the two formulations are assessed based on the expected utility over a set of historically bootstrapped test scenarios. In this case an underlying stochastic model is used to generate the scenario trees for the optimization problems, and a different historical bootstrap simulator is used to generate the test scenarios. In an ideal test the historically bootstrapped scenarios would be replaced by actual historical scenarios. Unfortunately, we do not have sufficient data to carry out the latter, but the comparison made here should give an indication of the sensitivity of the formulations with respect to differences in the true stochastics and those implied by the underlying stochastic model. In this comparison we consider the case where conditions C1 - C3 are violated all at once.

For simplicity we consider a two stage problem with monthly rebalancing. The only assets are US stock, US cash and US bond, and the initial portfolio is 1 in cash. All scenario trees are generated using a mean-covariance matching algorithm and a branching factor of 20 [4].

5.2.1 First Comparison

In this comparison we first generate a two stage scenario tree as in Figure 1. For the multistage formulation we solve the multistage problem and record the first stage portfolio and expected utility. For the rolling one stage formulation we first solve the t=1 one stage problem as depicted in Figure 4 and record the portfolio and expected utility. We then solve the one stage problems for the nodes at t=2 given the initial conditions implied by the t=1 portfolio and the asset returns in the appropriate state as depicted in Figure 5. The expected utilities of these problems are also recorded and the expected utility for the rolling one stage formulation can then be obtained from the expected utility of the individual one stage problems.

Tables 7, 8, 9, 10 and 11 give the t=1 portfolio weights and expected utilities of the

[4] The initial conditions of the problems are those in February 2001.

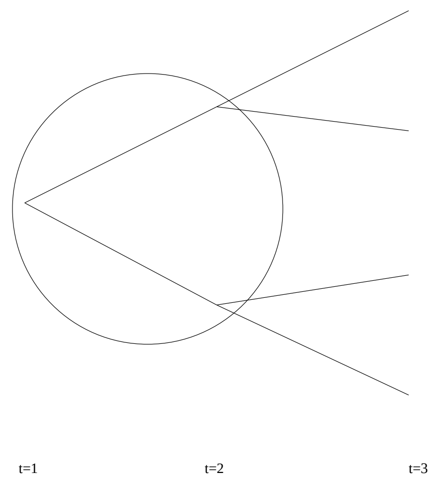

Figure 4: t=1 rolling one stage problem

two formulations for a number of experiments. In the first experiment (Table 7) none of the conditions (C1 - C3) that may make the rolling one stage formulation multistage suboptimal are violated. A linear period utility function is used, there are no transaction costs and the asset returns are assumed to be given by the following system of independent discretized geometric Brownian motions:

$$\boldsymbol{rsus_t} = .0123 + .0374\boldsymbol{\epsilon_t^{rsus}} \tag{41}$$

$$\boldsymbol{rcus_t} = .0042 + .00009\boldsymbol{\epsilon_t^{rcus}} \tag{42}$$

$$\boldsymbol{rbus_t} = .0067 + .0100\boldsymbol{\epsilon_t^{rbus}}, \tag{43}$$

where the $\boldsymbol{\epsilon}$ terms are uncorrelated. In this case the rolling one period formulation is multistage optimal.

In the second experiment (Table 8) condition C1 is violated by incorporating .05% proportional transaction costs on buying and selling all assets. In this case the rolling one period formulation does not see that the loss due to transaction costs is offset by the return on

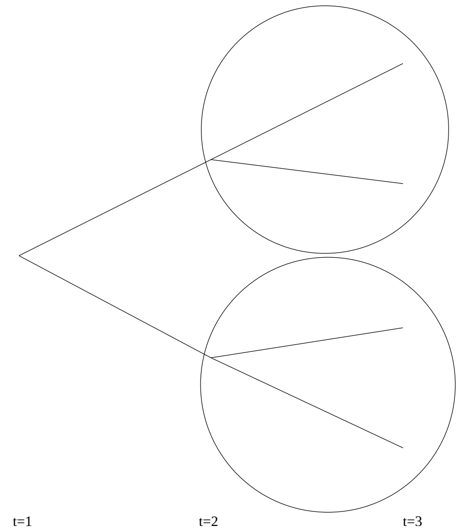

Figure 5: t=2 rolling one stage problems

stock if it is held for two stages and is therefore not multistage optimal.

In the third experiment (Table 9) condition C2 is violated by using a downside-linear ($\gamma_{1t} = 1, \gamma_{2t} = 1000, \tilde{w} = 1.0016^t$) period utility function. In this case the rolling one stage formulation invests too much in cash and is not multistage optimal.

In the fourth experiment (Table 10) condition C3 is violated by incorporating autocorrelated returns by setting the mean return on stock in the second stage to be the negative of the mean return on stock in the first stage. It also incorporates a 15% turnover constraints on all assets. In this case the rolling one stage formulation does not hedge against the negative return in stock in the second stage and its inability to rebalance significantly due to the turnover constraints and is not multistage optimal.

In the last experiment (Table 11) all three conditions (C1 - C3) are violated. This problem incorporates .05% transaction costs on all assets, 15% turnover constraints on all assets, 50% position limits on all assets, a downside-linear ($\gamma_{1t} = 1, \gamma_{2t} = 1000, \tilde{w} = 1.0016^t$) period utility function and asset returns generated by USMACRO [5]. Here the rolling one stage formulation invests too little in stock and is not multistage optimal. Compared to the previous experiments, the difference in expected utility between the two formulations is now more pronounced.

	stock	cash	bond	utility
multistage	1	0	0	2.0371
rolling one stage	1	0	0	2.0371

Table 7: No violations of conditions C1 - C3

	stock	cash	bond	utility
multistage	1	0	0	2.0168
rolling one stage	0	1	0	2.0126

Table 8: Condition C1 violated - transaction costs

	stock	cash	bond	utility
multistage	.03	.77	.20	2.0156
rolling one stage	.01	.91	.08	2.0145

Table 9: Condition C2 violated - downside utility function

[5] This instance of USMACRO was estimated from February 1988 to February 1999.

	stock	cash	bond	utility
multistage	.15	0	.85	2.0218
rolling one stage	1	0	0	2.0150

Table 10: Condition C3 violated - time dependent returns and turnover constraints

	stock	cash	bond	utility
multistage	.1	.5	.4	-6.2324
rolling one stage	.05	.5	.45	-7.4160

Table 11: All three conditions (C1 - C3) violated

5.2.2 Second Comparison

In this comparison we first generate 1000 two month *historically bootstrapped* test scenarios [6]. We then implement both formulations on each of the bootstrapped scenarios and record the realized utilities. Both formulations use USMACRO [7] as the underlying dynamic model, a downside-linear ($\gamma_{1t} = 1, \gamma_{2t} = 1000, \tilde{w} = 1.0016^t$) period utility function, .05% transaction costs on all assets, 50% position limits on all assets and 15% turnover constraints on all assets. Statistics of the utility distribution over the bootstrapped scenarios for the two formulations are given in Table 12.

	Mean	St. Dev.	Min	Max
multistage	-.1139	4.8128	-45.4002	2.0868
rolling one stage	-.4013	5.0111	-43.1712	2.0816

Table 12: Second comparison summary statistics

[6] These scenarios were generated by randomly drawing monthly returns from the time period February 1988 - February 1999.

[7] This instance of USMACRO was estimated from February 1988 to February 1999.

With the exception of the minimum utility, the multistage formulation produces better statistics than the rolling one stage formulation. To test whether or not the mean utility for the multistage formulation is statistically significantly higher than the mean utility for the rolling one stage formulation we use a t test. Specifically we use a paired (by scenario), two sample, one tailed (multistage > rolling one stage) test where the null hypothesis is that the means of the two formulations are equal. The p value for the resulting test is 4.4592e-33 suggesting definitive rejection of the null and a statistically significantly higher mean for the multistage formulation as compared to the rolling one stage formulation.

6 Historical Backtest

In this section we implement the stochastic optimization framework on historical data to judge how it would have done in practice. The *backtest* was carried out using a mean-covariance matching version of VARSIM 2.1 implemented on historical data between November 2000 and February 2001 with monthly rebalancing. The planning horizon was fixed at February 2001 and at each trading time, t, VARSIM 2.1 was estimated from February 1988 up to t. The proportional transaction cost was .5%, and the initial portfolio was 1 in cash in the home currency. A branching factor of 20 was used for each problem along with a downside-quadratic ($\gamma_{1t} = 1, \gamma_{2t} = 10000, \tilde{w}_t = 1.01^t$) period utility function and T3 constraint structure (see Appendix B).

Table 13 gives the portfolio, expected first stage scenario tree asset returns, first stage scenario tree standard deviations of asset returns, realized asset returns, realized portfolio return and period utility for each month in the backtest as well as the total return and utility over the backtest period [8].

[8]Returns are in the form of gross returns, i.e. $\frac{value_t}{value_{t-1}}$.

Nov-00	EUstock	EUcash	EUbond	USstock	JPstock
Weight	0.06	0.25	0.67	0.02	0.00
Expected Return	1.01	1.00	1.00	1.01	1.01
St. Dev. Return	0.05	0.00	0.01	0.05	0.07
Historical Return	0.99	1.00	1.01	0.93	0.84
Portfolio Return	1.00				
Period Utility	0.13				
Dec-00					
Weight	0.05	0.25	0.68	0.02	0.00
Expected Return	1.01	1.00	1.01	1.01	1.00
St. Dev. Return	0.05	0.00	0.01	0.05	0.07
Historical Return	1.01	1.00	1.01	1.04	1.00
Portfolio Return	1.01				
Period Utility	-0.52				
Jan-01					
Weight	0.05	0.25	0.68	0.02	0.00
Expected Return	1.01	1.00	1.01	1.01	1.00
St. Dev. Return	0.05	0.00	0.01	0.05	0.07
Historical Return	0.93	1.00	1.01	0.92	0.96
Portfolio Return	1.00				
Period Utility	-4.62				
Total Return	1.01				
Total Utility	-5.01				

Table 13: Backtest results

As a benchmark, the total utility and return of the MSCI Europe stock index over this time period were -121.36 and 0.92 respectively. The table shows the stochastic optimization framework produces well diversified portfolios that do not change drastically from one time to the next and results in a better total utility and return than the benchmark.

7 Conclusions

This paper has described a method of managing the portfolio of a global fund using a stochastic optimization framework. Downside-quadratic and downside-linear period utility functions were used to represent the preferences of the fund while short sale constraints, position limits and turnover constraints were used to represent the fund's regulatory and performance constraint structure. Two VAR models of assets, exchange rates and macroeconomic variables were presented to describe the underlying dynamics of the problem. Optimization based methods of generating moment matching scenario trees were shown to drastically improve both stability and accuracy and a method for generating arbitrage free scenario trees was given. The one stage stochastic optimization approach was shown to be more appropriate to and to subsume mean-variance analysis, and it was shown that a multistage formulation was able to outperform a rolling one stage formulation with respect to two separate comparisons. Lastly, a historical backtest was carried out which showed that the framework would have produced well diversified portfolios and a relatively good return and utility had it been implemented in practice.

Appendix A

Let:

$$\boldsymbol{y_t} = \mu + \sum_{i=1}^{3} \phi_i y_{t-i} + \boldsymbol{\eta_t}, \tag{44}$$

be a d dimensional VAR(3) and let I_d denote the $d \times d$ identity matrix, $\bar{0}_d$ denote the d dimensional vector of 0s and $\tilde{0}_d$ denote the $d \times d$ matrix of 0s. Then a d dimensional VAR(3) can be written as the VAR(1):

$$\boldsymbol{Y_t} = M + \Phi Y_{t-1} + \boldsymbol{N_t}, \tag{45}$$

where:

$$Y_t := \begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \end{pmatrix} \quad M := \begin{pmatrix} \mu \\ \bar{0}_d \\ \bar{0}_d \end{pmatrix} \quad N_t := \begin{pmatrix} \eta_t \\ \bar{0}_d \\ \bar{0}_d \end{pmatrix} \tag{46}$$

$$\Phi := \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ I_d & \tilde{0}_d & \tilde{0}_d \\ \tilde{0}_d & I_d & \tilde{0}_d \end{pmatrix}. \tag{47}$$

The process is stationary if all the eigenvalues of Φ lie within the unit circle (see e.g. Hamilton [13], Chapter 11). Assuming stationarity (44) implies:

$$E[\boldsymbol{y_t}] = \mu + \sum_{i=1}^{3} \phi_i E[\boldsymbol{y_t}], \tag{48}$$

where $E[\boldsymbol{y_t}]$ is the unconditional mean of the process. Thus:

$$E[\boldsymbol{y_t}] = (I - \sum_{i=1}^{3} \phi_i)^{-1} \mu. \tag{49}$$

The conditional covariance of the process is given by:

$$Cov[\boldsymbol{y_t} | y_{t-1}, y_{t-2}, y_{t-3}] = Cov[\boldsymbol{\eta_t}] = \Sigma. \tag{50}$$

Appendix B

The T3 constraint stucture is defined by the short sale limits, position limits and annual turnover constraints given in in Table 14.

Asset	**Short**	**Position**	**Turnover**
USstock	0	0.40	0.15
EUstock	0	0.80	0.15
EUbond	0	0.80	0.15
JPstock	0	0.15	0.15
EUcash	0	0.25	-

Table 14: T3 constraint structure

Assuming that the returns in Ω are ≥ 1 (prices are non-negative), the solvency constraints are automatically enforced in this constraint structure.

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