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Abstract: In this paper, two structural models where firms have stationary capital structures and endogenous default barriers are extended to allow the principal value of a firm's debt to grow at a constant rate. This allows firms to have a dynamic capital structure. These two models are then used in conjunction with observable equity data to calculate the implied asset volatilities of a sample of fifty firms. Unit root tests are applied to the implied asset volatility and equity volatility processes to determine whether the processes are mean-reverting. Evidence that asset volatility is mean-reverting is found for forty-six of the fifty firms in the sample, regardless of which structural model is used to calculate the asset volatility, while the number of firms whose equity volatility is mean-reverting is in general lower for the poorer credit classes, consistent with the leverage effect. The mean-reversion of asset volatility has implications for the modelling of both equity and debt, and for the pricing of equity options, corporate bonds and credit derivatives.

Keywords: structural credit models, asset volatility, equity volatility.

1. Introduction

There is now widespread empirical evidence of the leverage effect, which says that a firm's equity volatility is positively related to its leverage (see, for example, Black (1976), Christie (1982) and Duffee (1995)). Therefore, a firm's equity volatility may appear to be mean-reverting over a long period of time if its leverage remains steady. However, if the credit quality of the firm worsens dramatically and the firm's leverage increases, its equity volatility is likely to rise and may not fall again unless the credit quality of the firm improves. As a result, equity volatility is not, in general, mean-reverting over the complete life of a firm.

It has been noted widely, for example in Leland (1994), that structural models where the asset volatility of a firm is assumed to be constant give equity volatilities that are positively related to a firm's leverage, consistent with the leverage effect. However, there has been no empirical research into the asset volatilities calculated using structural models in conjunction with a time series of equity data. In this paper, the observed equity volatility and calculated asset volatility processes between December 1993 and December 2003 are studied for a sample of fifty firms. In particular, it will focus on whether the asset volatility of a firm is mean-reverting, and remains reverting to the same constant value as the credit quality of the firm changes.

The rest of this paper is structured as follows. In Section 2, two structural models where firms have a stationary capital structure and endogenous default boundaries are extended to allow the principal value of a firm's outstanding debt to grow at a constant rate. This allows firms to have a dynamic capital structure. These two models are then used in Section 3 together with observed equity data to calculate a time series of asset volatilities for each firm in the sample. The time series of equity and asset volatilities are illustrated for a single firm, Delta Air Lines Inc. Unit root tests are then performed on the time series of both equity and asset volatilities for all fifty firms in the sample to determine whether the volatility processes revert to a constant value. Conclusions are drawn in Section 4.

2. Extending Structural Models with Endogenous Default Boundaries

Two structural models with endogenous default boundaries, those models developed by Leland (1994) and Leland & Toft (1996), are extended in this section to allow the principal value of a firm's debt to grow at a constant rate over time. Although the two models share a number of features, different debt structures are assumed: in one case, a firm's debt is assumed to consist of finite-maturity coupon bonds, while in the second case, the firm's debt consists of perpetual coupon bonds.

Seven assumptions are common to both of the models that are developed in this section, and these are described below:

ASSUMPTION 1: The term structure of default-free interest rates is flat and known with certainty, i.e. the time-t_0 price of a default-free bond that promises a payment of one unit at a future time t_1 is $P(t_0,t_1) = \exp[-r(t_1 - t_0)]$, where r is the (instantaneous) default-free rate of interest, which is constant over time.

ASSUMPTION 2: Let $\boldsymbol{V_t}$ be the market value of a firm's total assets at time t. It is assumed that in the risk-neutral measure, the value of a firm's assets follows the lognormal process

$$\frac{d\boldsymbol{V_t}}{\boldsymbol{V_t}} = (r - d)dt + s\,d\boldsymbol{W_t}\,. \tag{1}$$

Both the asset volatility s and the fraction d of the value of the assets paid out to holders of the firm's debt and equity are assumed to be constant.

ASSUMPTION 3: The principal value of the outstanding debt of a firm F_t satisfies the non-random process,

$$\frac{dF_t}{F_t} = m_F\,dt\,, \tag{2}$$

where m_F is a constant. Therefore, the principal value of a firm's outstanding debt at time t is $F_t = F_0 e^{m_F t}$, where F_0 is the principal value of outstanding debt at time 0.

ASSUMPTION 4: A firm defaults on all of its outstanding debt when the process $\boldsymbol{V_t}$ hits a default boundary V_t^B, which is taken to be a fixed proportion of the principal value of the firm's debt[1], i.e. $V_t^B = b\,F_t$ for some constant b .

ASSUMPTION 5: If the firm defaults at a time t , the asset value of the firm reduces by aV_t^B due to the costs of default.

ASSUMPTION 6: The equity value of the firm is zero at the default boundary, i.e. equity-holders do not receive a rebate upon default by the firm.

ASSUMPTION 7: All debt issued by the firm has the same seniority. Furthermore, debt-holders receive the same fraction of par at the time of default, regardless of remaining maturity, while the recovery rate of future coupon payments is zero.

All of these assumptions apart from Assumption 3 are consistent with the models developed by Leland (1994) and Leland & Toft (1996). However, both of these models assume that the principal value of a firm's outstanding debt is fixed. The non-random process used in this paper to model the principal value of a firm's debt has been used before in Nickell, Perraudin & Varotto (2001) and Ericsson & Reneby (2003).

Assumptions 4 and 5 imply that if the firm defaults at time t , debt-holders receive a total of $(1-a)bF_t$ upon default, i.e. the recovery rate averaged over the firm's debts is $(1-a)b$. As all of the firm's debts have the same recovery rate as a consequence of Assumption 7, this implies that each debt has a recovery rate of $(1-a)b$.

Define the distance to default $\boldsymbol{X_t}$ to be the ratio of the value of the firm's total assets to the default boundary V_t^B, i.e.

$$\boldsymbol{X_t} = \frac{\boldsymbol{V_t}}{bF_t} = \frac{\boldsymbol{V_t}}{b\,F_0 e^{m_F t}}. \tag{3}$$

[1] Later it will be shown that the optimal default boundary is of this form for both of the capital structures that are investigated in this paper.

Default therefore occurs when $\boldsymbol{X}_t$ hits one. As a consequence of Itô's Lemma, the distance to default satisfies the stochastic differential equation

$$\frac{d\boldsymbol{X}_t}{\boldsymbol{X}_t} = (r - m_F - d)dt + s\, d\boldsymbol{W}_t\,, \tag{4}$$

while results from Harrison (1990) show that if the firm has a distance to default of X_{t_0} at time t_0, then the risk-neutral probability that the firm defaults in the period $[t_0, t_1]$ is given by

$$Q(X_{t_0}, t_1 - t_0) \;=\; \Phi\left(\frac{-\log X_{t_0} - a(t_1 - t_0)}{s\sqrt{t_1 - t_0}}\right) + X_{t_0}^{-\frac{2a}{s^2}}\Phi\left(\frac{-\log X_{t_0} + a(t_1 - t_0)}{s\sqrt{t_1 - t_0}}\right), \tag{5}$$

where

$$a = r - m_F - d - \frac{1}{2}s^2. \tag{6}$$

Differentiating (5), it can be seen that the probability density function of the first passage time of $\boldsymbol{X}_t$ to 1 (or equivalently of the first passage time of $\boldsymbol{V}_t$ to V_t^B) is equal to

$$q(X_{t_0}, t_1 - t_0) = X_{t_0}^{\frac{-2a}{s^2}} \frac{1}{\sqrt{2p}} \frac{\log X_{t_0}}{s\,(t_1 - t_0)^{3/2}} \exp\left[-\frac{1}{2}\left(\frac{a(t_1 - t_0) - \log X_{t_0}}{s\sqrt{t_1 - t_0}}\right)^2\right]. \tag{7}$$

2.1 Debt Structure Consists of Finite-Maturity Bonds

First, it is assumed that a firm continuously offers bonds with an initial time-to-maturity of $T < \infty$, so that the capital structure of a firm is dynamic and consists of equity and finite-maturity coupon bonds. This capital structure was originally analysed by Leland & Toft (1996), and therefore, the model developed in this subsection will be referred to as the Extended Leland & Toft model (or ELT for short). It is assumed here that the firm issues bonds with a principal value of

$$\tilde{F}e^{m_F s}ds \tag{8}$$

during the interval $[s, s + ds]$, where

$$\tilde{F} = \frac{m_F}{(1 - e^{-m_F T})}F_0\,. \tag{9}$$

Then the principal value of all of the outstanding bonds at time t is given by

$$F_t \;=\; \tilde{F}\int_{t-T}^{t} e^{m_F s}ds \;=\; \frac{\tilde{F}}{m_F}(1-e^{-m_F T})e^{m_F t} \;=\; F_0 e^{m_F t}\,, \tag{10}$$

consistent with Assumption 3. Leland & Toft (1996) analysed the special case where $m_F = 0$.

It is assumed that the bonds pay a continual stream of coupons at a rate of c. Then the time-t market value of the bonds that were issued in the interval $[s, s+ds]$ (where $t - T \le s \le t$) is

$$\begin{aligned} SB_t^s ds = c\tilde{F}e^{m_F s}ds \int_t^{s+T} e^{-r(u-t)}[1-Q(X_t, u-t)]du + \tilde{F}e^{m_F s}ds e^{-r(s+T-t)}[1-Q(X_t, s+T-t)] \\ +(1-a)b\tilde{F}e^{m_F s}ds\int_t^{s+T} e^{-r(u-t)}q(X_t, u-t)du. \end{aligned} \tag{11}$$

The first term on the right-hand side of (11) corresponds to the time-t value of the remaining coupons, the second term relates to the value of the principal payment made at time $s+T$, while the third term corresponds to the value of the recovery payment if the firm defaults before time $s+T$.

Define $G_1(X_{t_0}, t)$ to be

$$G_1(X_{t_0}, t) = \int_0^t e^{-(r-m_F)u} q(X_{t_0}, u)du\,. \tag{12}$$

Using results from Rubenstein & Reiner (1991), the function $G_1(X_{t_0}, t)$ is given by

$$G_1(X_{t_0}, t) = X_{t_0}^{\frac{-a+b_1}{s^2}} \Phi\left(\frac{-\log X_{t_0} - b_1 t}{s\sqrt{t}}\right) + X_{t_0}^{\frac{-a-b_1}{s^2}} \Phi\left(\frac{-\log X_{t_0} + b_1 t}{s\sqrt{t}}\right), \tag{13}$$

where

$$b_1 = \sqrt{a^2 + 2(r-m_F)s^2}\,. \tag{14}$$

Similarly, define $G_2(X_{t_0}, t)$ to be

$$G_2(X_{t_0}, t) = \int_0^t e^{-ru} q(X_{t_0}, u)du\,, \tag{15}$$

which can be shown to be equal to

$$G_2(X_{t_0}, t) = X_{t_0}^{\frac{-a+b_2}{s^2}} \Phi\left(\frac{-\log X_{t_0} - b_2 t}{s\sqrt{t}}\right) + X_{t_0}^{\frac{-a-b_2}{s^2}} \Phi\left(\frac{-\log X_{t_0} + b_2 t}{s\sqrt{t}}\right), \tag{16}$$

where

$$b_2 = \sqrt{a^2 + 2r s^2} \,. \tag{17}$$

A change of variable and an integration by parts shows that (11) is equal to

$$\begin{aligned} SB_t^s ds = \frac{c}{r}\tilde{F}e^{m_F s}ds + \left(1-\frac{c}{r}\right)\tilde{F}e^{m_F s}ds e^{-r(s+T-t)}[1-Q(X_t,\, s+T-t)] \\ + \left((1-a)b - \frac{c}{r}\right)\tilde{F}e^{m_F s}ds G_2(X_t,\, s+T-t). \end{aligned} \tag{18}$$

The market value of a firm's debt at time t is found by integrating the time-t value of all bonds that are outstanding at time t, i.e. bonds that were issued during the period $[t-T,t]$. Therefore, the time-t market value of a firm's debt is given by

$$\begin{aligned} \int_{t-T}^{t} SB_t^s ds \;=\; \frac{c}{r}\tilde{F}\int_{t-T}^{t} e^{m_F s}ds + \left(1-\frac{c}{r}\right)\tilde{F}\int_{t-T}^{t} e^{m_F s}e^{-r(s+T-t)}[1-Q(X_t, s+T-t)]ds \\ + \left((1-a)b - \frac{c}{r}\right)\tilde{F}\int_{t-T}^{t} e^{m_F s}G_2(X_t, s+T-t)ds. \end{aligned} \tag{19}$$

After performing the first integral, using the definition of $\tilde{F}$ from (9), and applying a change of variable to the second and third integrals, it is seen that (19) is equivalent to

$$\frac{c}{r}F_0 e^{m_F t} + \left(1-\frac{c}{r}\right)F_0 e^{m_F t} I(X_t,T) + \left((1-a)b - \frac{c}{r}\right)F_0 e^{m_F t} J(X_t,T), \tag{20}$$

where

$$I(X_t,T) = \left(\frac{m_F e^{-m_F T}}{1-e^{-m_F T}}\right)\left[\frac{1-e^{-(r-m_F)T}}{(r-m_F)} - \int_0^T e^{-(r-m_F)u}Q(X_t,u)du\right] \tag{21}$$

and

$$J(X_t,T) = \left(\frac{m_F e^{-m_F T}}{1-e^{m_F T}}\right)\int_0^T e^{m_F u}G_2(X_t,u)du \,. \tag{22}$$

As $F_t = F_0 e^{m_F t}$ (see Assumption 3), the market value of a firm's debt at time t can be written as

$$\frac{c}{r}F_t + \left(1-\frac{c}{r}\right)F_t I(X_t,T) + \left((1-a)b - \frac{c}{r}\right)F_t J(X_t,T), \tag{23}$$

while performing integration by parts on the integrals in (21) and (22) shows that $I(X_t,T)$ and $J(X_t,T)$ are given by

$$I(X_t,T)=\frac{m_F}{(r-m_F)(1-e^{-m_F T})}\left(e^{-m_F T}[1-G_1(X_t,T)]-e^{-rT}[1-Q(X_t,T)]\right) \quad (24)$$

and

$$J(X_t,T)=\frac{1}{1-e^{-m_F T}}[G_2(X_t,T)-e^{-m_F T}G_1(X_t,T)]. \quad (25)$$

L'Hôpital's rule is needed to calculate the value of $I(X_t,T)$ in the case when $m_F=r$ and the values of $I(X_t,T)$ and $J(X_t,T)$ when $m_F=0$.

Firms may receive tax benefits associated with debt financing. As is explained in Leland (1994), the tax benefit of a coupon of $\bar{c}$ is equal to $(tax)\bar{c}$, where tax is the effective tax rate, as long as the firm is solvent[2]. It is proposed here that the total value of the firm only includes the tax benefit of future coupon payments on the debt that is currently outstanding. From an examination of (20), it can be seen that the time-t value of the remaining coupon payments on bonds that are outstanding at time t is

$$\frac{c}{r}F_t\left[1-I(X_t,T)-J(X_t,T)\right]. \quad (26)$$

Therefore, the time-t value of the tax benefits to the firm is given by

$$TB_t=\frac{(tax)c}{r}F_t\left[1-I(X_t,T)-J(X_t,T)\right]. \quad (27)$$

According to Assumption 5, the value of the firm's assets is assumed to fall by a proportion a at the time of default. The value of the default costs of the firm at time t is taken to be the reduction in the value of the debt that is outstanding at time t due to the costs of default. Again from an examination of (20), the time-t value of the default costs of the firm can be seen to equal

$$DC_t=ab\,F_tJ(X_t,T). \quad (28)$$

The total value of the firm at time t, v_t, is taken to be

$$v_t = V_t+TB_t-DC_t = V_t+\frac{(tax)c}{r}F_t[1-I(X_t,T)]-\left(ab+\frac{(tax)c}{r}\right)F_tJ(X_t,T). \quad (29)$$

[2] Leland (1994) also explains that under US tax codes, a firm must have earnings before interest and taxes that are at least as large as the coupon payment if they are to receive the tax benefits. However, this technicality is not considered here.

The market value of equity is given by the total value of the firm minus the market value of the debt that is currently outstanding:

$$S_t = V_t - \frac{(1-tax)c}{r} F_t - \left(1 - \frac{(1-tax)c}{r}\right) F_t I(X_t, T) - \left(b - \frac{(1-tax)c}{r}\right) F_t J(X_t, T). \tag{30}$$

Default can be viewed as a decision by the managers of a firm: if a firm is struggling to make payments to debt-holders, managers can choose whether to liquidate assets to make the debt payments or to default on the debt. As equity-holders own the firm, one of the aims of managers is to maximise the value of a firm's equity. Therefore, it is assumed that managers choose the default boundary optimally so that the firm's equity value is maximised. Following Leland & Toft, the optimal default boundary is assumed to satisfy the smooth-pasting condition,

$$\left.\frac{\partial S_t}{\partial V_t}\right|_{V_t = V_t^B} = 0. \tag{31}$$

Therefore, the optimal boundary is given by $V_t^B = \hat{b} F_t$, where

$$\hat{b} = \frac{\frac{(1-tax)c}{r}(C_1 - C_2) - C_1}{1 - e^{-m_F T} - C_2} \tag{32}$$

and

$$\begin{aligned} C_1 = \frac{m_F e^{-m_F T}}{(r - m_F)} \left[\left(\frac{-a + b_1}{s^2} \right) - \left(\frac{2b_1}{s^2} \right) \Phi\left(\frac{b_1}{s} \sqrt{T} \right) - \frac{2}{s\sqrt{T}} f\left(\frac{b_1}{s} \sqrt{T} \right) \right] \\ + \frac{m_F e^{-rT}}{(r - m_F)} \left[\frac{2a}{s^2} \Phi\left(\frac{a}{s} \sqrt{T} \right) + \frac{2}{s\sqrt{T}} f\left(\frac{a}{s} \sqrt{T} \right) \right] \end{aligned} \tag{33}$$

and

$$\begin{aligned} C_2 = \left[\left(\frac{-a + b_2}{s^2} \right) - \left(\frac{2b_2}{s^2} \right) \Phi\left(\frac{b_2}{s} \sqrt{T} \right) - \frac{2}{s\sqrt{T}} f\left(\frac{b_2}{s} \sqrt{T} \right) \right] \\ - e^{-m_F T} \left[\left(\frac{-a + b_1}{s^2} \right) - \left(\frac{2b_1}{s^2} \right) \Phi\left(\frac{b_1}{s} \sqrt{T} \right) - \frac{2}{s\sqrt{T}} f\left(\frac{b_1}{s} \sqrt{T} \right) \right]. \end{aligned} \tag{34}$$

L'Hôpital's rule is needed to calculate the value of C_1 in the case when $m_F = r$ and the value of $\hat{b}$ when $m_F = 0$.

The expression for the equity value, and therefore the form of the optimal boundary, for the $m_F = 0$ case is different to that derived by Leland & Toft (1996). Leland & Toft assume that the total value of the firm includes the tax benefit of all future coupon payments, including coupons on bonds that are to be issued in the future. Similarly, they assume that the default costs of the debt that is currently outstanding and the debt that will be issued in the future are included in the calculation of the total value of the firm. One consequence of this is that the expression for the equity value derived by Leland & Toft was dependent on the costs of default, while the expression derived in this paper, (30), is independent of a .

As an example, Figure 1 shows the optimal boundary as calculated by Leland & Toft (1996) together with the optimal boundary given by (32) for different values of the asset volatility. The default-free interest rate r and the coupon rate c are both set at 6%, the net payout rate d is 5%, the effective tax rate is 15%, the proportion a of the asset value that is lost due to default costs is 40%, the principal value of debt is assumed to be constant over time, and the firm is assumed to issue bonds with an initial time-to-maturity T of 5 years.

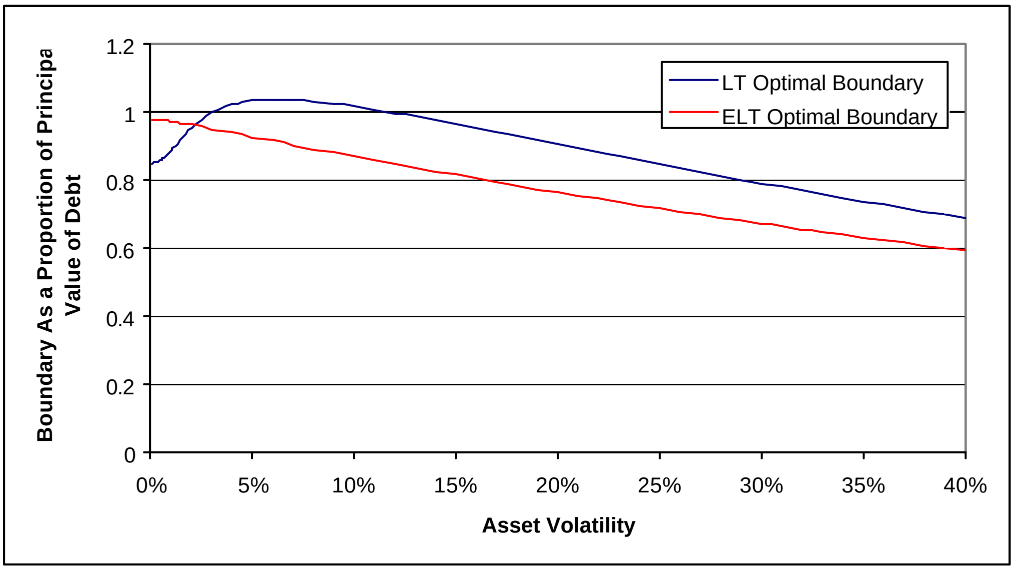

Figure 1: Optimal Default Boundary

Further tests on the two optimal default boundaries were performed using a wide range of input parameters. For all realistic values of the input parameters that were tested, the optimal default boundary given by (32) was found to be a monotonic decreasing function of the asset volatility for asset volatilities above 1% (for a small set of values of the input parameters, the boundary increases very slightly with asset volatility if the asset volatility is below 1%). On the other hand, Figure 1 shows that the boundary proposed by Leland & Toft is not a monotonic function of asset volatility for some realistic values of the input parameters[3]. Also, for the particular values used in Figure 1, it is optimal for a firm with an asset volatility of between 3% and 12% to default when the value of its total assets is above the principal value of the firm's debt if the Leland & Toft boundary is used. On the other hand, the optimal default boundary proposed in this section is below the principal value of debt for all values of the asset volatility.

Although the optimal default boundary derived by Leland & Toft was higher than the boundary given by (32) for most values of the asset volatility in the example above, it should be noted that there are many values of the input parameters where the optimal boundary proposed in this paper gives the higher default boundary. As an example, if d is reduced to 1%, a falls to 5% and all other parameters are kept the same as for Figure 1, the optimal boundary derived in this paper is higher than the default boundary derived by Leland & Toft for all asset volatilities less than 100%.

2.2 Debt Structure Consists of Perpetual Bonds

As in Leland (1994), it is now assumed that the debt structure of a firm consists of perpetual bonds. As a result, the model developed in this subsection will be referred to as the Extended Leland model (or EL for short). However, unlike in Leland, it is assumed that the firm continuously issues bonds, with the bonds issued in the interval $[s, s+ds]$ having a principal value of $m_F F_0 e^{m_F s} ds$. As a result, the principal value of all outstanding bonds at time t is

$$F_t = \int_{-\infty}^{t} m_F F_0 e^{m_F s} ds = F_0 e^{m_F t}, \tag{35}$$

consistent with Assumption 3. The time-t market value of the bonds that were issued in the interval $[s, s+ds]$, where $s \le t$, is equal to

$$\begin{aligned} \overline{SB}_t^s ds = c m_F F_0 e^{m_F s} ds \int_t^{\infty} e^{-r(u-t)} [1 - Q(X_t, u-t)] du \\ + (1-a) b m_F F_0 e^{m_F s} ds \int_t^{\infty} e^{-r(u-t)} q(X_t, u-t) du. \end{aligned} \tag{36}$$

[3] The numerical tests revealed that there are many sets of input parameters where the default boundary proposed by Leland & Toft first increases and then decreases with asset volatility in the manner illustrated in Figure 1.

A change of variable and an integration by parts on the first integral reveals that (36) is equal to

$$\overline{SB}_t^s ds = \frac{c}{r} m_F F_0 e^{m_F s} ds + \left((1-a)b - \frac{c}{r} \right) m_F F_0 e^{m_F s} ds G_2(X_t, \infty) \ . \tag{37}$$

The function $G_2(X_t, u)$ was given earlier by (16), and therefore, the time-t value of bonds that were issued in the interval $[s, s+ds]$ is

$$\overline{SB}_t^s ds = \frac{c}{r} m_F F_0 e^{m_F s} ds + \left((1-a)b - \frac{c}{r} \right) m_F F_0 e^{m_F s} ds X_t^{\frac{-a-b_2}{s^2}} , \tag{38}$$

where a and b_2 are given by (6) and (17) respectively. Note that the expression (38) is equal to the limit of (18) as $T \to \infty$, i.e. the time-t value of a perpetual bond is equal to the time-t value of a finite-maturity bond as its initial time-to-maturity tends to infinity.

The market value of a firm's debt at time t is found by integrating over the time-t value of all bonds that have been issued up to time t. As $F_t = F_0 e^{m_F t}$, the time-t value of a firm's debt is

$$\int_{-\infty}^{t} \overline{SB}_t^s ds \ = \ \frac{c}{r} F_t + \left((1-a)b - \frac{c}{r} \right) F_t X_t^{\frac{-a-b_2}{s^2}} \ . \tag{39}$$

As with the ELT model, it is assumed that the total value of the firm includes only the costs of default and the tax benefits of coupons on the debt that is currently outstanding. From an examination of (39), it is seen that the time-t value of the tax benefits and default costs are equal to

$$TB_t = \frac{(tax)c}{r} F_t \left(1 - X_t^{\frac{-a-b_2}{s^2}} \right) \tag{40}$$

and

$$DC_t = a\, b\, F_t X_t^{\frac{-a-b_2}{s^2}} \ . \tag{41}$$

Therefore, the total value of the firm at time t, v_t, is taken to be

$$v_t \ = \ V_t + TB_t - DC_t \ = \ V_t + \frac{(tax)c}{r} F_t \left(1 - X_t^{\frac{-a-b_2}{s^2}} \right) - a\, b\, F_t X_t^{\frac{-a-b_2}{s^2}} , \tag{42}$$

and the value of equity is given by the total value of the firm minus the market value of debt,

$$S_t \ = \ V_t - \frac{(1-tax)c}{r} F_t - \left(b - \frac{(1-tax)c}{r} \right) F_t X_t^{\frac{-a-b_2}{s^2}} \ . \tag{43}$$

Again, it should be noted that (43) is equal to the limit as $T \to \infty$ of (30), the expression for the value of equity in the ELT model. Further, if m_F is set equal to zero in (43), the expression for the value of equity derived by Leland (1994) is recovered.

As in the previous subsection, the optimal default boundary is assumed to satisfy the smooth-pasting condition

$$\left.\frac{\partial S_t}{\partial V_t}\right|_{V_t = V_t^B} = 0\,. \tag{44}$$

As a result, the optimal boundary is given by

$$V_t^B = \hat{b} F_t\,, \tag{45}$$

where

$$\hat{b} \;=\; \frac{\left(\frac{a+b_2}{s^2}\right)}{1+\left(\frac{a+b_2}{s^2}\right)}\left[\frac{(1-tax)c}{r}\right]. \tag{46}$$

2.3 Calibration using Equity Data

In both of the models developed in this section, the expressions for the value of a firm's equity contained three unobservable variables: the market value of the firm's assets V_t, the asset volatility s, and the net payout rate to security holders d. To find estimates for the values of these three variables, three equations are needed that link the unobservable variables to observable variables. One equation is provided by the expression for the firm's equity value, so two more equations are required.

In both models, the equity value is a function of the firm's asset value and the principal value of its outstanding debt,

$$\boldsymbol{S}_t = S(\boldsymbol{V}_t, F_t)\,. \tag{47}$$

An application of Itô's Lemma to (47) reveals that the market value of a firm's equity follows the process[4]

[4] Although Itô's Lemma can be used to derive the form of the drift term in (48), it is not necessary for the work here.

$$d\boldsymbol{S_t} = m_S(V_t, F_t)dt + \left(s V_t \frac{\partial S_t}{\partial V_t} \right) d\boldsymbol{W_t}. \tag{48}$$

This can be written as

$$d\boldsymbol{S_t} = \left(\frac{m_S(V_t, F_t)}{\boldsymbol{S_t}} \right) \boldsymbol{S_t} dt + s_t^S \boldsymbol{S_t} d\boldsymbol{W_t}, \tag{49}$$

so that by comparing (48) and (49), the equity volatility at time t, s_t^S, can be seen to equal

$$s_t^S = \frac{s V_t}{S_t} \frac{\partial S_t}{\partial V_t}. \tag{50}$$

In the case of the ELT model, a firm makes three sets of payments to security holders: dividend payments to equity-holders, and coupon and principal payments to debt-holders. However, the firm receives two sets of payments: the tax-sheltering value of the coupon payments made to debt-holders, and the money received from issuing new debt. Therefore, the net payout rate to security-holders is taken to be

$$d\,dt = \frac{d_t^S S_t + (1 - tax)cF_t + \tilde{F}e^{m_F(t-T)} - SB_t^t}{V_t} dt\ , \tag{51}$$

where d_t^S is the dividend yield at time t, $\tilde{F}e^{m_F(t-T)}dt$ is the principal payment for bonds that were issued in the interval $[t-T, t-T+dt]$, and $SB_t^t dt$ is the time-t market value of the bonds that are issued in the interval $[t, t+dt]$, which is given by (18). Therefore, the right-hand side of (51) provides an estimate of the net payout rate of the firm at time t. The net payout rate d is then fixed at this value as d is assumed to be constant over time (see Assumption 2).

In the case of the EL model, the firm does not make principal payments, and thus the net payout rate to security-holders is taken to be

$$d\,dt = \frac{d_t^S S_t + (1 - tax)cF_t - \overline{SB}_t^t}{V_t} dt, \tag{52}$$

where the time-t market value of the bonds that are issued in the interval $[t, t+dt]$, $\overline{SB}_t^t dt$, is given by (38). Note that (52) is the limit of (51) as $T \to \infty$.

Three equations linking the unobservable variables to observable variables have now been derived. Therefore, given values for the equity value S_t, equity volatility s_t^S, and the principal value of outstanding debt F_t of the reference firm as well as r, d_t^S, c, T, tax and a, the time-t market value of the reference firm's assets, the asset volatility and the net payout rate can be found by solving (30), (50) and (51) simultaneously in the case of the ELT model, and by solving (43), (50) and (52) simultaneously in the case of the EL model.

3. Mean-Reversion of Volatility

The two models developed in the previous section are now used in conjunction with equity data to derive a time series of calculated asset volatilities for a sample of firms. Unit root tests will then be performed on the time series of asset and equity volatilities to determine whether the processes are mean-reverting.

The equity price data that is used in this section was collected by Reuters and made publicly available through the Yahoo.com website. In this work, equity volatility is calculated using a simple moving average on the previous 250 days of price data. Thus, equal weighting is placed on all of the data points that are used in the calculation of equity volatility. Data that needs to be obtained from the balance sheet of a firm (the level of debt and the number of outstanding shares) was obtained from the Mergent Online database. The principal value of a firm's debt F_t is taken to be the book value of its total liabilities. All reference firms studied in this section are based in the US, where balance sheets are published quarterly. To calculate the book value of a firm's liabilities and the number of outstanding shares at intermediate months, it is assumed that the growth rate of both processes is constant during each quarter, e.g. if the debt of a firm is known at times t_0 and t_1, the debt at time $t \in [t_0, t_1]$ is taken to be

$$F_t = F_{t_0} \times \left(\frac{F_{t_1}}{F_{t_0}} \right)^{\frac{t-t_0}{t_1-t_0}}. \tag{53}$$

In this section, the ten-year US Treasury rate that is calculated by the Federal Reserve and published in their Statistical Release H.15 is converted into a continuously-compounded rate and then used as the default-free interest rate r. In the absence of historical data about the coupon rates that firms pay on their debt, there are two main approaches that can be taken.

The first is to use the coupon rate where the market value of a firm's debt is equal to the principal value of the debt. Therefore, firms with a low credit quality pay a higher coupon rate, so that the net payout rate to security holders d is greater for these firms *ceteris paribus*. Although this method may be valid for investment-grade firms, this can lead to unrealistically high coupon rates in the case of firms with a very low credit quality. A second approach, which is taken here, is to assume that the coupon rate is equal to $r+y$, where y is a constant. Therefore, the coupon rate is not affected by the credit quality of the firm, but firms with a low credit quality receive less money from issuing new debt. Again, this causes low credit quality firms to have a higher net payout rate to security holders than high-quality firms *ceteris paribus*. In this section, y is taken to be 0%, so that the coupon rate is equal to the default-free rate. The dividend yield at time t is taken to be

$$d_t^S = \sum_{i=1}^{n} \frac{p_{t_i}}{S_{t_i}}, \tag{54}$$

where the sum is over all dividend payments made in the previous year and p_{t_i} is the dividend payment made at time t_i. One-off special dividends are not included as they can lead to artificially high dividend yields, and therefore high payout rates, for short periods of time. Also, the growth rate of a firm's debt, m_F, is taken to be equal to the default-free interest rate r. In the case of the ELT model, the initial time-to-maturity of a firm's bonds, T, is assumed to be 8.5 years. Suppose that the remaining time-to-maturity of a firm's outstanding bonds is calculated by an average that is weighted by the principal value of the bonds, i.e.

$$\frac{1}{F_t}\int_{t-T}^{t}(s+T-t)\tilde{F}e^{m_F s}ds \quad = \quad \frac{T}{1-e^{-m_F T}} - \frac{1}{m_F}. \tag{55}$$

The mean 10-year default-free rate between December 1993 and December 2003 was 5.61%. When m_F is set equal to 5.61%, a value of $T = 8.5\text{ years}$ implies that a firm's outstanding bonds have a weighted-average remaining time-to-maturity of 4.59 years, close to the values reported by Stohs & Mauer (1996) for many credit classes. Following Leland (2004), the effective tax rate is taken to be 15%. Finally, empirical studies such as Andrade & Kaplan (1998) have shown that the costs of default are usually between 10% and 20% of the firm's asset value. Therefore, the default costs are taken to be 15% of the firm's asset value at the time of default, i.e. $a = 15\%$.

As an illustration, the volatility processes of a single firm, Delta Air Lines Inc., are shown. Figure 2 shows the observed equity volatility, and the asset volatility calculated using the ELT model, at monthly intervals between December 1993 and December 2003.

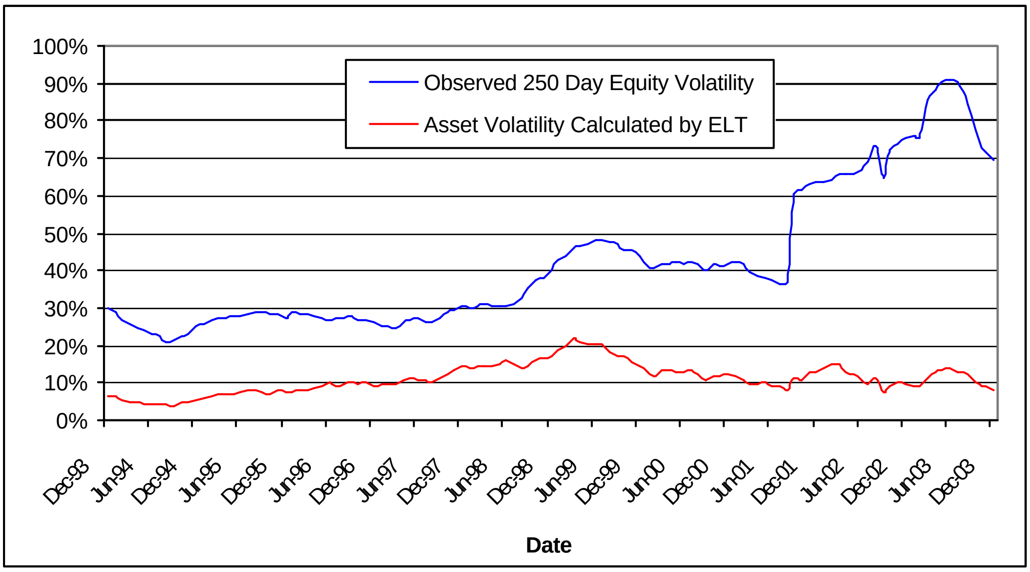

Figure 2: Asset Volatility and Equity Volatility of Delta Air Lines Inc.

The above graph has a number of interesting features. The equity volatility of Delta Air Lines increased sharply in September 2001 and then continued to rise throughout 2002 and early 2003. Even though the equity volatility fell in the second half of 2003, the level in December 2003 was still double that seen in August 2001. However, asset volatility behaved very differently. The increase in asset volatility in September 2001 is far less noticeable, and the asset volatility remained at a level throughout 2002 and 2003 that was similar to that seen in the 1990s.

To determine whether the volatility processes are mean-reverting, it is assumed that the volatility (either equity or asset) of firm i at time t, s_t^i, can be written as

$$\log(s_t^i) = a^i + e_t^i, \tag{56}$$

where the residuals e_t^i follow an AR(d+1) process,

$$e_t^i = \sum_{j=1}^{d+1} r_j^i e_{t-j}^i + u_t^i. \tag{57}$$

It will further be assumed that

$$u_t^i \sim N\left(0,(h^i)^2\right), \tag{58}$$

and that u_t^i is independent of u_{t-j}^i, where $j \geq 1$. By letting

$$h_j^i = \sum_{k=j+1}^{d+1} r_k^i, \quad 0 \leq j \leq d, \tag{59}$$

the relation (57) can be rewritten as

$$e_t^i = h_0^i e_{t-1}^i - \sum_{j=1}^{d} h_j^i (e_{t-j}^i - e_{t-j-1}^i) + u_t^i. \tag{60}$$

By substituting (56) into (60) and subtracting $\log(s_t^i)$ from both sides, the following relation is obtained

$$\log(s_t^i) - \log(s_{t-1}^i) = p^i + (h_0^i - 1)\log(s_{t-1}^i) - \sum_{j=1}^{d} h_j^i [\log(s_{t-j}^i) - \log(s_{t-j-1}^i)] + u_t^i, \tag{61}$$

where p^i is a constant. To test the null and alternative hypotheses,

$$H_0 : h_0^i = 1 \qquad\qquad H_1 : h_0^i < 1, \tag{62}$$

an augmented Dickey-Fuller test, as developed in Dickey & Fuller (1981), could be performed on (61). A rejection of the null hypothesis would indicate that the volatility process (or more precisely, the logarithm of the volatility process) reverts to a constant value.

However, this test is notorious for being very weak. To improve the power of the unit root test, the approach proposed by Elliott, Rothenberg & Stock (1996) is adopted. The null and alternative hypotheses are given by

$$H_0 : h_0^i = 1 \qquad\qquad H_1 : h_0^i = 1 - \frac{k}{N}, \tag{63}$$

where k is a constant and N is the number of observations in the time series. Elliott, Rothenberg & Stock showed that a value of $k = 7$ is often close to the value of k that optimises the power of the test, and thus this value of k is used here. The procedure proposed by Elliott, Rothenberg & Stock is to first detrend the volatility time series s^i, and then perform an augmented Dickey-Fuller test on the relation

$$\log(\bar{s}_t^i) - \log(\bar{s}_{t-1}^i) = \hat{h}_0^i \log(\bar{s}_{t-1}^i) - \sum_{j=1}^{d} h_j^i [\log(\bar{s}_{t-j}^i) - \log(\bar{s}_{t-j-1}^i)] + u_t^i, \tag{64}$$

where $\hat{h}_0^i = h_0^i - 1$ and $\bar{s}^i$ is the detrended volatility time series. This unit root test is often referred to as the DF-GLS test. The number of lag terms, d, is calculated by minimising the modified information criterion proposed by Ng & Perron (2001). The t-statistic of $\hat{h}_0^i$ is then compared to critical values determined by Cheung & Lai (1995) to determine whether the null hypothesis is rejected.

The volatility processes between December 1993 and December 2003 of a sample of fifty firms that are based in the US are investigated. To ensure that firms with a broad range of credit qualities are studied, ten firms that were rated Aa or above by Moody's Investors Services on December 31 2003 are chosen, and ten firms from each of the A, Baa, Ba and B ratings classes are also selected. All of the investment-grade firms in the sample are in the S&P 500. However, there are few B-rated firms in the S&P 500 with an equity history dating back to December 1992. Thus, some of the speculative-grade firms in the sample are not in the S&P 500, but instead come from the wider S&P Composite 1500[5].

First, at the end of each month in the period being studied, the equity volatility of each firm is measured (using 250 days of historical price data) and the asset volatility is calculated using the ELT and EL models. As a result, there are 121 data points in each of the three time series for each firm. Tables 1 and 2 give the t-statistics of $\hat{h}_0^i$ for the asset volatilities calculated by the two structural models and the equity volatility process for the fifty firms in the sample. The number of lag terms, *d*, that minimises the modified information criterion is given in parentheses in the tables. Then the equity volatility is measured, and the asset volatility is calculated using the two models, at the end of each year in the period being studied, so that there are eleven data points in the time series for each firm. The t-statistics of $\hat{h}_0^i$ for each firm are given in Table 3 and 4, and as before, the optimal number of lag terms is given in parentheses. In all four tables, if the p-value of $\hat{h}_0^i$ is less than 1%, i.e. the null hypothesis that the volatility process contains a unit root is rejected at the 1% significance level, the relevant t-statistic is denoted by three asterisks. The t-statistic is denoted by two asterisks if the p-value is between 1% and 5%, while a p-value of between 5% and 10% is shown by denoting the t-statistic with one asterisk. If the t-statistic of $\hat{h}_0^i$ is denoted by no asterisks, then the p-value of $\hat{h}_0^i$ is greater than 10%, i.e. the null hypothesis is not rejected at the 10% significance level.

[5] The S&P Composite 1500 is an equity index of 1500 US firms that combines the S&P 500, S&P MidCap 400 and the S&P SmallCap 600.

Firm Name	Rating	ELT Asset Volatility	EL Asset Volatility	Equity Volatility
Exxon Mobil Corp.	Aaa	-1.851 (1) *	-1.841 (1) *	-1.768 (1) *
General Electric Co.	Aaa	-1.961 (0) **	-1.787 (0) *	-1.175 (1)
Johnson & Johnson	Aaa	-1.422 (0)	-1.421 (0)	-1.428 (0)
Merck & Co. Inc.	Aaa	-1.021 (0)	-1.486 (1)	-1.603 (1)
Pfizer Inc.	Aaa	-1.058 (1)	-1.114 (1)	-1.352 (1)
3M Co.	Aa1	-1.714 (0) *	-1.645 (0) *	-1.543 (1)
Microsoft Corp.	Aa2	-1.449 (1)	-1.464 (0)	-1.538 (1)
Coca-Cola Co.	Aa3	-1.475 (1)	-1.477 (1)	-1.429 (1)
Colgate-Palmolive Co.	Aa3	-1.361 (1)	-1.372 (1)	-1.420 (1)
Home Depot Inc.	Aa3	-1.588 (2)	-1.645 (2) *	-1.332 (1)
BellSouth Telecommunications Inc.	A1	-1.800 (0) *	-1.879 (1) *	-1.910 (1) *
International Business Machines Corp.	A1	-1.301 (0)	-1.142 (0)	-1.492 (1)
Alcoa Inc.	A2	-1.655 (0) *	-1.576 (0)	-1.584 (1)
Baker Hughes Inc.	A2	-1.432 (1)	-1.331 (1)	-1.060 (1)
Caterpillar Inc.	A2	-2.202 (0) **	-2.010 (0) **	-1.228 (0)
McDonald's Corp.	A2	-1.969 (1) **	-1.952 (1) **	-1.812 (1) *
Target Corp.	A2	-1.857 (2) *	-1.818 (2) *	-1.688 (1) *
The Boeing Co.	A3	-1.627 (0) *	-1.819 (0) *	-2.216 (0) **
Campbell Soup Co.	A3	-1.294 (1)	-1.365 (1)	-1.579 (1)
Schering-Plough Corp.	A3	-1.205 (1)	-1.226 (1)	-1.166 (1)
Ford Motor Co.	Baa1	-1.561 (0)	-1.633 (0) *	-1.313 (1)
May Department Stores Co.	Baa1	-1.750 (0) *	-1.590 (0)	-1.523 (1)
Ryder System Inc.	Baa1	-2.510 (0) **	-1.923 (0) *	-1.280 (0)
Altria Group Inc.	Baa2	-2.037 (1) **	-1.991 (1) **	-1.850 (1) *
Black & Decker Corp.	Baa2	-3.809 (0) ***	-3.526 (0) ***	-1.812 (0) *
Kellogg Co.	Baa2	-1.541 (1)	-1.483 (1)	-1.262 (1)
Mattel Inc.	Baa2	-1.369 (0)	-1.252 (0)	-1.030 (0)
Clear Channel Communications Inc.	Baa3	-1.853 (0) *	-1.702 (0) *	-2.238 (1) **
Computer Associates International Inc.	Baa3	-1.816 (0) *	-1.496 (0)	-1.243 (0)
Eastman Kodak Co.	Baa3	-2.443 (0) **	-2.730 (0) ***	-1.863 (0) *

Table 1: t-statistics for Investment-Grade Firms (Monthly Frequency)

Firm Name	Rating	ELT Asset Volatility	EL Asset Volatility	Equity Volatility
Bowater Inc.	Ba1	-2.454 (0) **	-2.199 (0) **	-2.329 (1) **
Hilton Hotels Corp.	Ba1	-2.593 (0) ***	-2.460 (0) **	-1.308 (0)
Unisys Corp.	Ba1	-1.324 (0)	-1.355 (0)	-1.284 (0)
Apple Computer Inc.	Ba2	-1.124 (0)	-1.063 (0)	-0.771 (0)
Cummins Inc.	Ba2	-2.854 (0) ***	-2.333 (0) **	-1.169 (0)
Edison International	Ba2	-1.679 (0) *	-1.367 (0)	-1.287 (1)
Smithfield Foods Inc.	Ba2	-1.578 (0)	-1.318 (0)	-0.636 (0)
Westar Energy Inc.	Ba2	-1.473 (0)	-1.526 (1)	-1.103 (1)
Dana Corp	Ba3	-2.611 (0) ***	-2.333 (0) **	-1.895 (1) *
Georgia-Pacific Corp.	Ba3	-1.837 (0) *	-1.659 (0) *	-1.488 (1)
Delta Air Lines Inc.	B1	-2.591 (0) ***	-2.817 (0) ***	-1.154 (0)
Goodyear Tire & Rubber Co.	B1	-1.639 (0) *	-1.790 (0) *	-0.448 (1)
Xerox Corp.	B1	-2.210 (0) **	-1.978 (0) **	-1.264 (1)
Dillard's Inc.	B2	-1.800 (0) *	-1.982 (0) **	-1.299 (1)
Pep Boys - Manny, Moe & Jack	B2	-1.968 (0) **	-1.678 (0) *	-1.151 (0)
Advanced Micro Devices Inc.	B3	-1.278 (0)	-1.044 (0)	-0.966 (0)
CMS Energy Corp.	B3	-1.186 (0)	-0.661 (0)	-0.608 (1)
Kulicke & Soffa Industries Inc.	B3	-2.082 (0) **	-1.846 (0) *	-2.223 (1) **
Milacron Inc.	B3	-4.120 (0) ***	-2.776 (0) ***	-0.568 (1)
Williams Cos. Inc.	B3	-1.736 (0) *	-1.405 (0)	-1.180 (1)

Table 2: t-statistics for Speculative-Grade Firms (Monthly Frequency)

Firm Name	Rating	ELT Asset Volatility	EL Asset Volatility	Equity Volatility
Exxon Mobil Corp.	Aaa	-2.016 (0) *	-2.041 (0) *	-1.531 (1)
General Electric Co.	Aaa	-1.542 (0)	-1.524 (0)	-1.519 (0)
Johnson & Johnson	Aaa	-1.719 (1) *	-1.696 (1) *	-1.566 (1)
Merck & Co. Inc.	Aaa	-1.899 (0) *	-1.978 (0) *	-2.112 (0) *
Pfizer Inc.	Aaa	-1.639 (0)	-1.743 (0) *	-1.263 (1)
3M Co.	Aa1	-1.354 (0)	-1.351 (0)	-1.307 (0)
Microsoft Corp.	Aa2	-1.744 (0) *	-1.762 (0) *	-1.862 (0) *
Coca-Cola Co.	Aa3	-2.022 (0) *	-2.035 (0) *	-2.055 (0) *
Colgate-Palmolive Co.	Aa3	-1.392 (0)	-1.432 (0)	-1.652 (0)
Home Depot Inc.	Aa3	-1.755 (0) *	-1.785 (0) *	-1.174 (1)
BellSouth Telecommunications Inc.	A1	-2.263 (1) **	-2.184 (1) **	-1.773 (1) *
International Business Machines Corp.	A1	-1.691 (0) *	-1.706 (0) *	-2.110 (0) *
Alcoa Inc.	A2	-1.787 (0) *	-1.735 (0) *	-1.689 (0) *
Baker Hughes Inc.	A2	-1.399 (0)	-1.297 (0)	-1.124 (0)
Caterpillar Inc.	A2	-1.884 (0) *	-1.919 (0) *	-1.689 (0) *
McDonald's Corp.	A2	-2.204 (0) **	-1.702 (1) *	-1.438 (1)
Target Corp.	A2	-2.065 (0) *	-2.035 (0) *	-2.043 (0) *
The Boeing Co.	A3	-2.121 (0) **	-2.286 (0) **	-2.468 (0) **
Campbell Soup Co.	A3	-1.858 (0) *	-1.939 (0) *	-2.101 (0) *
Schering-Plough Corp.	A3	-1.975 (0) *	-2.030 (0) *	-2.092 (0) *
Ford Motor Co.	Baa1	-1.975 (0) *	-2.125 (0) **	-0.920 (1)
May Department Stores Co.	Baa1	-1.918 (0) *	-1.843 (0) *	-1.561 (0)
Ryder System Inc.	Baa1	-3.451 (0) ***	-2.931 (0) **	-2.173 (0) **
Altria Group Inc.	Baa2	-2.304 (0) **	-2.244 (0) **	-2.072 (0) *
Black & Decker Corp.	Baa2	-2.421 (0) **	-2.190 (1) **	-1.302 (1)
Kellogg Co.	Baa2	-2.467 (0) **	-2.478 (0) **	-2.008 (0) *
Mattel Inc.	Baa2	-1.936 (0) *	-1.821 (0) *	-1.590 (0)
Clear Channel Communications Inc.	Baa3	-2.603 (1) **	-2.550 (1) **	-1.881 (1) *
Computer Associates International Inc.	Baa3	-1.951 (0) *	-2.114 (0) *	-1.369 (1)
Eastman Kodak Co.	Baa3	-2.347 (0) **	-2.932 (0) **	-1.910 (0) *

Table 3: t-statistics for Investment-Grade Firms (Annual Frequency)

Firm Name	Rating	ELT Asset Volatility	EL Asset Volatility	Equity Volatility
Bowater Inc.	Ba1	-3.649 (0) ***	-3.540 (0) ***	-3.136 (0) **
Hilton Hotels Corp.	Ba1	-3.876 (0) ***	-3.551 (0) ***	-1.999 (0) *
Unisys Corp.	Ba1	-2.325 (0) **	-2.247 (0) **	-1.858 (0) *
Apple Computer Inc.	Ba2	-1.146 (0)	-1.127 (0)	-1.363 (0)
Cummins Inc.	Ba2	-2.987 (0) **	-2.072 (0) *	-1.800 (0) *
Edison International	Ba2	-2.195 (0) **	-1.864 (0) *	-1.599 (0)
Smithfield Foods Inc.	Ba2	-2.322 (0) **	-2.488 (0) **	-1.216 (1)
Westar Energy Inc.	Ba2	-2.200 (0) **	-1.945 (0) *	-1.541 (0)
Dana Corp	Ba3	-2.466 (1) **	-2.285 (1) **	-1.600 (0)
Georgia-Pacific Corp.	Ba3	-1.791 (0) *	-1.785 (0) *	-1.634 (0)
Delta Air Lines Inc.	B1	-2.534 (0) **	-3.224 (0) ***	-1.087 (0)
Goodyear Tire & Rubber Co.	B1	-1.770 (0) *	-2.088 (0) *	-0.475 (0)
Xerox Corp.	B1	-2.306 (0) **	-2.018 (0) *	-1.323 (0)
Dillard's Inc.	B2	-2.200 (0) **	-3.289 (0) ***	-1.457 (0)
Pep Boys - Manny, Moe & Jack	B2	-2.065 (1) *	-1.594 (1)	-1.519 (0)
Advanced Micro Devices Inc.	B3	-1.791 (0) *	-1.876 (0) *	-2.007 (0) *
CMS Energy Corp.	B3	-2.220 (0) **	-1.493 (0)	-0.697 (0)
Kulicke & Soffa Industries Inc.	B3	-2.251 (0) **	-2.241 (1) **	-1.370 (1)
Milacron Inc.	B3	-4.534 (0) ***	-3.017 (0) **	-0.212 (0)
Williams Cos. Inc.	B3	-2.015 (0) *	-1.858 (0) *	-0.420 (1)

Table 4: t-statistics for Speculative-Grade Firms (Annual Frequency)

Looking at the sample of fifty firms as a whole, the null hypothesis that the asset volatility process contains a unit root is rejected at the 10% significance level either at the monthly or annual frequency for 46 of the firms, regardless of whether the ELT or EL model is used. As a rejection of the null hypothesis indicates that the volatility process reverts to a constant value, this provides strong evidence that the asset volatility of many firms is mean-reverting. On the other hand, the hypothesis that the equity volatility process of a firm contains a unit root is rejected at the 10% significance level for 26 firms.

The weakest evidence of volatility being mean-reverting is for firms with a Aaa or Aa rating. If the ELT model is used to calculate asset volatility, the null hypothesis is rejected at the 10% significance level for three firms if monthly data is used and for six firms if annual data is used. On the other hand, if the EL model is used, slightly more firms are found to have an asset volatility process that is mean-reverting: the hypothesis of a unit root is rejected at the 10% significance level for four firms if monthly data is used and for seven firms if the data is of an annual frequency. However, at the 10% significance level, only one firm with a Aa rating or higher (Exxon Mobil) had an equity volatility that was mean-reverting if monthly data is used, while the equity volatility of three firms was seen to be mean-reverting when annual data is used. Therefore, even though the group of ten firms with a rating of Aaa or Aa provides the weakest evidence of asset volatility being mean-reverting, the evidence is far stronger for asset volatility to be mean-reverting than for equity volatility to be mean-reverting.

One reason that the long-term mean asset volatility of a firm may change is if the nature of the firm changes, for instance, through a merger or acquisition. As an example, if a large firm with a low asset volatility buys a smaller firm with a high asset volatility, it is likely that the asset volatility of the large firm will increase. Out of the firms in the sample, those with Aaa or Aa ratings went through the greatest amount of merger and acquisition activity between 1993 and 2003, and this may explain why this group of firms shows the weakest evidence of asset volatility being mean-reverting.

For the forty firms with a rating of A or lower, the evidence that firm-specific asset volatility is mean-reverting is seen to be strong, with the null hypothesis of a unit root being rejected at the 10% significance level for thirty-eight firms if annual data is used together with the ELT model (the null hypothesis is not rejected for Baker Hughes and Apple Computer only).

However, the evidence that equity volatility is mean-reverting becomes weaker as the credit quality of firms decreases: the null hypothesis is rejected at the 10% significance level for eight, five, four and one of the groups of firms with an A, Baa, Ba and B rating respectively. This is a consequence of the leverage effect mentioned in the introduction. The credit quality of many of the firms with a B-rating had fallen in recent years and their leverages had increased substantially. Consistent with the leverage effect, their equity volatilities had risen noticeably during this period, so that the hypothesis of a unit root could not be rejected. The one firm with a B-rating whose equity volatility is seen to be mean-reverting is Advanced Micro Devices, whose senior unsecured debt had a rating of Ba or B throughout the period being studied. As a result, the leverage of this firm did not vary as much as the other nine firms with a B-rating, and thus it is not too surprising that the null hypothesis that its equity volatility process contains a unit root is rejected at the 10% significance level.

The asset volatilities implied by the EL model provide slightly weaker evidence for mean-reversion of asset volatilities, with the null hypothesis that the asset volatility process contains a unit root being rejected at the 10% significance level for thirty-six of the forty firms with a rating of A or below when annual data is used. Although the t-statistics of $\hat{\eta}_0^i$ for the ELT and EL models were similar for investment-grade firms, there was a greater differential for the speculative-grade firms. For many values of the input parameters, the optimal default boundary is lower for the EL model than for the ELT model *ceteris paribus*[6]. Hence, for firms with a very low equity value, the value of the firm's assets will be lower if the EL model is used. As a consequence of (50), the asset volatility implied by the EL model will therefore be higher than that implied by the ELT model for firms of a poor credit quality. However, far from the default boundary, the asset volatilities given by the two structural models are similar.

Although the evidence of volatility being mean-reverting is weaker when monthly data is used, the evidence is stronger for the mean-reversion of asset volatility than for the mean-reversion of equity volatility. Out of the sample of fifty firms, the null hypothesis that the asset volatilities implied by the ELT and EL models contains a unit root is rejected at the

[6] In the ELT model, the debt structure of a firm consists of finite-maturity bonds, and therefore the firm has to pay coupons and the principal of debt that matures. However, the debt structure in the EL model consists of perpetual bonds, so that the firm only has to pay coupons. Therefore, a firm is more likely to remain solvent if the value of its assets is far below the principal value of outstanding debt in the EL model than in the ELT model. As a result, the optimal default boundary in the EL model is often below that in the ELT model.

10% significance level for thirty and twenty-seven firms respectively. However, the null hypothesis that the equity volatility process contains a unit root is rejected at the 10% significance level for only twelve of the fifty firms when monthly data is used. Focusing on the forty firms with a rating of A or below, the asset volatilities implied by the ELT and EL models were found to be mean-reverting at the 10% significance level for twenty-seven and twenty-three firms respectively, while equity volatility was mean-reverting for only eleven firms.

4. Conclusions

In this paper, two structural models with endogenous default barriers were extended to allow the principal value of a firm's debt to grow at a constant rate. The value of the tax benefits and the costs of default were calculated in such a way that for realistic values of the input parameters, the optimal default boundary is a monotonic decreasing function of the asset volatility. Also, the Extended Leland model was seen to be a special case of the Extended Leland & Toft model, so that all expressions in the EL model are equal to the limit of the corresponding expressions in the ELT model as the initial time-to-maturity of the bonds tends to infinity.

These two models were then used in conjunction with observable equity data to calculate implied asset volatilities. Unit root tests were applied to the implied asset volatility and equity volatility processes to determine whether the processes are mean-reverting. Evidence that asset volatility is mean-reverting was found for forty-six of the fifty firms in the sample, regardless of which of the two structural models were used. Further, the number of firms whose asset volatility is mean-reverting was approximately the same for each credit class, apart from firms with a rating of Aa or above, which provided the weakest evidence. However, the number of firms whose equity volatility is mean-reverting was in general lower for the poorer credit classes, consistent with the leverage effect.

As mentioned in the introduction, structural models with a constant asset volatility give equity volatilities that are positively related to a firm's leverage, consistent with empirical evidence. This paper suggests a slightly stronger result: the equity volatilities that are observed in the market are consistent with a mean-reverting asset volatility for the majority of firms.

The consequences of this result are many and varied, both in terms of implications for current models and for suggesting a possible direction for future research. First, this is further evidence that a structural model with a constant asset volatility provides benefits to both equity and debt modelling, and is more consistent with empirical evidence than, say, assuming that the equity volatility of a firm is constant. In the structural approach, changes in equity volatility can be split into two components: change due to variations in a firm's leverage (through the leverage effect) and change due to variations in a firm's asset volatility. The result that this second component is mean-reverting has consequences for the modelling of equity and the pricing of equity options, e.g. if a firm's asset volatility is below its long-term mean, then its equity volatility is likely to rise in the future provided that the firm's leverage does not fall significantly. Also, the mean-reversion of asset volatility has implications for the calibration of structural models, the modelling of debt and the pricing of corporate bonds and credit derivatives, e.g. that a long-term measure of asset volatility should be used when pricing long-dated credit products. Furthermore, this paper suggests that a structural credit model with a stochastic asset volatility that is mean-reverting would be a powerful tool. As the model would be consistent with equity data, it could be used to model equity prices and price equity derivatives. Being a structural model, it could also be used to price debt and credit derivatives; depending on how consistent these debt prices are with empirical data, the model could be used in the integration of equity and debt modelling.

This paper looked at equity volatility calculated using 250 days of historical price data. For firms where there is a long history of option prices available, the implied equity volatilities could be used to calculate asset volatilities. It would be interesting to note whether the implied equity volatilities are also consistent with a mean-reverting asset volatility process.

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