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VOLATILITY-INDUCED  
FINANCIAL GROWTH

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WP 10/2004

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# VOLATILITY-INDUCED FINANCIAL GROWTH\*

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## Abstract

We show that the volatility of the price process, which is usually regarded as an impediment for financial growth, can serve as an endogenous factor of its acceleration.

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1. Can price volatility, which is present in virtually every financial market and usually thought of as a risky investment’s downside, serve as an “engine” of financial growth? Paradoxically, the answer to this question turns out to be positive.

To demonstrate this paradox, we examine the long-run performance of constant proportions investment strategies in a securities market. Such strategies prescribe to rebalance the investor’s portfolio, depending on price fluctuations, so as to keep fixed proportions of wealth in all the portfolio positions. In the basic model we deal with, it is assumed that asset returns form stationary ergodic processes and asset prices grow (or decrease)

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at a common asymptotic rate  $\rho$ . It is shown that if an investor employs a constant proportions strategy, then the value of his or her portfolio grows almost surely at a rate strictly greater than  $\rho$ , provided the investment proportions are strictly positive and the stochastic price process is in a sense non-degenerate. The very mild assumption of non-degeneracy we impose requires some randomness, or volatility, of the price process. If this assumption is violated, then the market is essentially deterministic, and the result ceases to hold. Thus, in the present context, the price volatility may be viewed as an endogenous source of acceleration of financial growth. This phenomenon might seem counterintuitive, especially in stationary markets [3], [4], where the asset prices themselves, and not only their returns, are stationary. In this case,  $\rho = 0$ , i.e. each asset grows at zero rate, while any constant proportions strategy exhibits growth at a strictly positive exponential rate with probability one.

In this introductory discussion, we focus primarily on the case where all the assets have the same growth rate  $\rho$ . The results, however, are extended to a model with different growth rates  $\rho^1, \dots, \rho^K$ . In this setting, a constant proportions strategy with proportions  $\lambda^1 > 0, \dots, \lambda^K > 0$  grows almost surely at a rate strictly greater than  $\sum_k \lambda^k \rho^k$ .

The effect highlighted in this article is demonstrated in the framework of a conventional, well-studied model of a financial market. Constant proportions strategies are involved in many practical financial computations, cf. [9]. However, to our knowledge, the phenomenon examined here has not been clearly described and systematically investigated in the literature. The only reference we can indicate in this connection is [8], Section 15.2, where some examples related to the topic under consideration are discussed. No general results showing that the effect of volatility-induced growth can be established in *any* securities market with stationary non-degenerate asset returns are available in the literature. The aim of this note is to fill this gap.

It should be noted that our approach is a partial equilibrium one: the stochastic process of asset prices is given, and investors' decisions do not influence it. It is certainly of interest, but outside the scope of this paper, to build a general equilibrium model to explore the impact of volatility on asset prices in the presence of self-financing constant proportions strategies. At the same time, we note that the approach used in this work is standard and commonly accepted in finance. In particular, by passing to the limit from discrete to continuous time in the model at hand (see e.g. [5]), one can derive the Black-Scholes formula. As is well-known, the Black-Scholes methodology

is not only widely used for practical computations but determined in many respects the structure of existing derivative securities markets. This exceptional role of the model under consideration in financial theory and practice enhances the significance of conclusions inferred from it.

We first present the results outlined above in the context of an ideal (frictionless) market where there are no transaction costs. Then we show that the main conclusions remain valid if transaction costs are present but are small enough. The case of an ideal market is considered in Sections 2, 3 and 4, in which we describe the model, formulate the assumptions and state the main results, respectively. Section 5 contains proofs of the assertions formulated in Section 4. In the remainder of the paper, we show how the main results can be extended to the case of small transaction costs.

**2.** Consider a financial market with  $K \geq 2$  securities (assets). Let  $S_t = (S_t^1, \dots, S_t^K)$  be the vector of security prices at time  $t = 0, 1, 2, \dots$ . Assume that  $S_t^k > 0$  for each  $t$  and  $k$ , and denote by

$$R_t^k = \frac{S_t^k}{S_{t-1}^k} \quad (k = 1, 2, \dots, K, t = 1, 2, \dots) \quad (1)$$

the (gross) return on asset  $k$  over the time period  $(t-1, t]$ . Define  $R_t = (R_t^1, \dots, R_t^K)$ .

At each time period  $t$ , an investor chooses a portfolio  $h_t = (h_t^1, \dots, h_t^K)$ , where  $h_t^k$  is the number of units of asset  $k$  in the portfolio  $h_t$ . Generally,  $h_t$  might depend on the observed values of the price vectors  $S_0, S_1, \dots, S_t$ . A sequence  $H = \{h_0, h_1, \dots\}$  specifying a portfolio  $h_t = h_t(S_0, \dots, S_t)$  at each time  $t$  as a measurable function of  $S_0, S_1, \dots, S_t$  is called a trading strategy. We will deal only with those trading strategies for which  $h_t^k \geq 0$ , thus excluding short sales of assets.

Let  $\lambda = (\lambda^1, \dots, \lambda^K)$  be a vector such that

$$\lambda^k \geq 0 \quad (k = 1, 2, \dots, K) \quad \text{and} \quad \sum_{k=1}^K \lambda^k = 1. \quad (2)$$

A trading strategy  $H$  is called a constant proportions strategy with vector of proportions  $\lambda = (\lambda^1, \dots, \lambda^K)$  if

$$S_t^k h_t^k = \lambda^k S_t h_{t-1}^k \quad (k = 1, 2, \dots, K, t = 1, 2, \dots). \quad (3)$$

If  $\lambda^k > 0$  for each  $k$ , then  $H$  is said to be completely mixed. The scalar product  $S_t h_{t-1} = \sum_{k=1}^K S_t^k h_{t-1}^k$  expresses the value of the portfolio  $h_{t-1}$  in terms of

the prices  $S_t^k$  at time  $t$ . An investor following the strategy (3) rebalances (without transaction costs) the portfolio  $h_{t-1}$  at time  $t$  so that the available wealth  $S_t h_{t-1}$  is distributed across the assets  $k = 1, 2, \dots, K$  according to the proportions  $\lambda^1, \dots, \lambda^K$ . It is immediate from (2) and (3) that

$$S_t h_t = S_t h_{t-1}, \quad t = 1, 2, \dots, \quad (4)$$

and, consequently, the strategy  $H$  is *self-financing*. To specify a constant proportions strategy it is sufficient to specify the vector  $\lambda$  and an initial portfolio  $h_0 = h_0(S_0)$ ; then  $h_t$  for each  $t$  is recursively determined by formula (3).

We fix  $\lambda$  and  $H$  satisfying (2) and (3) and denote by  $V_t = S_t h_t$  the value of the portfolio  $h_t$  at time  $t = 0, 1, 2, \dots$  expressed in terms of the current prices  $S_t^k$ . We will suppose that the price vectors  $S_t$ , and hence the return vectors  $R_t$ , are random, i.e., they evolve in time as stochastic processes. Then the trading strategy  $h_t$ ,  $t = 0, 1, 2, \dots$ , generated by the investment rule (3) and the value  $V_t = S_t h_t$ ,  $t = 0, 1, 2, \dots$ , of the portfolio  $h_t$  are stochastic processes as well. We are interested in the asymptotic behavior of  $V_t$  as  $t \rightarrow \infty$ .

**3.** We will assume:

(R) *The vector stochastic process  $R_t$  is stationary and ergodic. The expected values  $E|\ln R_t^k|$  are finite.*

Recall that a stochastic process  $R_1, R_2, \dots$  is called *stationary* if, for any  $m = 0, 1, 2, \dots$  and any measurable function  $\phi(x_0, x_1, \dots, x_m)$ , the distribution of the random variable  $\phi_t := \phi(R_t, R_{t+1}, \dots, R_{t+m})$  ( $t = 0, 1, \dots$ ) does not depend on  $t$ . According to this definition, all probabilistic characteristics of the process  $R_t$  are time invariant. If  $R_t$  is stationary, then for any measurable function  $\phi$  for which  $E|\phi(R_t, R_{t+1}, \dots, R_{t+m})| < \infty$ , the averages

$$\frac{\phi_1 + \dots + \phi_t}{t} \quad (5)$$

converge almost surely (a.s.) as  $t \rightarrow \infty$  (Birkhoff's ergodic theorem – see, e.g., [2]). If the limit of all averages of the form (5) is equal to a constant a.s., then the process  $R_t$  is called *ergodic*. In this case, the above limit is equal a.s. to the expectation  $E\phi_t$  (which does not depend on  $t$ ).

It follows from (R) that  $S_t^k = S_0^k R_1^k \dots R_t^k$ , where the random sequence  $R_t^k$  is stationary. This assumption on the structure of the price process is a fundamental hypothesis commonly accepted in mathematical finance. Moreover, it is quite often assumed that the random variables  $R_t^k, t = 1, 2, \dots$

are independent, i.e., the price process  $S_t^k$  forms a *geometric random walk*. This postulate, which is much stronger than the hypothesis of stationarity of  $R_t^k$ , lies in the basis of the Black-Scholes formula, see e.g. [8].

By virtue of Birkhoff's ergodic theorem, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln S_t^k = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{j=1}^t \ln R_j^k = E \ln R_t^k \text{ (a.s.)} \quad (6)$$

for each  $k = 1, 2, \dots, K$ . This means that each asset  $k$  has almost surely a well-defined and finite asymptotic (exponential) *growth rate*, which turns out to be equal a.s. to the expectation

$$\rho^k := E \ln R_t^k,$$

the *drift* of this asset. The drift can be positive, zero or negative. It does not depend on  $t$  in view of the stationarity of  $R_t$ .

4. We formulate central results of this article – Theorems 1 and 2. In these theorems,  $H = \{h_0, h_1, \dots\}$  is a constant proportions strategy with some vector of proportions  $\lambda = (\lambda^1, \dots, \lambda^K)$  satisfying (2). It is assumed that  $V_0 = S_0 h_0 > 0$ , which implies  $V_t = S_t h_t > 0$  for each  $t$  (see formula (8) below). If the limit

$$\lim_{t \rightarrow \infty} \frac{\ln V_t}{t}$$

exists, it is called the *growth rate of the strategy  $H$* . In Theorems 1 and 2, we assume that the following condition holds:

(V) With strictly positive probability,

$$S_t^k / S_t^m \neq S_{t-1}^k / S_{t-1}^m \text{ for some } 1 \leq k, m \leq K \text{ and } t \geq 1.$$

**Theorem 1.** *If all the coordinates  $\lambda^k$  of the vector  $\lambda$  are strictly positive, i.e. the strategy  $H$  is completely mixed, then the growth rate of the strategy  $H$  is almost surely strictly greater than  $\sum_k \lambda^k \rho^k$ , where  $\rho^k$  is the drift of asset  $k$ .*

We recall that, since the process  $R_t$  is stationary, the expectation  $E \ln(R_t \lambda)$  involved in the statement of the theorem does not depend on  $t$ . Condition (V) is a very mild assumption of volatility of the price process. This condition does not hold if and only if, with probability one, the ratio  $S_t^k / S_t^m$

of the prices of any two assets  $k$  and  $m$  does not depend on  $t$ . Clearly, the ratios  $S_t^k/S_t^m$  do not depend on  $t$  if and only if, for every  $t \geq 1$ , the return  $R_t^k = S_t^k/S_{t-1}^k$  on each asset  $k$  is equal to the same number  $\alpha_t$ :

$$R_t^1 = R_t^2 = \dots = R_t^K = \alpha_t.$$

If  $R_t^k = \alpha_t$  for all  $k$  (a.s.), then  $\rho^k = E \ln \alpha_t$ , and the growth rate of  $H$  is equal to  $E \ln(R_t \lambda) = E \ln \alpha_t = \rho^k = \sum_k \lambda^k \rho^k$ . Consequently, if (V) fails to hold, then the assertion of Theorem 1 is not valid.

**Example.** Consider the case where there are two assets, i.e.,  $K = 2$  and  $k = 1, 2$ . Suppose the first asset ( $k = 1$ ) is riskless, i.e., its return  $R_t^1$  is a constant:  $R_t^1 = R > 0$ . Then *condition (V) is fulfilled if the stationary process  $R_t^2$ , describing the returns of the risky asset  $k = 2$ , is not constant a.s.* Indeed, if (V) does not hold, then, as has been noticed above,  $R_t^2 = R_t^1 = R = \text{const}$  with probability one.

We are primarily interested in the situation when all the assets under consideration have *the same* drift, and hence a.s. the same growth rate:

(R1) *There exists a number  $\rho$  such that, for each  $k = 1, \dots, K$ , we have  $E \ln R_t^k = \rho$ .*

Assumption (R1) allows to concentrate, for example, on those assets in the market that grow at the maximum rate. One may think that all the others, growing slower, will eventually be driven out of the market. As long as we deal with an infinite time horizon, we may exclude such assets from consideration.

From Theorem 1, we immediately obtain the following result.

**Theorem 2.** *Under assumption (R1), the growth rate of the strategy  $H$  is almost surely strictly greater than the growth rate of each individual asset.*

In the context of Theorem 2, the volatility of the price process appears to be the only cause for any completely mixed constant proportions strategy to grow at a rate strictly greater than  $\rho$  – the growth rate of each particular asset. This result looks at first glance unexpected, since the volatility of asset prices is usually regarded as an impediment for financial growth, while here it serves as a factor of its acceleration. In a stationary market, where the process  $S_t$  (and not only  $R_t$ ) is ergodic and stationary and where  $E|\ln S_t^k| < \infty$ , the growth rate of each asset is zero,

$$E \ln R_t^k = E \ln S_t^k - E \ln S_{t-1}^k = 0,$$

while any completely mixed constant proportions strategy grows at a strictly positive exponential rate.



Common intuition suggests that if the market is stationary, then the portfolio value  $V_t$  for a constant proportions strategy must converge in one sense or another to a stationary process. (This is the most common first guess regarding the asymptotic behaviour of  $V_t$ .) The usual intuitive argument in support of this conjecture appeals to the self-financing property (4). The self-financing constraint seems to exclude possibilities of unbounded growth. The truth, however, lies in the opposite direction: unbounded exponential growth is not only compatible with self-financing, but is characteristic for any completely mixed constant proportions strategy.

We have seen how one might arrive at a wrong conclusion about the behaviour of a constant proportions strategy. But what is the intuition for the right conclusion? It is quite simple. It lies in the fact that the constant proportions always force one to "buy low and sell high" - the common sense dictum of all trading. Indeed, those assets whose prices have risen from the last rebalance date will be overweighted in the portfolio and their holdings must be reduced to meet the required proportions and to be replaced in part by assets whose prices have fallen and whose holdings must therefore be increased. (Obviously, for this mechanism to work the prices must change in time; if they are constant, one cannot get any profit from trading.) Further, under suitable conditions the specified portfolio proportions can be optimised to maximize the exponential growth rates of the portfolio returns implied by Theorems 1 and 2. Such maximization problems are considered in the theory of log-optimal investments (see [1] and [6]). In this paper we do not touch this topic: our focus is on the analysis of any, not necessarily log-optimal, constant proportions strategies.

Our result bears some similarity with the concept of asymptotic arbitrage, see e.g. [7]. Three features, however, stand out: growth is exponentially fast, unbounded wealth is achieved with probability one, and the effect of growth is demonstrated for specific (constant proportions) strategies. None of these properties can directly be deduced from asymptotic arbitrage.

5. The proof of Theorem 1 relies upon the following two lemmas.

**Lemma 1.** *The growth rate of the strategy  $H$  is equal to  $E \ln(R_t \lambda)$  (a.s.).*

*Proof.* We have

$$V_t = S_t h_t = \sum_{m=1}^K S_t^m h_{t-1}^m = \sum_{m=1}^K \frac{S_t^m}{S_{t-1}^m} S_{t-1}^m h_{t-1}^m =$$

$$\sum_{m=1}^K \frac{S_t^m}{S_{t-1}^m} \lambda^m S_{t-1} h_{t-1} = V_{t-1} \sum_{m=1}^K R_t^m \lambda^m = (R_t \lambda) V_{t-1}. \quad (7)$$

Thus

$$V_t = V_0 (R_1 \lambda) (R_2 \lambda) \dots (R_t \lambda), \quad (8)$$

and so

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln V_t = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{j=1}^t \ln(R_j \lambda) = E \ln(R_t \lambda) \text{ (a.s.)}, \quad (9)$$

which proves the lemma.

**Lemma 2.** *If condition (V) holds, then  $E \ln(R_t \lambda) > \sum_{k=1}^K \lambda^k \rho^k$ .*

*Proof.* Observe that condition (V) is equivalent to the following one:

(V1) For some  $t \geq 1$  (and hence for each  $t \geq 1$ ), the probability

$$P\{R_t^k \neq R_t^m \text{ for some } 1 \leq k, m \leq K\}$$

is strictly positive.

Indeed, we have  $S_t^k/S_t^m \neq S_{t-1}^k/S_{t-1}^m$  if and only if  $S_t^k/S_{t-1}^k \neq S_t^m/S_{t-1}^m$ , which can be written as  $R_t^k \neq R_t^m$ . Denote by  $\delta_t$  the random variable that is equal to 1 if the event  $\{R_t^k \neq R_t^m \text{ for some } 1 \leq k, m \leq K\}$  occurs and 0 otherwise. Condition (V) means that  $P\{\max_{t \geq 1} \delta_t = 1\} > 0$ , while (V.1) states that, for some  $t$  (and hence for each  $t$ ),  $P\{\delta_t = 1\} > 0$ . The latter property is equivalent to the former because

$$\{\max_{t \geq 1} \delta_t = 1\} = \cup_{t=1}^{\infty} \{\delta_t = 1\}.$$

By using Jensen's inequality and (V1), and we find that

$$\ln \sum_{k=1}^K R_t^k \lambda^k > \sum_{k=1}^K \lambda^k (\ln R_t^k)$$

with strictly positive probability, while the non-strict inequality holds always. Consequently,

$$E \ln(R_t \lambda) > \sum_{k=1}^K \lambda^k E(\ln R_t^k) = \sum_{k=1}^K \lambda^k \rho^k, \quad (10)$$

which proves the lemma.

*Proof of Theorem 1.* The result is immediately obtained by combining Lemmas 1 and 2.

**6.** We now assume that, in the market under consideration, there are transaction costs for buying and selling assets. When selling  $x$  units of asset  $k$  at time  $t$ , one gets the amount  $(1 - \varepsilon_-^k)S_t^k x$  (rather than  $S_t^k x$  as in the frictionless case). To buy  $x$  units of asset  $k$ , one has to pay  $(1 + \varepsilon_+^k)S_t^k x$ . The numbers  $\varepsilon_-^k, \varepsilon_+^k \geq 0$ ,  $k = 1, 2, \dots, K$  (the *transaction cost rates*) are supposed to be given. In this market, portfolio rebalancing might lead to a loss of wealth, therefore *self-financing trading strategies*  $H = \{h_0, h_1, \dots\}$  are defined as those satisfying the condition

$$\sum_{k=1}^K (1 + \varepsilon_+^k) S_t^k (h_t^k - h_{t-1}^k)_+ \leq \sum_{k=1}^K (1 - \varepsilon_-^k) S_t^k (h_{t-1}^k - h_t^k)_+, \quad (11)$$

where  $x_+ = \max\{x, 0\}$ . Inequality (11) means that asset purchases can be made only at the expense of asset sales.

In the current context, we say that  $H = \{h_0, h_1, \dots\}$  is a *constant proportions strategy* with vector of proportions  $\lambda = (\lambda^1, \dots, \lambda^K)$  if

$$S_t^k h_t^k = (1 - \delta) \lambda^k S_t h_{t-1} \quad (k = 1, 2, \dots, K, t = 1, 2, \dots), \quad (12)$$

where  $\delta \in (0, 1)$  is some constant. We include into this definition the requirement that  $V_0 = S_0 h_0 > 0$ , which guarantees that  $V_t = S_t h_t > 0$  (and hence  $h_t \neq 0$ ) for all  $t$ . Strategies of the form (12), as well as those defined by (3), prescribe to keep constant shares of wealth in all the portfolio positions, but in contrast with (3), the equality  $S_t h_t = S_t h_{t-1}$  does not hold. We have  $S_t h_t = (1 - \delta) S_t h_{t-1} < S_t h_{t-1}$ , and so the value of the portfolio  $h_t$ , expressed in terms of the prices  $S_t^k$ , is strictly less than the value of  $h_{t-1}$ .

Theorem 3 below extends the results of Theorems 1 and 2 to the model with transaction costs. As before, we assume that hypotheses (R) and (V) hold.

**Theorem 3.** *Let  $\lambda = (\lambda^1, \dots, \lambda^K)$  be a strictly positive vector with  $\lambda^1 + \dots + \lambda^K = 1$ . If  $\delta \in (0, 1)$  is small enough, then a constant proportions strategy  $H$  of the form (12) has a growth rate strictly greater than  $\sum_{k=1}^K \lambda^k \rho^k$  (a.s.), and if  $\rho^1 = \dots = \rho^K = \rho$ , then the growth rate of  $H$  is strictly greater than  $\rho$  (a.s.). Further, if the transaction cost rates  $\varepsilon_-^k, \varepsilon_+^k \geq 0$ ,  $k = 1, 2, \dots, K$ , are small enough (in particular, if they do not exceed  $\delta/2$ ), then the strategy  $H$  is self-financing.*

*Proof.* We first observe that the growth rate of the strategy  $H$  is equal to  $E \ln[(1 - \delta)R_t\lambda]$ . This fact is proved exactly like Lemma 1 (replace in (7), (8) and (9)  $\lambda$  by  $(1 - \delta)\lambda$ ). By virtue of Lemma 2,  $E \ln(R_t\lambda) > \sum_{k=1}^K \lambda^k \rho^k$ . This inequality will remain valid if  $\lambda$  is replaced by  $(1 - \delta)\lambda$ , provided  $\delta \in (0, 1)$  is small enough. Fix any such  $\delta \in (0, 1)$ . Denote by  $\varepsilon$  the greatest of the numbers  $\varepsilon_-^k, \varepsilon_+^k$ . It remains to show that  $H$  is self-financing when  $\varepsilon \geq \delta/2$ . To this end we note that inequality (11) is implied by

$$\sum_{k=1}^K (1 + \varepsilon) S_t^k (h_t^k - h_{t-1}^k)_+ \leq \sum_{k=1}^K (1 - \varepsilon) S_t^k (h_{t-1}^k - h_t^k)_+,$$

which is equivalent to

$$\varepsilon \sum_{k=1}^K |S_t^k h_t^k - S_t^k h_{t-1}^k| \leq S_t (h_{t-1} - h_t). \quad (13)$$

If formula (12) holds, then the right-hand side of the last inequality is equal to  $\delta S_t h_{t-1}$ , and the left-hand side can be estimated as follows:

$$\begin{aligned} \varepsilon \sum_{k=1}^K |(1 - \delta)\lambda^k S_t h_{t-1} - S_t^k h_{t-1}^k| &\leq \varepsilon \sum_{k=1}^K (1 - \delta)\lambda^k S_t h_{t-1} + \varepsilon \sum_{k=1}^K S_t^k h_{t-1}^k = \\ &\varepsilon(1 - \delta)S_t h_{t-1} + \varepsilon S_t h_{t-1} \leq 2\varepsilon S_t h_{t-1}. \end{aligned}$$

Consequently, if  $0 \leq \varepsilon \leq \delta/2$ , the strategy  $H$  is self-financing.

The proof is complete.

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