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Asset Pricing and Hedging in Financial Markets with Transaction Costs: An Approach Based on the von Neumann-Gale Model *

M.A.H. Dempster [a], I.V. Evstigneev [b] and M.I. Taksar [c]

Abstract

The paper develops a general discrete-time framework for asset pricing and hedging in financial markets with proportional transaction costs and trading constraints. The framework is suggested by analogies between dynamic models of financial markets and (stochastic versions of) the von Neumann-Gale model of economic growth. The main results are hedging criteria stated in terms of "dual variables"—consistent prices and consistent discount factors. It is shown how these results can be applied to a number of specialized models involving transaction costs and portfolio restrictions.

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1 Introduction

Von Neumann's [63] model of economic growth, generalised by Gale [26], was one of the first models in mathematical economics that served as the basis for a rich and interesting theory. This theory, in its classical form, was developed in the 1950s and 1960s. It was purely deterministic: it did not reflect the influence of random factors on economic growth. First attempts aimed at the construction of stochastic analogues of the von Neumann–Gale model were undertaken in the 1970s by Dynkin [17], [18, Chapter 9], Radner [52] and their research groups. However, the initial attack on the problem left many questions unanswered, and a substantial progress in the field was made only in the 1990s (see the survey by Evstigneev and Schenk-Hoppé [21]).

The main idea of this work is the observation that the stochastic version of the von Neumann–Gale model can serve as a convenient vehicle for analyzing the dynamics of financial markets. We reinterpret in financial terms the basic notions related to the model: commodity vectors—portfolios of assets; technology constraints—self-financing (solvency) constraints; paths in the model—self-financing trading strategies, etc. The study aims at the systematic development of this approach with the view to its application to asset pricing and hedging in incomplete financial markets with transaction costs and trading constraints.

The well-known theory of asset pricing, based on the concept of arbitrage, goes back to Black, Scholes, Merton, Kreps, Harrison, Pliska and others, whose work is now regarded as the classics of Mathematical Finance (for introductory expositions see e.g. Pliska [51], Björk [6] and Föllmer and Schied [24]). This theory, in its standard form, deals with the case of a frictionless market, where there are no transaction costs and no portfolio constraints—in particular, short sales of all assets are allowed. Moreover, a clear and unique solution to the asset pricing problem can be obtained only under the assumption that the market under consideration is complete. In this case, the "fair" (no-arbitrage) price of a derivative security is equal to the expectation with respect to the unique martingale measure of the security's discounted payoff.

Attempts to build asset pricing models for real financial markets with transaction costs and trading constraints, generalizing the above idealized scheme, have been undertaken by many researchers (e.g., Bensaid, Lesne, Pagès and Scheinkman [4], Cvitanić and Karatzas [12, 13], Soner, Shreve and Cvitanić [60], Jouini and Kallal [35–37], Jouini [33, 34], Föllmer and Kramkov [25], Carassus, Pham and Touzi [10], Napp [48], Jouini and Napp [38], Kabanov and Stricker [42], Kabanov [39], Stettner [62], Schachermayer [58], Cherny [11] and others, see further references below in the paper). In spite of the large amount of work done, the previous studies have not led to a complete and convenient framework unifying the results available. The main difficulties appearing in this field are of a conceptual nature. In particular, the notion of an equivalent martingale measure—playing a central role in the case of a frictionless, and especially complete, market—appears to be inadequate in the general context. It has to be replaced in more realistic models by other, more complex notions. Moreover, transaction costs and market incompleteness lead to the failure of the no-arbitrage pricing principle in general: it does not provide a justified algorithm for asset valuation in the presence of market frictions. Instead of no arbitrage, the principle of hedge pricing—according to which the price of a contingent claim is defined as the minimum level of initial wealth needed to hedge[1] the contingent claim—comes to the fore.

We develop the hedge pricing principle using the parallelism between paths of economic dynamics in the von Neumann–Gale model and hedging strategies in a dynamic model of a financial market. We show that the problem of characterization of the set of initial states of an economic system from which the given state at the end of the planning horizon can be attained is analogous to the problem of characterization of the set of initial endowments sufficient for hedging the given contingent claim. Dual paths in the von Neumann–Gale model are counterparts of consistent valuation systems—the notion which replaces in the context of a financial market with frictions that of an equivalent martingale measure. By using this notion, we give a general solution to the hedging problem and then show what specific forms this solution takes on in various specialized models.

A consistent valuation system includes (a) a pair of discount factors providing a relative valuation of initial endowments and contingent claims at the beginning and at the end of the time period under consideration and (b) a sequence of consistent asset prices defined for each moment of time within this period. We show that the existence of discount factors mentioned in (a) is equivalent to the no-arbitrage hypothesis. This result has essentially the same nature as various results in economic theory involving the characterization of Pareto-optimal states in terms of positive linear functionals (see e.g. Aliprantis, Brown and Burkinshaw [2, Section 3.5]). Having constructed consistent discount factors, we construct consistent prices as dual variables (Kuhn–Tucker multipliers) relaxing intertemporal balance constraints in a dynamic optimization problem. Such dual variables are analogous to "shadow prices", well-known in mathematical economics (e.g. Birchenhall and Grout [5, Chap-

[1]The notion of hedging we deal with (see the definitions in Section 2) covers what is often referred to as "superhedging" or "superreplication"—e.g. Föllmer and Schied [24].

ter 9]).

The paper is organized as follows. In Section 2 we describe the basic data of the model. In Section 3, we state the general hedging problem and define the notion of consistent discount factors. In Section 4, we formulate the basic assumptions of the model. Section 5 provides no-arbitrage and hedging criteria in terms of consistent discount factors. Section 6 defines and discusses the notion of a consistent valuation system (combining consistent prices and discount factors). No-arbitrage and hedging criteria based on this notion are obtained in Section 7. Section 8 contains some auxiliary material needed for the further analysis of specialized models. Section 9 examines a conventional model with proportional transaction costs and trading constraints. In Section 10, we apply these results to the special case of a market without transaction costs but with portfolio restrictions. Section 11 discusses the classical case of a frictionless market. Sections 12 and 13 analyze two different versions of a currency market model. In Section 14, we discuss some basic concepts related to the von Neumann–Gale model and their links to the theory developed in the paper. The Appendix assembles some general mathematical facts (basically, from convex analysis) used in this work.

2 Dynamic securities market model with transaction costs and trading constraints

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T = \mathcal{F}$ a sequence of σ-algebras. Sets in the σ-algebra $\mathcal{F}_t$ are interpreted as events observable prior to date t. For any natural number $n \geq 1$ and any $t = 0, ..., T$, we denote by $\mathcal{L}_t(n)$ the linear space of $\mathcal{F}_t$-measurable random vectors with values in an n-dimensional Euclidean space $\mathbb{R}^n$. We normally do not distinguish between random vectors if they coincide almost surely (a.s.). We omit "a.s." when this does not lead to ambiguity.

For each $t \geq 0$ and $n \geq 1$, vectors $y(\omega) = (y^1(\omega), ..., y^n(\omega))$ in $\mathcal{L}_t(n)$ represent *contingent portfolios* of n *assets* (*securities*). The ith coordinate $y^i(\omega)$ of the vector $y(\omega)$ indicates the amount (the number of "physical units") of asset i in the portfolio $y(\omega)$. The assumption of $\mathcal{F}_t$-measurability of a vector function $y(\omega)$ in $\mathcal{L}_t(n)$ expresses the fact that the portfolio $y(\omega)$ can be chosen by an investor based on observation of events occurring at date t or earlier.

Trading on the market is possible at any of the dates $t = 0, 1, ..., T$. For each $t = 0, ..., T-1$, a natural number n_t is given. At date 0 an investor can purchase n_0 kinds of assets; at any next date $t = 1, ..., T-1$, n_{t-1} kinds of assets can be sold and n_t kinds of assets can be purchased; and at date T, an investor can sell n_{T-1} kinds of assets. Two natural numbers m_0 and m_T are given. Elements of the spaces $\mathcal{L}_0(m_0)$ and $\mathcal{L}_T(m_T)$ are interpreted as *initial endowments* and *contingent claims*, respectively.

In the model we study, the sets

$$Y_0(\omega) \subseteq \mathbb{R}^{n_0}, \tag{1}$$

and

$$Z_t(\omega) \subseteq \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{n_t},\ t = 1, ..., T-1, \tag{2}$$

are given for each $\omega \in \Omega$. A sequence

$$(y_0, y_1, ..., y_{T-1}),\ y_t \in \mathcal{L}_t(n_t),\ t = 0, ..., T-1, \tag{3}$$

is called a (feasible) *trading strategy* if, with probability one,

$$y_0(\omega) \in Y_0(\omega), \tag{4}$$

$$(y_{t-1}(\omega), y_t(\omega)) \in Z_t(\omega),\ t = 1, ..., T-1. \tag{5}$$

An investor implementing a trading strategy $(y_0, y_1, ..., y_{T-1})$ starts at time 0 with the *initial portfolio* $y_0 \in \mathcal{L}_0(n_0)$. The set of admissible portfolios at time 0 is specified by the constraint $y_0(\omega) \in Y_0(\omega)$ (a.s.). At the beginning of each date $t = 1, 2, ..., T-1$, the investor possesses the portfolio y_{t-1}, that was created at the previous date. The dimension of the vector y_{t-1} is n_{t-1}, and y_{t-1} depends $\mathcal{F}_{t-1}$-measurably on ω, so that $y_{t-1} \in \mathcal{L}_{t-1}(n_{t-1})$. Trading on the market, the investor *rebalances* the portfolio y_{t-1} into y_t, which is possible if and only if $(y_{t-1}(\omega), y_t(\omega)) \in Z_t(\omega)$ almost surely (a.s.). There are n_t assets available for purchase at date t, and the choice of y_t is made based on the observation of events in $\mathcal{F}_t$, hence $y_t \in \mathcal{L}_t(n_t)$. The process of trading terminates at date $T-1$, when the investor constructs the *terminal portfolio* y_{T-1}.

When speaking of portfolio rebalancing, we always presume that the investor can buy assets for the new portfolio y_t only at the expense of selling some assets contained in the old one, y_{t-1}, i.e. we mean rebalancing under the assumption of *self-financing*. Generally, the operations of buying and selling assets involve transaction costs. Furthermore, not all portfolios might be admissible. For example, it might be required that admissible portfolio vectors must be non-negative: when *borrowing* and *short sales* are ruled out. All such restrictions—the portfolio *admissibility constraints* and *self-financing constraints*—are specified by the sets $Z_t(\omega)$, $t = 1, ..., T-1$.

Further, in the model under consideration, we are given sets

$$V_0(\omega) \subseteq \mathbb{R}^{m_0} \times \mathbb{R}^{n_0}, \tag{6}$$

and

$$V_T(\omega) \subseteq \mathbb{R}^{n_{T-1}} \times \mathbb{R}^{m_T}, \tag{7}$$

describing possibilities of constructing initial portfolios and liquidating terminal ones. A sequence

$$(v_0, y_0, ..., y_{T-1}, v_T),$$

where $v_t \in \mathcal{L}_t(m_t),\ t = 0, T$, and $(y_0, ..., y_{T-1})$ is a feasible trading strategy, is called a *hedging strategy* if

$$(v_0, y_0) \in V_0(\omega), \tag{8}$$

$$(y_{T-1}, v_T) \in V_T(\omega). \tag{9}$$

An investor implementing the hedging strategy $(v_0, y_0, ..., y_{T-1}, v_T)$ starts at date 0 with the initial endowment v_0 and constructs the initial portfolio y_0. The set of feasible pairs (v_0, y_0) is specified by (8) and (1). At each of the dates $t = 1, 2, ..., T-1$, the investor rebalances his/her old portfolio y_{t-1} into the new one, y_t. At date T, the investor *liquidates* the terminal portfolio y_{T-1} with the view of *hedging* the contingent claim v_T. The set of those pairs (y_{T-1}, v_T) for which this is possible is specified by (9).

Generally, the dimensions m_0 and m_T of vectors v_0 and v_T, specifying initial endowments and contingent claims, may be greater than one, but in the classical case, we have $m_0 = m_T = 1$. Then initial endowments and contingent claims are measured in terms of a numeraire (e.g. cash). Cases where m_0 and m_T are greater than one correspond to situations when there are several reference currencies in the market under consideration (for example, euro, dollar and the national currency, if different from the former two). An important special case is when all the traded assets are currencies—see Sections 12 and 13.

A contingent claim $v_T \in \mathcal{L}_T(m_T)$ may be thought of as a contract prescribing the delivery of some specified amount of each of the m_T assets, e.g. reference currencies, depending on the random situation ω at date T. If the dimension m_T of the random vector v_T is equal to one, the contract requires the payment of a specified amount of cash contingent on the random situation

ω. An investor who follows a hedging strategy $(v_0, y_0, ..., y_T, v_T)$ liquidates at date T the portfolio $y_{T-1} \in \mathcal{L}_{T-1}(n_{T-1})$ obtaining a sum of cash which hedges the contingent claim v_T.

To illustrate the above general definitions, consider the classical model of a frictionless market—without transaction costs and trading constraints. In this model, we have $m_0 = m_T = 1$ and $n_0 = ... = n_{T-1} = n$. For each $t = 0, ..., T$, we are given an $\mathcal{F}_t$-measurable n-dimensional random vector $S_t = S_t(\omega) \geq 0$ specifying the market prices of the n assets at date t. The portfolio rebalancing constraints are defined by the sets

$$Z_t(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n : S_t(\omega)a = S_t(\omega)b\},\ t = 1, ..., T-1. \tag{10}$$

There are no restrictions on the initial portfolios, so that

$$Y_0(\omega) := \mathbb{R}^n. \tag{11}$$

The sets characterizing possibilities of creating initial portfolios from initial endowments and possibilities of liquidating a terminal portfolio with the view to hedging contingent claims are as follows:

$$V_0(\omega) := \{(a, b) \in \mathbb{R}^1 \times \mathbb{R}^n : a = S_0(\omega)b\}, \tag{12}$$

$$V_T(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^1 : S_T(\omega)a = b\}. \tag{13}$$

Alternatively, one can define $V_T(\omega)$ by

$$V_T(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^1 : S_T(\omega)a \geq b\}, \tag{14}$$

replacing the equality $S_T(\omega)a = b$ by the inequality $S_T(\omega)a \geq b$. The latter definition corresponds to what is often called *superhedging.*[2] An investor liquidating a portfolio y_T at date T can (super-)hedge a contingent claim v_T if $S_T(\omega)y_T \geq v_T$ (a.s.).

In this paper, we will consider various examples of the data of the model Y_0, Z_t and V_t taking into account transaction costs and trading constraints.

[2] A contingent claim is conventionally called superhedgeable, if there exists a self-financing trading strategy (with the given initial endowment) which obtains a final payoff that is not less almost surely than the amount of the contingent claim. In the model under consideration, this notion is a special case of the general notion of hedging defined in terms of the set $V_T(\omega)$. It corresponds to those cases when $V_T(\omega)$ contains for each pair of vectors (a, b) any pair (a, b') with $b' \leq b$.

3 Hedging problem and consistent discount factors

We shall say that an *initial endowment* $v_0 \in \mathcal{L}_0(m_0)$ *allows the hedging of a contingent claim* $v_T \in \mathcal{L}_T(m_T)$ if there exists a hedging strategy of the form $(v_0, y_0, ..., y_{T-1}, v_T)$. The main question we are going to consider is as follows. Suppose a contingent claim $v_T \in \mathcal{L}_T(m_T)$ is given. How can we characterize the set of initial endowments $v_0 \in \mathcal{L}_0(m_0)$ allowing the hedging of v_T? Let $\mathcal{H}$ denote the set of pairs $(v_0, v_T) \in \mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T)$ such that v_0 allows the hedging of v_T. We are interested in the characterization of the set $\mathcal{H}$.

The above question is of fundamental importance for asset pricing. Suppose $m_0 = 1$, initial endowments $v_0 \in \mathcal{L}_0(1)$ are constants (the initial σ-algebra $\mathcal{F}_0$ is trivial) and the set $\{v_0 : (v_0, v_T) \in \mathcal{H}\}$ is non-empty and contains a smallest element. Then this element—the minimum initial endowment sufficient to hedge the contingent claim v_T—is called the *hedging price* of the contingent claim v_T.

We introduce a key concept in terms of which a solution of the general hedging problem will be given. Let $q_0 \in \mathcal{L}_0(m_0)$ and $q_T \in \mathcal{L}_T(m_T)$ be strictly positive vector functions such that, for all $(v_0, v_T) \in \mathcal{H}$, the expectations Eq_0v_0 and Eq_Tv_T with respect to the measure P are well-defined and finite. We shall say that q_0, q_T is a pair of *consistent discount factors* if

$$Eq_0v_0 \geq Eq_Tv_T, \ (v_0, v_T) \in \mathcal{H}. \tag{15}$$

If $m_0 = m_T = 1$, i.e. initial endowments and contingent claims are scalars, then q_0 and q_T are scalars as well. The discount factors q_0 and q_T provide relative valuation of initial endowments at date 0 and contingent claims at date T. According to (15), this relative valuation is such that the *expected discounted profit* $Eq_Tv_T - Eq_0v_0$ is non-positive for any pair (v_0, v_T), as long as v_T can be hedged starting from the initial endowment v_0. If initial endowments v_0 and/or contingent claims v_T are vectors, e.g. when there are several reference currencies in the market, then q_0 and q_T are vectors as well. Their coordinates are discount factors providing relative valuations of the currencies at dates 0 and T. By a convenient abuse of terminology, we refer to q_0 and q_T as consistent discount factors rather than "consistent vector discount factors".

We note that if $\mathcal{H}$ is a linear space (which is so in the classical model of a frictionless market—see (10)–(13)), then condition (15) is equivalent to the following:

$$Eq_0v_0 = Eq_Tv_T, \ (v_0, v_T) \in \mathcal{H}. \tag{16}$$

4 Basic assumptions

We will analyze the model at hand under the assumption that, for each ω, the sets $Z_t(\omega)$ $(t = 1, ..., T-1)$, $Y_0(\omega)$ and $V_t(\omega)$ $(t = 0, T)$ are closed *cones*[3]. In view of the assumption imposed, the model takes into account *proportional* transaction costs. This is reflected by the property that if (a, b) is a pair of portfolios in $Z_t(\omega)$, then $(\lambda a, \lambda b) \in Z_t(\omega)$ for all $\lambda \geq 0$. Furthermore, if portfolios b and b' can be obtained by rebalancing a and a', respectively, then $b + b'$ can be obtained by rebalancing $a + a'$. Various models involving proportional transaction costs considered in the literature lead to constraint sets $Z_t(\omega)$ $(t = 1, ..., T-1)$ possessing these properties. Recall that $Z_t(\omega)$ (as well as $Y_0(\omega)$) also incorporates restrictions on admissible portfolios. The assumption that the sets $Y_0(\omega), Z_1(\omega), ..., Z_{T-1}(\omega)$ are cones reflects the fact that the admissibility constraints (such as short sales constraints) in our model are expressed in terms of proportions between holdings of different assets in admissible portfolios. The analogous considerations apply to the cones $V_0(\omega)$ and $V_T(\omega)$ describing the possibilities of portfolio creation and liquidation, respectively.

In this work, we will concentrate on the modeling issues and avoid technicalities as much as possible. Therefore we will assume throughout the paper that the *probability space* Ω *is finite*. Generalizations of some of the results to the case of a general Ω will be discussed in a subsequent publication.

We will assume that the sets $Z_t(\omega)$ $\mathcal{F}_t$-measurably depend on ω, which means—in the case of a finite Ω—that there exists a partition of Ω into $\mathcal{F}_t$-measurable sets on each of which $Z_t(\omega)$ is constant. We will also assume that $V_t(\omega)$ are $\mathcal{F}_t$-measurable $(t = 0, T)$ and $Y_0(\omega)$ is $\mathcal{F}_0$-measurable.

As long as Ω is finite, all the linear spaces of vector functions of ω we consider are finite-dimensional. We will assume that they are endowed with the conventional Euclidean topology. We will suppose that the following condition holds.

(C) The set $\mathcal{H}$ is closed.

Recall that $\mathcal{H}$ consists of those pairs $(v_0, v_T) \in \mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T)$ for which there exists a hedging strategy $(v_0, y_0, ..., y_{T-1}, v_T)$. Thus $\mathcal{H}$ is the projection on $\mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T)$ of the set $\mathcal{T}$ of all hedging strategies $(v_0, y_0, ..., y_{T-1}, v_T) \in \mathcal{L}_0(m_0) \times \mathcal{L}_0(n_0) \times ... \times \mathcal{L}_{T-1}(n_{T-1}) \times \mathcal{L}_T(m_T)$. If the cones $Y_0(\omega)$, $Z_t(\omega)$ and $V_t(\omega)$ are *polyhedral* for all t and ω, [4] then $\mathcal{T}$ and $\mathcal{H}$ are polyhedral and hence closed. The cones mentioned are polyhedral in all the specialized models we

[3] A set in a linear space is called a (convex) cone if it contains with each vectors x and y the vector $\alpha x + \beta y$, where α and β are any nonnegative numbers.

[4] A set in a linear space is called polyhedral if it can be represented as an intersection of a finite family of closed half-spaces.

consider in this paper.

5 Consistent discount factors, no-arbitrage and hedging

We are going to give a general solution to the hedging problem in terms of consistent discount factors. First of all, we have to examine the question of their existence. We will show that, under the assumptions on the model introduced in the previous section, the existence of consistent discount factors is equivalent to the *no-arbitrage hypothesis:*

(NA) If $(v_0, v_T) \in \mathcal{H}$, $v_0 \leq 0$ and $v_T \geq 0$, then $v_0 = 0$ and $v_T = 0$.

Here and in what follows, all equalities between random vectors are understood coordinatewise and a.s. By defining

$$\mathcal{K} := \{(v_0, v_T) \in \mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T) : v_0 \leq 0 \text{ and } v_T \geq 0\}, \tag{17}$$

we can restate (NA) as follows:

$$\mathcal{H} \cap \mathcal{K} = \{0\}. \tag{18}$$

Under quite general assumptions (see Section 7, Proposition 7.2), hypothesis (NA) admits the following equivalent reformulation:

(NA_0) If $(v_0, v_T) \in \mathcal{H}$, $v_0 = 0$ and $v_T \geq 0$, then $v_T = 0$.

A criterion for the existence of consistent discount factors is provided by the following theorem.

Theorem 5.1. *The existence of consistent discount factors is equivalent to the no-arbitrage hypothesis (NA).*

Proof. Let q_0, q_T be consistent discount factors. Suppose $(v_0, v_T) \in \mathcal{H} \cap \mathcal{K}$. Then $-v_0 \geq 0$, $v_T \geq 0$ and $Eq_0(-v_0) + Eq_T v_T \leq 0$ by virtue of (15). The last inequality, combined with the fact that the functions q_0, q_T are strictly positive, implies $v_0 = 0$ and $v_T = 0$. (Recall that we identify functions of ω coinciding a.s.) Thus the existence of consistent discount factors implies (18).

To prove the converse, we apply Theorem A.2 (see the Appendix) to the cones $\mathcal{H}$ and $\mathcal{K}$. Since $\mathcal{H}$ and $\mathcal{K}$ are closed and since $\mathcal{K}$ is proper (i.e. $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$), all the requirements needed for the validity of Theorem A.2 are satisfied. The cones $\mathcal{H}$ and $\mathcal{K}$ are contained in the linear space $\mathcal{U} := \mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T)$. Any linear functional $l(u)$, $u \in \mathcal{U}$, on this space can be represented as

$$l(u) = -Eq_0 v_0 + Eq_T v_T \ \ [u = (v_0, v_T)], \tag{19}$$

where $q_0 \in \mathcal{L}_0(m_0)$ and $q_T \in \mathcal{L}_T(m_T)$. We have $l \in \mathcal{K}^+$ (i.e., $l(k) > 0$ for $k \in \mathcal{K}\backslash\{0\}$) if and only if $q_0, q_T > 0$. Let (18) hold. Then by virtue of Theorem A.2, there exists a linear functional l of the form (19) such that $q_0, q_T > 0$ and $l(u) \le 0$ for $u = (v_0, v_T) \in \mathcal{H}$. We can see that the last inequality is equivalent to (15). □

We now will give an answer to the general question posed in the previous section and provide criteria for an initial endowment v_0 to be sufficient to hedge a contingent claim v_T. These criteria will be formulated in terms of consistent discount factors. The following two results, Theorem 5.2 and Theorem 5.3, correspond to two different cases important for the applications. In the first case, we assume that the following requirement is fulfilled:

(SH) ("Superhedging hypothesis".) We have

$$-\mathcal{K} \subseteq \mathcal{H}.$$

Hypothesis (SH) covers those situations when we are interested in questions of "superhedging"—obtaining at date T payoff that is not less with probability one that the amount of contingent claim. Condition (SH) is equivalent to the requirement

$$\mathcal{H} - \mathcal{K} \subseteq \mathcal{H}, \tag{20}$$

which means that *if an initial endowment* v_0 *is sufficient to hedge a contingent claim* v_T*, then any initial endowment* $\mathcal{L}_0(m_0) \ni v_0' \ge v_0$ *is sufficient to hedge any contingent claim* $\mathcal{L}_T(m_T) \ni v_T' \le v_T$ (see the definition of $\mathcal{K}$ in (17)). Assumptions on the data of the model guaranteeing the validity of (SH) will be discussed in Section 7, Proposition 7.1.

Theorem 5.2. *Let hypotheses (NA) and (SH) hold. Then, for any* $(v_0, v_T) \in \mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T)$*, the following conditions are equivalent.*

(a) $(v_0, v_T) \in \mathcal{H}$.

(b) For all consistent discount factors q_0, q_T*, we have* $Eq_Tv_T \le Eq_0v_0$.

Proof. Put

$$\mathcal{N} := \{l \in \mathcal{K}^+ : l(u) \le 0 \text{ for all } u \in \mathcal{H}\}.$$

Denote by u the given pair (v_0, v_T) of random vectors. In view of (20), assertion (a) holds if and only if $u \in \mathcal{H} - \mathcal{K}$. Assertion (b) means that, for each linear functional $l = (q_0, q_T) \in \mathcal{N}$, we have $l(u) = Eq_Tv_T - Eq_0v_0 \le 0$. Thus (a) and (b) are equivalent by virtue of Theorem A.4. This theorem is applicable because $\mathcal{K}$ is proper and closed, $\mathcal{H}$ is closed and $\mathcal{H} \cap \mathcal{K} = \{0\}$, the latter property being postulated in (NA). □

In the second of the two cases we consider, $\mathcal{H}$ is supposed to be a linear space. This situation is characteristic for the classical model of a frictionless market defined by (10)–(13). The result below is proved exactly as the previous one with the only difference that instead of Theorem A.4 we have to use Theorem A.5.

Theorem 5.3. *Let hypothesis (NA) hold. Let $\mathcal{H}$ be a linear space. Then, for any $(v_0, v_T) \in \mathcal{L}_0(m_0)\times\mathcal{L}_T(m_T)$, the following conditions are equivalent.*

(a) $(v_0, v_T) \in \mathcal{H}$.

(b) For all consistent discount factors q_0, q_T, we have $Eq_Tv_T = Eq_0v_0$.

6 Consistent valuation systems: the definition

The results obtained in the previous section provide hedging criteria based on the concept of consistent discount factors. We now will introduce another important concept—that of a consistent valuation system. In the next section, we will establish refinements of the previous results allowing a solution of the hedging problem in terms of consistent valuation systems.

Denote by $\mathcal{P}_t$ the set of non-negative (a.s.) vector functions in $\mathcal{L}_t(n_{t-1})$ ($t = 1, ..., T$) and put $\mathcal{P}_0 := \mathcal{L}_0(n_0)$. Let us write $E_t(\cdot) := E(\cdot|\mathcal{F}_t)$ for the conditional expectation given $\mathcal{F}_t$. A sequence of vector functions

$$(p_0, p_1, ..., p_T),\ p_t \in \mathcal{P}_t,\ t = 0, ..., T, \tag{21}$$

is called a *consistent price system,* if for almost all ω, we have

$$(\bar{p}_1 - p_0)a \leq 0 \text{ for all } a \in Y_0(\omega), \tag{22}$$

$$\bar{p}_{t+1}b - p_ta \leq 0 \text{ for all } (a, b) \in Z_t(\omega),\ t = 1, ..., T-1, \tag{23}$$

where

$$\bar{p}_{t+1} := E_t(p_{t+1}),\ t = 0, ..., T-1.$$

Requirements (23) state that the prices and discount factors under consideration are such that *the conditional expectation of profit* one can get in the course of asset trading cannot be greater than zero. If $y_0, ..., y_T$ is a trading strategy, then by virtue of (22) and (23), we have

$$p_0y_0 \geq \bar{p}_1y_0 = E_0(p_1y_0), \tag{24}$$

$$p_ty_{t-1} \geq \bar{p}_{t+1}y_t = E_t(p_{t+1}y_t),\quad t = 1, ..., T-1, \tag{25}$$

and so the process

$$p_0 y_0,\ p_1 y_0,\ p_2 y_1,\ \dots, p_T y_{T-1} \tag{26}$$

is a supermartingale with respect to the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_T$.

It will be convenient to introduce the following notation. For every $\omega \in \Omega$, put

$$Z_t^{\times}(\omega) := \{(c, d) \in \mathbb{R}_+^{n_{t-1}} \times \mathbb{R}_+^{n_t} : db \le ca \text{ for all } (a, b) \in Z_t(\omega)\}. \tag{27}$$

Elements of the cone $Z_t^{\times}(\omega)$ (*cross-dual cone*) can be interpreted as pairs of non-negative price vectors (c, d) such that, for every pair of portfolios $(a, b) \in Z_t(\omega)$, the value db of portfolio b is not greater than the value ca of portfolio a. Condition (23) involved in the definition of a consistent price system can be written equivalently as

$$(p_t(\omega), \bar{p}_{t+1}(\omega)) \in Z_t^{\times}(\omega) \tag{28}$$

(a.s.).

If $(q_0, q_T) \in \mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T)$ is a pair of strictly positive functions (discount factors) and $(p_0, p_1, \dots, p_T)$ is a consistent price system, the sequence $(q_0, p_0, p_1, \dots, p_T, q_T)$ is called a *consistent valuation system* if, with probability one,

$$p_0 b - q_0 a \le 0,\ (a, b) \in V_0(\omega), \tag{29}$$

$$q_T b - p_T a \le 0,\ (a, b) \in V_T(\omega). \tag{30}$$

According to (29) and (30), the prices and discount factors under consideration are such that, with probability one, the profit one can get in the course of portfolio creation or liquidation cannot be strictly positive.

It follows from (24), (25), (29) and (30) that if $(q_0, p_0, p_1, \dots, p_T, q_T)$ is a consistent valuation system, then q_0, q_T are consistent discount factors. Conversely, it can be proved under some general assumptions (see Section 7, Theorem 7.1) that if q_0, q_T are consistent discount factors, then there exists a consistent price system $(p_0, \dots, p_T)$ such that $(q_0, p_0, p_1, \dots, p_T, q_T)$ is a consistent valuation system.

We observe that if each of the cones $Y_0(\omega)$, $Z_t(\omega)$ $(t = 1, \dots, T-1)$ and $V_t(\omega)$ $(t = 0, T)$ is a linear space, then we can replace in formulas (24), (25), (29) and (30) inequalities by equalities. This implies that the sequence (26) is a martingale.

Why do we need to consider consistent valuation systems in place of (or in addition to) consistent discount factors? The main advantage of the former notion comparative to the latter lies in the following. It might be difficult to verify directly that some (q_0, q_T) is a pair of consistent discount factors. To this end one has to check inequality (15) involving the cone $\mathcal{H}$ which might generally have quite a complex structure. The analysis of consistent valuation systems simplifies the problem. It allows one to "decompose" it—to reduce it to a family of simpler problems over each of the time periods $t-1, t$. Indeed, to check that $(p_0, ..., p_T)$ is a consistent price system, we have to verify the inequalities in (22) and (23) separately for every t. The analysis of each of these inequalities requires a separate consideration of each of the given cones $Y_0(\omega), Z_1(\omega), ..., Z_{T-1}(\omega)$. Analogous considerations pertain to inequalities (29) and (30).

7 Consistent valuation systems, no-arbitrage and hedging

For the analysis of consistent valuation systems, it is convenient to introduce the following notion. Let

$$\xi := \{v_0, x_0, y_0, x_1, y_1, \; ..., x_{T-1}, y_{T-1}, x_T, v_T\} \tag{31}$$

be a sequence of vector functions

$$v_0 \in \mathcal{L}_0(m_0), \; y_t \in \mathcal{L}_t(n_t), \; t = 0, ..., T-1, \tag{32}$$

$$x_t \in \mathcal{L}_t(n_{t-1}), \; t = 0, ..., T, \; v_T \in \mathcal{L}_T(m_T), \tag{33}$$

(where $n_{-1} := n_0$) such that

$$(v_0, x_0) \in V_0(\omega), \; y_0 \in Y_0(\omega), \tag{34}$$

$$(x_t, y_t) \in Z_t(\omega), \; t = 1, ..., T-1, \; (x_T, v_T) \in V_T(\omega) \tag{35}$$

(a.s.). We shall say that ξ is a *generalized hedging strategy* (or a *hedging strategy with consumption*) if

$$x_0 \geq y_0, \, y_0 \geq x_1, \, y_1 \geq x_2, \; ..., \; y_{T-1} \geq x_T. \tag{36}$$

If the above inequalities hold as equalities, we obtain an ordinary hedging strategy.

Let us introduce the following conditions.

(S) There exists a generalized hedging strategy

$$\check{\xi} := \{(\check{v}_0, \check{x}_0, \check{y}_0, \check{x}_1, \check{y}_1, ..., \check{x}_{T-1}, \check{y}_{T-1}, \check{x}_T, \check{v}_T\}$$

such that

$$\check{x}_0 > \check{y}_0,\ \check{y}_{t-1} > \check{x}_t,\ t = 1, ..., T.$$

(M_0) If $(a, c) \in V_0(\omega)$, $c \geq b \in Y_0(\omega)$ and $a' \geq a$, then there exists $b' \geq b$ such that $b' \in Y_0(\omega)$ and $(a', b') \in V_0(\omega)$.

For $t = 1, ..., T-1$, consider the following hypothesis:

(M_t) If $(a, b) \in Z_t(\omega)$ and $a' \geq a$, then there exists $b' \geq b$ such that $(a', b') \in Z_t(\omega)$.

(M_T) If $(a, b) \in V_T(\omega)$ and $a' \geq a$, then $(a', b') \in V_T(\omega)$ for some $b' \geq b$.

Hypothesis (S) (an analogue of "Slater's constraint qualification", see the Appendix) says that there exists a trading strategy with consumption such that an investor following it can sell, with a view to consumption, strictly positive amounts of assets of each type at every date.

Conditions (M_t) (properties of "monotonicity") are supposed to hold for each ω and $t = 0, ..., T$. Their meaning is, roughly speaking, "the more the better". According to (M_t) ($t = 1, ..., T-1$), if a portfolio a can be rebalanced into an admissible portfolio b at date t then any portfolio a' that is not less (in each position) than a can be rebalanced into some admissible portfolio b' containing not less assets of each type than b. Similar properties are reflected in hypotheses (M_0) and (M_T), dealing with the constraints on portfolio construction at date 0 and liquidation at date T. Additionally, hypothesis (M_0) imposes restrictions on the set $Y_0(\omega)$ of initial portfolios.

Let us say that the model under consideration, specified by the cones $Y_0(\omega)$, $Z_t(\omega)$, $t = 1, ..., T-1$, and $V_t(\omega)$, $t = 0, T$, is *regular* if either condition (S) is fulfilled or the cones are polyhedral for each t and ω. We shall say that the model is *monotone* if it possesses the properties described in (M_t) for each ω and $t = 0, ..., T$. The following theorem establishes links between the concepts of consistent discount factors and consistent valuation systems.

Theorem 7.1. *(a) If $(q_0, p_0, ..., p_T, q_T)$ is a consistent valuation system, then q_0, q_T are consistent discount factors.*

(b) Let the model be regular and monotone. If q_0, q_T are consistent discount factors, then there exists a consistent price system $(p_0, ..., p_T)$ such that $(q_0, p_0, ..., p_T, q_T)$ is a consistent valuation system.

This result is similar to various assertions on the existence of "shadow prices" (or "supporting prices") in stochastic models of economic dynamics—see, e.g. Arkin and Evstigneev [3, Section 2.4]. It is especially close to one of the first results of this kind obtained by Dynkin [17], [18, Section 9.5]. Before proving Theorem 7.1, the following remark is in order.

Remark 7.1. If the model is monotone, than for any generalized hedging strategy (31), there is a hedging strategy $(v'_0, y'_0, ..., y'_{T-1}, v'_T)$ such that $v'_0 = v_0$, $y'_t \geq y_t$ $(t = 0, ..., T-1)$ and $v'_T \geq v_T$. This is easily proved by induction using (M_t), $t = 0, ..., T$, and a measurable selection argument—see the Appendix, Theorem A.7.

Proof of Theorem 7.1. The first assertion of the theorem is immediate from the definition of a consistent valuation system. To prove the second, consider consistent discount factors (q_0, q_T). By virtue of their definition, we have $Eq_Tv_T - Eq_0v_0 \leq 0$ for all $(v_0, v_T) \in \mathcal{H}$. Consequently, $Eq_Tv'_T - Eq_0v'_0 \leq 0$ for all hedging strategies $(v'_0, y'_0, ..., y'_{T-1}, v'_T)$. Denote by Ξ the set of all generalized hedging strategies (31) and for each $\xi \in \Xi$, define $F(\xi) = Eq_Tv_T - Eq_0v_0$. In view of Remark 7.1, we have

$$F(\xi) = Eq_Tv_T - Eq_0v_0 \leq Eq_Tv'_T - Eq_0v_0 \leq 0$$

for some hedging strategy $(v'_0, y'_0, ..., y'_{T-1}, v'_T)$ with $v'_0 = v_0$ and $v'_T \geq v_T$. Consequently, $F(\xi) \leq 0$ for all $\xi \in \Xi$. Thus, the maximum of the functional $F(\xi)$ over all sequences (31) satisfying (32) – (36) is equal to zero. By applying the Kuhn–Tucker theorem (see Theorem A.5 in the Appendix) to this maximization problem, we relax constraints (36). The theorem is applicable because the model at hand is regular. Consequently, there exist vector functions $p_t \in \mathcal{P}_t$ $(t = 0, ..., T)$ such that

$$Eq_Tv_T - Eq_0v_0 + Ep_0(x_0 - y_0) + \sum_{t=1}^{T} Ep_t(y_{t-1} - x_t) \leq 0 \qquad (37)$$

for all sequences $\{v_0, x_0, y_0, x_1, y_1, ..., x_{T-1}, y_{T-1}, x_T, v_T\}$ satisfying (32) – (35).

Rearranging the summands in (37), we obtain

$$(Ep_0x_0 - Eq_0v_0) + (Ep_1y_0 - Ep_0y_0) +$$

$$[\sum_{t=1}^{T-1}(Ep_{t+1}y_t - Ep_tx_t)] + (Eq_Tv_T - Ep_Tx_T) \leq 0.$$

This inequality holds if and only if the inequalities

$$Ep_0x_0 - Eq_0v_0 \leq 0, \ (v_0, x_0) \in V_0(\omega), \qquad (38)$$

$$E\bar{p}_1y_0 - Ep_0y_0 \leq 0, \ y_0 \in Y_0(\omega), \qquad (39)$$

$$E\bar{p}_{t+1}y_t - Ep_tx_t \le 0,\ (x_t, y_t) \in Z_t(\omega),\ t = 1, ..., T-1, \tag{40}$$

$$Eq_Tv_T - Ep_Tx_T \le 0,\ (x_T, v_T) \in V_T(\omega), \tag{41}$$

hold. By using Theorem A.8 in the Appendix, we deduce (29), (22), (23) and (30) from (38), (39), (40) and (41), respectively. Thus $(q_0, p_0, ..., p_T, q_T)$ is the sought-for consistent valuation system. □

In Theorems 7.2 and 7.3 stated below, *we assume that the model under consideration is regular and monotone.*

Theorem 7.2. *The validity of the no-arbitrage hypothesis (NA) is equivalent to the existence of a consistent valuation system.*

This result is an immediate consequence of Theorems 5.1 and 7.1.

Theorems 5.1 and 7.2, providing criteria for the absence of arbitrage in terms of consistent discount factors and consistent price systems respectively, may be regarded as generalizations of the classical Fundamental Theorem of Asset Pricing—Kreps, Harrison, Pliska and others. A detailed discussion of the classical (frictionless) case will be given in Section 11.

By using Theorems 5.2 and 7.1, we arrive at the following hedging criterion stated in terms of consistent valuation systems.

Theorem 7.3. *Let hypotheses (NA) and (SH) hold. Then, for any* $(v_0, v_T) \in \mathcal{L}_0(m_0) \times \mathcal{L}_T(m_T)$*, the following conditions are equivalent*:

(a) $(v_0, v_T) \in \mathcal{H}$.

(b) For all consistent valuation systems $(q_0, p_0, ..., p_T, q_T)$*, we have* $Eq_Tv_T \le Eq_0v_0$.

Conditions guaranteeing the validity of hypothesis (SH) are provided in Proposition 7.1 below. Consider the following assumption ("superhedging hypothesis at time T"):

(SH_T) For each ω, if $(a, b) \in V_T(\omega)$ and $b \ge b' \in \mathbb{R}^{m_T}$, then $(a, b') \in V_T(\omega)$.

Proposition 7.1. *Hypothesis (SH) holds if the model is monotone and requirement* (SH_T) *is satisfied.*

Proof. To deduce (SH) from (SH_T) and the assumption of the monotonicity of the model, consider any $(v_0, v_T) \in -\mathcal{K}$. We have $v_0 \ge 0$ and $v_T \le 0$. Since $(0, 0) \in V_0(\omega)$, $0 \in Y_0(\omega)$ and $v_0 \ge 0$, hypothesis (M_0), combined with a measurable selection argument, implies the existence of $y_0 \in \mathcal{L}_0(n_0)$ such that $y_0 \in Y_0(\omega)$, $y_0 \ge 0$ and $(v_0, y_0) \in V_0(\omega)$. We have $(0, 0) \in Z_1(\omega)$ and $y_0 \ge 0$, and so, by using (M_1), we can construct $y_1 \in \mathcal{L}_1(n_1)$ such that $y_1 \ge 0$ and $(y_0, y_1) \in Z_1(\omega)$. Continuing this process and using (M_t), $t = 2, ..., T$, we arrive at a feasible hedging strategy $(v'_0, y'_0, ..., y'_{T-1}, v'_T)$ with $v'_0 = v_0$ such that $y'_0 \ge 0, ..., y'_{T-1} \ge 0, v'_T \ge 0$. Then, by virtue of ($\mathrm{SH}_T$), $(v_0, y'_0, ..., y'_{T-1}, v_T)$ is also a feasible hedging strategy, which proves (SH). □

In the case when $\mathcal{H}$ is a linear space (which is so when all the cones $Y_0(\omega)$, $Z_t(\omega)$, $V_t(\omega)$ are linear spaces), we obtain the following result.

Theorem 7.4. *Let hypothesis (NA) hold. Let $\mathcal{H}$ be a linear space. Then, for any $(v_0, v_T) \in \mathcal{L}_0(m_0)\times\mathcal{L}_T(m_T)$, the following conditions are equivalent.*

(a) $(v_0, v_T) \in \mathcal{H}$.

(b) For all consistent valuation systems $(q_0, p_0, ..., p_T, q_T)$, we have $Eq_T v_T = Eq_0 v_0$.

Theorem 7.4 is a direct consequence of Theorems 5.3 and 7.1.

We conclude this section with a proposition analyzing the question of equivalence of the no-arbitrage hypotheses (NA_0) and (NA). For each $t = 0, ..., T$, let us introduce a stronger version, (SM_t), of the monotonicity condition (M_t). *Hypothesis (SM_t) is stated exactly as (M_t) with the additional requirement that $b' \neq b$ when $a' \neq a$.* If (SM_t) holds for each $t = 0, ..., T$, the model is said to be *strictly monotone.*

Put $Z_0(\omega) := \{(a, b) : (a, b) \in V_0(\omega),\ b \in Y_0(\omega)\}$ and $Z_T(\omega) := V_T(\omega)$. From the definition of strict monotonicity of the model, we immediately obtain the following consequence:

(SM) For each t, ω, if $(a, b) \in Z_t(\omega)$, $a' \geq a$ and $a' \neq a$, then there exists b' such that $(a', b') \in Z_t(\omega)$, $b' \geq b$ and $b' \neq b$.

It is easily proved by means of induction that condition (SM) implies the following property of the cone $\mathcal{H}$.

($\mathrm{SM}_{\mathcal{H}}$) If $(v_0, v_T) \in \mathcal{H}$, $\mathcal{L}_0(m_0) \ni v_0' \geq v_0$ and $v_0' \neq v_0$, then there exists v_T' such that $(v_0', v_T') \in \mathcal{H}$, $v_T' \geq v_T$ and $v_T' \neq v_T$.

Proposition 7.2. *Under assumption ($SM_{\mathcal{H}}$), hypothesis (NA) is equivalent to (NA_0).*

Proof. Clearly, (NA) implies (NA_0). Suppose (NA) does not hold, while (NA_0) is valid. Then there are $v_0 \leq 0$ and $v_T \geq 0$ such that $(v_0, v_T) \in \mathcal{H}$ and $(v_0, v_T) \neq 0$. By virtue of (NA_0), we have $v_0 \neq 0$. Since $0 \geq v_0$ and $0 \neq v_0$, it follows from ($\mathrm{SM}_{\mathcal{H}}$), that there exists $v_T' \geq 0$ such that $(0, v_T') \in \mathcal{H}$, $v_T' \geq v_T$ and $v_T' \neq v_T$. We have $v_T' \neq 0$ (because otherwise, $v_T' = v_T = 0$), which leads to a contradiction with (NA_0). □

8 Solvency cones and portfolio values

Fix some t $(1 \leq t \leq T-1)$ and assume that $n_{t-1} = n_t = n$. Suppose that the cone $Z_t(\omega)$ involved in the model description (see (2)) is of the form

$$Z_t(\omega) = \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n : b - a \in M_t(\omega),\ b \in Y_t(\omega)\}, \tag{42}$$

where $M_t(\omega) \subseteq \mathbb{R}^n$ and $Y_t(\omega) \subseteq \mathbb{R}^n$ are some given closed cones depending $\mathcal{F}_t$-measurably on ω. According to (42), one can obtain a portfolio b by rebalancing a portfolio a if and only if the vector $b - a$ is an element of the *solvency cone* $M_t(\omega)$ (see Kabanov [39] and references therein). The cone $Y_t(\omega)$ specifies what portfolios are *admissible* at each moment of time and each random situation ω.

If for all $t = 1, ..., T-1$ the cones $Z_t(\omega)$ are given by formula (42), the rebalancing constraints $(y_{t-1}(\omega), y_t(\omega)) \in Z_t(\omega)$ (a.s.), involved in the definition of trading strategies, can be decomposed into two parts: $y_t(\omega) - y_{t-1}(\omega) \in M_t(\omega)$ (a.s.) and $y_t(\omega) \in Y_t(\omega)$ (a.s.). The former deals with the difference between two successive portfolios y_t and y_{t-1}. The latter imposes restrictions on the portfolio y_t obtained as a result of rebalancing y_{t-1}.

The structure of consistent price systems in models defined by (42) can be examined by using the following proposition.

Proposition 8.1. *If $Z_t(\omega)$ is given by (42), then*

$$Z_t^\times(\omega) = \{(c, d) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : -c \in M_t^*(\omega),\ c - d \in Y_t^*(\omega)\}. \tag{43}$$

Proof. We omit t and ω. According to the definition of $Z^\times$, a pair of vectors $(c, d) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ belongs to $Z^\times$ if and only if $db \le ca$ for all $(a, b) \in Z$. In the model at hand, $(a, b) \in Z$ if and only if $f := b - a \in M$ and $b \in Y$. Consequently, Z consists of pairs (a, b) of the form $(a, b) = (b - f, b)$ where f and b run through M and Y respectively. Thus $(c, d) \in Z^\times$ if and only if

$$db - c(b - f) \le 0 \text{ for all } f \in M \text{ and } b \in Y.$$

The last inequality is equivalent to the following two conditions:

$$cf \le 0,\ f \in M;\ (d - c)b \le 0,\ b \in Y.$$

These two conditions are equivalent to the requirements $-c \in M^*$ and $c - d \in Y^*$, respectively. □

From Proposition 8.1, we obtain the following consequence.

Proposition 8.2. *If the cone $Z_t(\omega)$ is of the form (42), then condition (28) involved in the definition of a consistent price system holds if and only if*

$$-p_t \in M_t^*(\omega),\ p_t - E_t p_{t+1} \in Y_t^*(\omega). \tag{44}$$

Sequences of random vectors p_t such that p_t is $\mathcal{F}_t$-measurable and $p_t - E_t p_{t+1} \in Y_t^*(\omega)$ (a.s.) are called *Y^*-martingales.*

The main results about consistent price systems are obtained under the assumption that the model under consideration is regular and monotone. The assumption of regularity is guaranteed if all the cones we deal with are polyhedral. Conditions on $M_t(\omega)$ and $Y_t(\omega)$ under which the monotonicity hypothesis (M_t) ($t = 1, ..., T-1$) holds are provided in the following proposition.

Proposition 8.3. *Let the cone $Z_t(\omega)$ be defined by (42). Then the hypothesis of monotonicity (M_t) holds if any of the following requirements is satisfied:*

(a) $M_t(\omega) \supseteq -\mathbb{R}^n_+$ for all ω;

(b) $Y_t(\omega) \supseteq \mathbb{R}^n_+$ for all ω;

(c) the vector $(\kappa, 0, ..., 0)$ belongs to $Y_t(\omega)$ for all $\kappa \geq 0$ and for every non-zero vector $c \in \mathbb{R}^n_+$ there is $\kappa > 0$ for which $(\kappa, 0, ..., 0) - c \in M_t(\omega)$.

Conditions (b) and (c) are sufficient for the validity of the hypothesis of strict monotonicity (SM_t).

Condition (a) means that the solvency constraints do not restrict short sales of any asset; such restrictions, if any, are imposed only by the portfolio admissibility constraints specified by the cones $Y_t(\omega)$. According to (b), all portfolios with non-negative positions are admissible. We will typically interpret the first position a^1 of a portfolio $(a^1, ..., a^n)$ as the (discounted) amount of cash in the bank account. Condition (c) states that any positive sum of cash can be held in the bank account and any non-zero portfolio with non-negative positions can be exchanged to some strictly positive amount of cash.

Proof of Proposition 8.3. We will fix t and ω and omit these symbols in formulas. Consider any $(a, b) \in Z$ such that $a' \geq a$ and $a' \neq a$. To verify (M_t) we have to show that $(a', b') \in Z$ for some $b' \geq b$. To check (SM_t), we need additionally that $b' \neq b$. By the definition of Z, we have $b - a \in M$ and $b \in Y$.

If (a) holds, put $b' = b$. Then $(a', b') \in Z$ because $b' - a' = b - a' \leq b - a \in M$ by virtue of (a).

If (b) is valid, define $b' = b + (a' - a)$. Then $b' \in Y$ in view of (b) and we have $b' - a' = b - a \in M$. Finally, $b' \geq b$ and $b' \neq b$.

Let (c) hold. Put $e_1 = (1, 0, ..., 0)$ and consider that number $\kappa > 0$ for which $\kappa e_i - (a' - a) \in M$. This number exists because $a' - a \geq 0$ and $a' - a \neq 0$. Define $b' = b + \kappa e_i$. Then $b' \in Y$ because $b \in Y$ and $\kappa e_i \in Y$. Furthermore, $b' \geq b$ and $b' \neq b$. Finally, $b' - a' \in M$ because

$$b' - a' = b + \kappa e_i - a' = (b - a) + \kappa e_i - (a' - a) \in M.$$

$\square$

Suppose the sets $V_t(\omega)$, $t = 0, T$, are defined by

$$V_0(\omega) := \{(a, b) \in \mathbb{R}^1 \times \mathbb{R}^n : \ a \geq W_0(\omega, b)\}, \tag{45}$$

$$V_T(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^1 : \ W_T(\omega, a) \geq b\}, \tag{46}$$

where the functions $W_0 : \Omega \times \mathbb{R}^{n_0} \to \mathbb{R}^1$ and $W_T : \Omega \times \mathbb{R}^{n_{T-1}} \to \mathbb{R}^1$ satisfy the following conditions:

(W1) if $c' \geq c$, then $W_t(\omega, c') \geq W_t(\omega, c)$;

(W2) the functions $W_0(\omega, \cdot)$ and $-W_T(\omega, \cdot)$ are convex and positively homogeneous;

(W3) $W_t(\omega, b)$ is $\mathcal{F}_t$-measurable with respect to ω $(t = 0, T)$.

Here $W_0(\omega, b)$ is the *acquisition value* of the portfolio b, while $W_T(\omega, b)$ is its *liquidation value.* According to (45), an initial endowment v_0 allows the construction of an initial portfolio y_0 if and only if $v_0 \geq W_0(\omega, y_0)$. By virtue of (46), a terminal portfolio y_{T-1} is sufficient for hedging a contingent claim v_T if and only if $W_T(\omega, y_{T-1}) \geq v_T$.

If condition (W1) is satisfied, the function $W_t(\omega, \cdot)$ is called *monotone.* It is called *strictly monotone* if $W_t(\omega, c') > W_t(\omega, c)$ when $c' \geq c$ and $c' \neq c$. In the following proposition, assumptions (W1) – (W3) are supposed to hold.

Proposition 8.4. *(a) The sets (45) and (46) are cones satisfying the following condition:*

(V) If $(a, b) \in V_t(\omega)$, $a' \geq a$ and $b \geq b'$, then $(a', b') \in V_t(\omega)$ $(t = 0, T)$.

(b) Condition (V) implies the validity of hypotheses (M_0) and (M_T).

(c) If the function $W_T(\omega, \cdot)$ is strictly monotone, then (SM_T) holds.

(d) If $Y_0(\omega)$ contains $(\kappa, 0, ..., 0)$ for each $\kappa > 0$, then hypothesis (SM_0) is valid.

Proof. We focus on the last assertion; all the others are straightforward. Since the function $W_0(\omega, \cdot)$ is convex on $\mathbb{R}^n$, it is continuous. Therefore if $a' > a \geq W_0(\omega, c)$ and $c \geq b$, then $b' := b + \kappa e_1 \in Y_0(\omega)$ and $a' > W_0(\omega, b + \kappa e_1)$ for all $\kappa > 0$ small enough, which yields (SM_0). □

Proposition 8.5. *The cone $V_0^\times(\omega)$ consists of those $(c, d) \in \mathbb{R}^n_+ \times \mathbb{R}^1_+$ for which*

$$cW_0(\omega, b) \geq db, \ b \in \mathbb{R}^n. \tag{47}$$

Elements of the cone $V_T^\times(\omega)$ are pairs $(c, d) \in \mathbb{R}^1_+ \times \mathbb{R}^n_+$ satisfying

$$ca \geq dW_T(\omega, a), \ a \in \mathbb{R}^n. \tag{48}$$

Proof. Straightforward. □

9 A model with proportional transaction costs and portfolio constraints

In the remainder of the paper, we will consider several specific models which can be included in the general framework we have developed. All the models we are going to examine are defined in terms of polyhedral cones, hence they are regular. We will be interested in the characterization of consistent valuation systems and in hedging criteria stated in terms of them. Recall that the existence of consistent valuation systems is equivalent to the no arbitrage hypothesis (NA). When analyzing hedging criteria, we will assume that (NA) holds.

The material of this section is based, in particular, on the work of Jouini and Kallal [35, 36] and Pham and Touzi [49].

Suppose that, for each $t = 0, ..., T$, we are given a vector $S_t(\omega) = (S_t^1(\omega), ..., S_t^n(\omega)) > 0$ specifying the market prices of n assets $i = 1, 2, ..., n$ at date t. Assume that $m_0 = m_T = 1$, so that initial endowments and contingent claims are measured in terms of cash. Fix some $\mathcal{F}_t$-measurable random variables $\alpha_t^i(\omega) \geq 0$ and $1 > \beta_t^i(\omega) \geq 0$ – *transaction cost rates* ($t = 0, ..., T$, $i = 1, ..., n$). By selling one unit of asset i at time t, one gets $(1 - \beta_t^i)S_t^i$, and in order to buy one unit of asset i, one has to pay $(1 + \alpha_t^i)S_t^i$. For any vector $a = (a^1, ..., a^n) \in \mathbb{R}^n$ and any $i = 1, 2, ..., n$, define

$$\tau_t^i(\omega, a) := (1 + \alpha_t^i(\omega))a_+^i + (1 - \beta_t^i(\omega))a_-^i, \tag{49}$$

where $a_+^i := \max\{a^i, 0\}$ and $a_-^i := \min\{a^i, 0\}$. The functions $\tau_t^i(\omega, a)$ are convex and positively homogeneous in a. Consider the mapping $\tau_t^i(\omega, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ acting by the formula

$$\tau_t(\omega, a) := (\tau_t^1(\omega, a), ..., \tau_t^n(\omega, a)). \tag{50}$$

For each $t = 0, 1, ..., T - 1$, let $Y_t(\omega)$ be a polyhedral cone depending $\mathcal{F}_t$-measurably on ω. Let $Y_0(\omega)$ specify constraints on initial portfolios and put

$$Z_t(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n : \ S_t(\omega)\tau_t(\omega, b - a) \leq 0, \ b \in Y_t(\omega)\} \tag{51}$$

$(t = 1, ..., T - 1)$,

$$V_0(\omega) := \{(a, b) \in \mathbb{R}^1 \times \mathbb{R}^n : \ a \geq S_0(\omega)\tau_0(\omega, b)\}, \tag{52}$$

$$V_T(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^1 : \ -S_T(\omega)\tau_T(\omega, -a) \geq b\}. \tag{53}$$

The inequality $S_t\tau_t(b-a)\le 0$ in involved in (51) can be written

$$\sum_{i=1}^{n}(1+\alpha_t^i)S_t^i(b^i-a^i)_+ \le -\sum_{i=1}^{n}(1-\beta_t^i)S_t^i(b^i-a^i)_- .$$

This relation expresses the *self-financing condition*: assets are purchased only at the expense of sales of other assets. In order to construct a portfolio b at time 0, one needs the amount

$$S_0\tau_0(b)=\sum_{i=1}^{n}(1+\alpha_0^i)S_0^i(b^i)_+ + \sum_{i=1}^{n}(1-\beta_0^i)S_0^i(b^i)_- ,$$

and when liquidating a portfolio a at time T, one gets

$$-S_T\tau_T(-a)=-\sum_{i=1}^{n}(1+\alpha_T^i)S_T^i(-a^i)_+ - \sum_{i=1}^{n}(1-\beta_T^i)S_T^i(-a^i)_- =$$

$$\sum_{i=1}^{n}(1+\alpha_T^i)S_T^i(a^i)_- + \sum_{i=1}^{n}(1-\beta_T^i)S_T^i(a^i)_+.$$

This motivates the definitions of the cones V_0 and V_T in (52) and (53).

Define $B_t := S_t^1$ and assume that $B_0=1$ and $B_t(\omega)>0$ for all t and ω. Suppose that the following condition is fulfilled.

(B) The vector $(\kappa,0,...,0)$ belongs to the cone $Y_t(\omega)$ for all $\kappa>0$ and $\omega\in\Omega$.

This assumption means that the bank account can contain any positive amount of cash. It follows from (B) and from the strict monotonicity of the function $-S_T(\omega)\tau_T(-a)$ that the model under consideration is strictly monotone—see Proposition 8.3, (c) and Proposition 8.4, (c) and (d). The property of strict monotonicity implies the equivalence of the no arbitrage hypotheses (NA) and (NA$_0$) (see Proposition 7.2). Condition (SH$_T$), following from (53), implies the validity of hypothesis (SH) (see Proposition 7.1). The model is regular, since all the cones involved in its description are polyhedral. Thus, by virtue of Theorem 7.2, *the absence of arbitrage opportunities in the model at hand is equivalent to the existence of consistent valuation systems*, and the hedging criterion stated in Theorem 7.3 is valid.

Let $(\lambda_0,...,\lambda_T)$ be a sequence of strictly positive scalar-valued random variables such that λ_t is $\mathcal{F}_t$-measurable ($t=0,...,T$). We say that $\lambda_0,...,\lambda_T$ is a *chain of consistent discount factors* if there exists a Y^*-martingale $p_0,...,p_T$ satisfying

$$\lambda_t S_t^i(1-\beta_t^i)\le p_t^i\le \lambda_t S_t^i(1+\alpha_t^i),\ t=0,...,T. \tag{54}$$

Recall that Y^*-martingales satisfy, by definition, the condition $p_t - E_t p_{t+1} \in Y_t^*(\omega)$ (a.s.).

Theorem 9.1. *The following assertions are equivalent.*

(a) A sequence $(q_0, p_0, ..., p_T, q_T)$ is a consistent valuation system in the model (51) – (53).

(b) There exists a chain of consistent discount factors $\lambda_0, ..., \lambda_T$ such that $q_0 = \lambda_0$, $q_T = \lambda_T$, and the sequence $p_0, p_1, ..., p_T$ is a Y^*-martingale satisfying (54).

Proof. By using Propositions 8.1, 8.2, 8.5 and the definitions of the cones Y_0, Z_t, V_t in the model under consideration, we obtain that a sequence $(q_0, p_0, ..., p_T, q_T)$ is a consistent valuation system if and only if $q_t > 0,\ t = 0, T$,

$$q_0 S_0 \tau_0(b) \geq p_0 b,\ b \in \mathbb{R}^n, \tag{55}$$

$$p_0 - \bar{p}_1 \in Y_0^*, \tag{56}$$

$$S_t \tau_t(b) \leq 0 \ \Rightarrow\ p_t b \leq 0,\ t = 1, ..., T-1, \tag{57}$$

$$p_t - E_t p_{t+1} \in Y_t^*,\ t = 1, 2, ..., T-1, \tag{58}$$

$$p_T a \geq -q_T S_T \tau_T(-a),\ a \in \mathbb{R}^n. \tag{59}$$

Condition (57) holds if and only if

$$l_t S_t^i (1 - \beta_t^i) \leq p_t^i \leq l_t S_t^i (1 + \alpha_t^i),\ i = 1, ..., n, \tag{60}$$

for some real-valued $\mathcal{F}_t$-measurable functions $l_t = l_t(\omega) \geq 0$. To show this, let us fix $\omega \in \Omega$ and omit it in the notation. Consider the following optimization problem: maximize $p_t b$ over $b \in \mathbb{R}^n$ subject to $-S_t \tau_t(b) \geq 0$. Property (57) means that $b = 0$ is a solution to this problem. By virtue of the Kuhn–Tucker theorem (see Theorem A.6 in the Appendix), this assertion holds if and only if there exists $l_t = l_t(\omega) \geq 0$ such that

$$p_t b - l_t S_t \tau_t(b) \leq 0 \text{ for all } b \in \mathbb{R}^n. \tag{61}$$

(The Kuhn–Tucker theorem can be applied because the function $-S_t \tau_t(b)$ is concave in b and the Slater condition $-S_t \tau_t(b) > 0$ is fulfilled for $b =$

$(-1,0,0,...,0)$.) Since $S_t\tau_t(b)=\sum_{i=1}^n S_t^i\tau_t^i(b)$, inequality (61) is valid if and only if the analogous "coordinatewise" inequality

$$p_t^i r - l_t S_t^i \tau_t^i(r) \le 0,\ i=1,...,n,\ r\in\mathbb{R}^1, \tag{62}$$

is valid. We observe that, for $r<0$, (62) is equivalent to the first inequality in (60), and for $r>0$, (62) is equivalent to the second inequality in (60). We have constructed the desired l_t with properties (60) for each ω. It remains to choose $l_t=l_t(\omega)$ in an $\mathcal{F}_t$-measurable way (see Theorem A.7).

We note that relations (55) and (59) hold if and only if, for $t=0,T$, we have

$$q_t S_t^i(1-\beta_t^i) \le p_t^i \le q_t S_t^i(1+\alpha_t^i),\ i=1,...,n,\ t=0,T. \tag{63}$$

This follows from the fact that (61) is equivalent to (60) (put $l_t=q_t$, $t=0,T$).

(a)⇒(b) Suppose $(q_0,p_0,...,p_T,q_T)$ is a consistent valuation system. Consider the functions l_t in (60) and define $\lambda_t=l_t$, $t=1,2,...,T-1$, $\lambda_t=q_t$, $t=0,T$. Then, as we have shown above, $p_0,...,p_T$ is a Y^*-martingale satisfying (54). By virtue of assumption (B), p_t^1 is a supermartingale, and so $p_t^1>0$ because $p_T^1\ge q_T S_T^1(1-\beta_T^1)>0$. Consequently, $\lambda_t>0$ since $p_t^1\le\lambda_t S_t^1(1+\alpha_t^1)$. Thus $\lambda_0,...,\lambda_T$ is a chain of consistent discount factors.

(b)⇒(a) We have $q_t=\lambda_t>0$, $t=0,T$. Since $p_0,...,p_T$ is a Y^*-martingale, conditions (56) and (58) hold. As we have seen, relations (54) imply (55), (57) and (59). Consequently, $(q_0,p_0,...,p_T,q_T)$ is a consistent valuation system. □

From Theorems 9.1 and 5.2, we obtain the following result.

Theorem 9.2. *In the model at hand, an initial endowment v_0 allows the hedging of a contingent claim v_T if and only if $E\lambda_0 v_0\ge E\lambda_T v_T$ for all chains of consistent discount factors $\lambda_0,...,\lambda_T$.*

Let us introduce the following assumptions (expressing the notion of an "ideal bank account") :

(IB) The vector $(\kappa,0,...,0)$ belongs to the cone $Y_t(\omega)$ for all $\kappa\in(-\infty,+\infty)$ and $\omega\in\Omega$.

(IB_1) We have $\alpha_t^1=\beta_t^1=0$.

According to (IB) and (IB_1), the bank account can be used for both lending and borrowing—with the same interest rate and without transaction costs.

Consider the *discounted asset prices* S_t^i/B_t $(i=1,...,n,\ t=0,...,T)$. Let us call a real-valued random variable $\lambda(\omega)>0$ a *consistent density* if $E\lambda=1$ and there exists a sequence of random vectors $f_0,...,f_T$ such that

$$\frac{S_t^i}{B_t}(1-\beta_t^i)\le f_t^i\le\frac{S_t^i}{B_t}(1+\alpha_t^i),\ t=0,...,T, \tag{64}$$

and the process $f_0, ..., f_T$ is a Y^*-martingale with respect to the probability measure $P^\lambda(d\omega) = \lambda(\omega)P(d\omega)$ equivalent to P with density $\lambda(\omega)$ (*equivalent consistent measure*).

Proposition 9.1. *Let $\lambda_0 > 0, ..., \lambda_T > 0$ be a sequence of random variables such that λ_t is $\mathcal{F}_t$-measurable. Under assumptions (IB) and (IB$_1$), the following assertions are equivalent:*

(a) The sequence $\lambda_0, ..., \lambda_T$ is a chain of consistent discount factors.

(b) There exists a constant $\kappa > 0$ and a consistent density λ such that

$$\lambda_t = \kappa \frac{E_t \lambda}{B_t}, \; t = 0, ..., T. \tag{65}$$

Proof. By using the formula $E_t^\lambda \xi = (E_t \lambda \xi)/E_t \lambda$, valid for the conditional expectation $E_t^\lambda \xi$ of a random variable ξ with respect to the measure P^λ, we obtain that a process $f_0, ..., f_T$ is a Y^*-martingale with respect to the probability measure $P^\lambda(d\omega) = \lambda(\omega)P(d\omega)$ if and only if

$$f_t - \frac{E_t(\lambda f_{t+1})}{E_t \lambda} \in Y_t^*(\omega). \tag{66}$$

Since $E_t(\lambda f_{t+1}) = E_t[(E_{t+1}\lambda) f_{t+1}]$ and since $Y_t^*(\omega)$ is a cone, relation (66) is equivalent to the following one:

$$(E_t \lambda) f_t - E_t[(E_{t+1}\lambda) f_{t+1}] \in Y_t^*(\omega). \tag{67}$$

Consequently, the process $f_t, \; t = 0, ..., T$, is a Y^*-martingale with respect to P^λ if and only if the process $(E_t \lambda) f_t, \; t = 0, ..., T$, is a Y^*-martingale with respect to the original measure P.

Suppose (b) holds. Let $\kappa > 0$ be a constant and let λ be a consistent density such that the functions $\lambda_0, ..., \lambda_T$ admit the representation (65). Then there exists a process $f_0, ..., f_T$ such that the sequence $(E_t \lambda) f_t, \; t = 0, ..., T$, is a Y^*-martingale with respect to the original measure P and inequalities (64) hold. Define $p_t = \kappa (E_t \lambda) f_t$. Then the process $p_0, ..., p_T$ is a Y^*-martingale and

$$\kappa \frac{(E_t \lambda)}{B_t} S_t^i (1 - \beta_t^i) \le p_t^i \le \kappa \frac{(E_t \lambda)}{B_t} S_t^i (1 + \alpha_t^i),$$

which implies that the sequence of functions $\lambda_0, ..., \lambda_T$ given by (65) is a chain of consistent discount factors.

Conversely, assume (a) is valid. Then there exists a Y^*-martingale $p_0, ..., p_T$ satisfying (54). By virtue of (IB) and (IB$_1$), the sequence $p_t^1 = \lambda_t B_t, \; t =$

$0, ..., T$, is a martingale, and so $\lambda_t = [E_t(\lambda_T B_T)]/B_t$. Put $\kappa = E(\lambda_T B_T)$ and $\lambda = (\lambda_T B_T)/\kappa$. Then formula (65) holds. From (54) we get

$$\kappa(E_t\lambda)\frac{S_t^i}{B_t}(1-\beta_t^i) \le p_t^i \le \kappa(E_t\lambda)\frac{S_t^i}{B_t}(1+\alpha_t^i),\ t = 0, ..., T.$$

Define $f_t = p_t(\kappa E_t\lambda)^{-1}$. Then the sequence $f_0, ..., f_T$ satisfies (64), and the process $(E_t\lambda)f_t,\ t = 0, ..., T$, is a Y^*-martingale (with respect to the original measure P). Thus λ is a consistent density. □

By using Proposition 9.1 and Theorems 7.2 and 9.1, we obtain the following necessary and sufficient condition for the absence of arbitrage opportunities in the model we study in this section.

Theorem 9.3. *Under assumptions (IB) and (IB$_1$), hypothesis (NA) is valid if and only if there exists an equivalent consistent measure.*

Proposition 9.1 and Theorem 9.2 allow to give a hedging criterion stated in terms of equivalent consistent measures.

Theorem 9.4. *Let the algebra $\mathcal{F}_0$ be trivial and let assumptions (IB) and (IB$_1$) be fulfilled. If (NA) holds, then an initial endowment v_0 is sufficient to hedge a contingent claim v_T if and only if*

$$v_0 \ge \sup_Q E^Q \frac{v_T}{B_T}, \tag{68}$$

where the supremum is taken with respect to all equivalent consistent measures Q.

We write E^Q for the expectation with respect to the equivalent consistent measure $Q(d\omega) = \lambda(\omega)P(d\omega)$, where $\lambda(\omega)$ is the corresponding consistent density.

Proof of Theorem 9.4. By virtue of Theorem 9.2, v_0 is sufficient to hedge v_T if and only if $E\lambda_0 v_0 \ge E\lambda_T v_T$ for all pairs (λ_0, λ_T), where λ_0 and λ_T are the initial and the final terms in a chain of consistent discount factors $(\lambda_0, ..., \lambda_T)$. Such pairs are, by virtue of Proposition 9.1, given by $\lambda_0 = \kappa E\lambda/B_0 = \kappa$, $\lambda_T = \kappa\lambda/B_T$, where λ is a consistent density and $\kappa > 0$. Thus v_0 is sufficient to hedge v_T if and only if $v_0 \ge E[(\lambda v_T)/B_T] = E^Q(v_T/B_T)$. □

10 Portfolio constraints, no transaction costs

Models with portfolio constraints, in discrete and continuous time, have been considered by many authors – see, in particular, Cvitanić and Karatzas [12], Karatzas and Kou [44], Jouini and Kallal [44], Schürger [44], Föllmer and Kramkov [25], Brannath [8], Pham and Touzi [49], Pham [50], Carassus, Pham and Touzi [10], Föllmer and Schied [24], Evstigneev, Schürger and Taksar [22] and Rokhlin [55].

We will examine a special case of the model analyzed in the previous section in which there are no transaction costs (i.e., $\alpha_t^i = \beta_t^i = 0$ for all i), but not all portfolios are admissible. Suppose we are given a vector $S_t(\omega) = (S_t^1(\omega), ..., S_t^n(\omega)) > 0$ specifying the market prices of n assets $i = 1, 2, ..., n$ at date t. Initial endowments and contingent claims are measured in terms of cash, so that $m_0 = m_T = 1$. For each $t = 0, 1, ..., T-1$, a polyhedral cone $Y_t(\omega)$ depending $\mathcal{F}_t$-measurably on ω and *satisfying condition (B)* is given. The set $Y_t(\omega)$ specifies the class of admissible portfolios at date t.

The cones $Z_t(\omega)$, $t = 1, ..., T-1$, are defined by

$$Z_t(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n : \ S_t(\omega)(b - a) \leq 0, \ b \in Y_t(\omega)\}. \tag{69}$$

Further, we have

$$V_0(\omega) := \{(a, b) \in \mathbb{R}^1 \times \mathbb{R}^n : \ a \geq S_0(\omega)b\}, \tag{70}$$

and the cone $V_T(\omega)$ is defined by formula (14). These definitions correspond to (51) – (53) in the special case where $\alpha_t^i = \beta_t^i = 0$ (then $\tau_t(b) = b$).

In the model at hand, *chains of consistent discount factors are sequences* $(\lambda_0, ..., \lambda_T)$ *of strictly positive real-valued functions such that* λ_t *is* $\mathcal{F}_t$*-measurable and the process* $\lambda_0 S_0, ..., \ \lambda_T S_T$ *is a* Y^**-martingale.* This is so because the inequalities in (54) turn into equalities as long as $\alpha_t^i = \beta_t^i = 0$ for all i and t. Thus we obtain, as a consequence of Theorems 9.1 and 9.2, the following result.

Theorem 10.1. *A sequence* $(q_0, p_0, ..., p_T, q_T)$ *is a consistent valuation system in the model given by (69), (70) and (14) if and only if there exists a sequence* $\lambda_0, ..., \lambda_T$ *of strictly positive real-valued functions such that* λ_t *is* $\mathcal{F}_t$*-measurable, the process* $\lambda_0 S_0, ..., \ \lambda_T S_T$ *is a* Y^**-martingale, and we have*

$$q_0 = \lambda_0, \ p_0 = \lambda_0 S_0, ..., \ p_T = \lambda_T S_T, \ q_T = \lambda_T \,. \tag{71}$$

An initial endowment v_0 *allows the hedging of a contingent claim* v_T *if and only if* $E\lambda_0 v_0 \geq E\lambda_T v_T$ *for all such sequences* $\lambda_0, ..., \lambda_T$.

Let us call a scalar function $\lambda(\omega) > 0$ a Y^**-martingale density* if $E\lambda = 1$ and the sequence S_t/B_t $(t = 0, ..., T)$ of the discounted price vectors is a Y^*-martingale with respect to the probability measure $P^\lambda(d\omega) = \lambda(\omega)P(d\omega)$ equivalent to P with density $\lambda(\omega)$ (*equivalent* Y^**-martingale measure*). If assumption (IB) holds, then by virtue of Proposition 9.1 we obtain that a sequence of strictly positive random variables $\lambda_0, ..., \lambda_T$ is a chain of consistent discount factors if there exists a constant $\kappa > 0$ and a Y^*-martingale density

λ such that equalities (65) are satisfied. By using this remark and Theorems 7.2 and 10.1, we obtain the following result.

Theorem 10.2. *Let assumption (IB) be fulfilled. Then the existence of an equivalent Y^*-martingale measure is a necessary and sufficient condition for the validity of hypothesis (NA).*

Theorem 9.4 implies the following hedging criterion in terms of equivalent Y^*-martingale measures.

Theorem 10.3. *Let the algebra $\mathcal{F}_0$ be trivial and let condition (IB) hold. If hypothesis (NA) is valid, then an initial endowment v_0 is sufficient to hedge a contingent claim v_T if and only if*

$$v_0 \geq \sup_Q E^Q \frac{v_T}{B_T}, \tag{72}$$

where the supremum is taken with respect to all equivalent Y^*-martingale measures Q.

Consider the model which is defined exactly as the previous one with the only difference being that in the definition of the cones $V_0(\omega)$ and $Z_t(\omega)$, $t = 1, ..., T-1$ (see (70) and (69)) inequalities are replaced by equalities:

$$V_0(\omega) := \{(a, b) \in \mathbb{R}^1 \times \mathbb{R}^n : \, a = S_0(\omega) b\}, \tag{73}$$

$$Z_t(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n : S_t(\omega)(b - a) \, = 0, \; b \in Y_t(\omega)\}. \tag{74}$$

All the results obtained for the previous model remain valid for this modification of it. The hypotheses of strict monotonicity (SM_t), $t = 1, ..., T-1$, and (SM_T) follow from Proposition 8.3, (c) and Proposition 8.4, (c), respectively. Condition (SM_0) can be easily verified by using (73) and (B). The monotonicity of the model and property (SH_T) (following from (14)) imply the validity of hypothesis (SH). Thus the no-arbitrage hypotheses (NA) and (NA_0) are equivalent to each other and to the existence of consistent valuation systems. Furthermore, consistent valuation systems for the original model and for its modification are the same. This follows from the fact that the cones $V_0^\times(\omega)$ and $Z_t^\times(\omega)$ ($t = 1, ..., T-1$) do not change if we define $V_0(\omega)$ and $Z_t(\omega)$ by (73) and (74) rather than by (70) and (69). Consequently, the hedging and no-arbitrage criteria in both models are the same.

We summarize the above comments in the following theorem.

Theorem 10.4. *All the assertions of Theorems 10.1 – 10.3 remain valid for the model defined by (73), (74) and (14).*

11 Frictionless market: strict hedging and superhedging

Suppose that, in the model discussed at the end of the previous section, $Y_t(\omega) = \mathbb{R}^n$, $t = 0, ..., T-1$, i.e. portfolio constraints are absent. Then we arrive at the version of the classical model of a frictionless market in which the self-financing constraints for asset trading are described in terms of the cones $Z_t(\omega)$ defined by (10) and possibilities of portfolio creation and liquidation are characterized by the cones $V_0(\omega)$ and $V_T(\omega)$ given by (12) and (14). Alternatively, we can define $V_T(\omega)$ by (13) instead of (14). This will lead to another version of the classical model and to another notion of hedging. We will refer to the former notion as *superhedging* and to the latter as *strict hedging*. According to the former (see (12), (10) and (14)) a contingent claim v_T can be superhedged starting from an initial endowment v_0 if there exists a trading strategy $y_0, ..., y_{T-1}$ ($y_t \in \mathcal{L}_t(n)$) such that $v_0 = S_0 y_0$, $S_t y_{t-1} = S_t y_t$, $t = 1, ..., T-1$, and $S_T y_{T-1} \geq v_T$ (a. s.). The latter notion—strict hedging—has the analogous meaning (see (12), (10) and (13)), with the only difference being that the last inequality is replaced by the equality $S_T y_{T-1} = v_T$. Here, we recall, $0 < S_t = (S_1^1, ..., S_t^n) \in \mathcal{L}_t(n)$ $(t = 0, ..., T)$ are the given vectors of asset prices. We write B_t for S_t^1 and assume that $B_0 = 1$.

Since the model defined by (12), (10) and (14) (let us call it *the model of superhedging*) is a special case of the one considered at the end of the previous section, we can directly apply Theorem10.4 (and hence Theorems 10.1 – 10.3) to it. We have only to observe that if $Y_t(\omega) = \mathbb{R}^n$, then Y^*-martingales are simply martingales. This leads to the following results.

Theorem 11.1. *In the model of superhedging, hypothesis (NA) is necessary and sufficient for the existence of an equivalent martingale measure. If the σ-algebra $\mathcal{F}_0$ is trivial and hypothesis (NA) holds, then an initial endowment v_0 allows the (super)hedging of a contingent claim v_T if and only if*

$$v_0 \geq \sup_Q E^Q \frac{v_T}{B_T}$$

for all equivalent martingale measures Q.

The first assertion of this theorem is the classical "Fundamental Theorem of Asset Pricing"—see Harrison and Kreps [27], Harrison and Pliska [28], Kreps [46], Dalang, Morton and Willinger [14], Schachermayer [57], Kabanov and Kramkov [41], Rogers [54], Jacod and Shiryaev [32], Kabanov and Stricker [43]. The latter is the well-known superhedging criterion—e.g. El Karoui and Quenez [19], Cvitanić and Karatzas [13], Pliska [51], and Föllmer and Schied [24].

Let us now turn to the model defined by (12), (10) and (13); let us call it *the model of strict hedging.* This model is regular and strictly monotone, but it does not satisfy hypothesis (SH), in contrast with all the others considered in this paper. Thus Theorem 7.3 which we used for deriving hedging criteria is not applicable. However, all the cones involved in the description of the model (12), (10) and (13) are linear spaces. Therefore we can apply Theorem 7.4. We also note that, for the models of superhedging and strict hedging, the cones $V_t^\times(\omega)$ $(t = 0, T)$, $Z_t^\times(\omega)$ $(t = 1, 2, ..., T-1)$ are the same, and so both models have the same consistent valuation systems. Thus we arrive at the following theorem.

Theorem 11.2. *In the model of strict hedging, hypothesis (NA) is necessary and sufficient for the existence of an equivalent martingale measure. If the σ-algebra $\mathcal{F}_0$ is trivial and hypothesis (NA) holds, then an initial endowment v_0 allows the (strict) hedging of a contingent claim v_T if and only if*

$$v_0 = E^Q \frac{v_T}{B_T} \tag{75}$$

for all equivalent martingale measures Q.

Relation (75) is the classical formula for the *no-arbitrage price* of a strictly hedgeable contingent claim (see, e.g. Pliska [51]). According to our definition, a contingent claim v_T is strictly hedgeable if there exists a feasible trading strategy of the form $(v_0, y_0, ..., y_{T-1}, v_T)$, allowing one to obtain at date T exactly the payoff v_T. In view of formula (75) (holding under the no-arbitrage hypothesis) the initial endowment v_0 of this strategy is determined uniquely and can be computed by formula (75). Note that the right-hand side of (75) does not depend on the particular choice of an equivalent martingale measure Q.

If the market is *complete*, i.e. any contingent claim is strictly hedgeable, then formula (75) gives a natural recipe for the "fair" price of each contingent claim. As is well-known, in the case of a complete market, the equivalent martingale measure is unique. Indeed, if λ and λ' are the densities of two such measures, we have $E(\lambda B_T^{-1} v_T) = E(\lambda' B_T^{-1} v_T)$ for each v_T, which implies $\lambda = \lambda'$ (a.s.).

In an incomplete market, not every contingent claim is strictly hedgeable. However, in our model, every contingent claim v_T can be superhedged (in particular, because Ω is finite). Therefore we can use formula (75) for determining the price of v_T. The supremum on the right-hand side of (75) (denote it by $\pi(v_T)$) is called the *superhedging price* of the contingent claim v_T. This is the minimum initial level of wealth needed to superhedge v_T.

Remark 11.1. It is important to note that the supremum in (75) is attained if and only if the contingent claim v_T is strictly hedgeable. In this case the set $\{E^Q(B_T^{-1}v_T) : \ Q \in \mathcal{Q}\}$, where $\mathcal{Q}$ is the set of all equivalent martingale measures, consists of one point. Otherwise, this set is an open interval[5]. Elements of this interval are *no-arbitrage prices* of the contingent claim v_T (see Föllmer and Schied [24]). Thus, if a contingent claim is not strictly hedgeable, then, by assigning the superhedging price $\pi(v_T)$ to it and treating it as a new traded asset, we obtain a market with an arbitrage opportunity. Arbitrage opportunities can be eliminated if we slightly increase the price $\pi(v_T)$—replace it by $\pi(v_T)+\varepsilon$, where $\varepsilon > 0$ is any sufficiently small number.

The phenomenon described in the above remark is essentially due to the fact that, in the classical model we deal with, borrowing and short sales of all assets are allowed. If borrowing and short sales are prohibited, then arbitrage is excluded (starting from zero, one can get nothing but zero), and then no-arbitrage considerations are not sufficient for determining the price of an asset. In that case, however, the hedge pricing principle remains applicable and consistent valuation systems can be used to price assets. A typical model illustrating this situation will be discussed in Section 13.

12 A currency market model

The model we examine in this section develops that studied in a series of papers by Kabanov and coauthors (see, e.g., [40], [42], [39], and [15]). Consider a financial market where n currencies $i = 1, 2, ..., n$ are traded. For each $t = 0, 1, ..., T$, we are given an $n \times n$ matrix $(\gamma_t^{ij}(\omega))$, where γ_t^{ij} are strictly positive $\mathcal{F}_t$-measurable random variables. The numbers γ_t^{ij}, $i \neq j$, specify the *exchange rates* of the currencies (including transaction costs) at date t: for a unit of i, one can get γ_t^{ij} units of j. The number $\gamma_t^{ii} \geq 1$ determines the *interest rate* $\gamma_t^{ii} - 1$ for currency i: one unit of i deposited with a bank account over the time period t yields the amount γ_t^{ii}. A portfolio of currencies $b = (b^1, ..., b^n)$ can be obtained from a portfolio $a = (a^1, ..., a^n)$ at date t in a random situation ω if and only if there exists a nonnegative $n \times n$ matrix

[5] To verify this it is sufficient to observe the following: (a) the set of martingale densitities $\lambda > 0$ is a (relatively) open convex set in the linear manifold of all the functions λ for which the sequence $\lambda_0 S_0, ..., \lambda_T S_T$, where $\lambda_t = E_t(B_t^{-1}\lambda)$, is a martingale; (b) the linear mapping $\lambda \mapsto E\lambda v_T$ maps the above open set either onto a point or onto an open convex set in $\mathbb{R}^1$.

(g^{ij}) such that

$$b^i \le a^i + \sum_{j=1}^{n} \gamma_t^{ji}(\omega) g^{ji} - \sum_{j=1}^{n} g^{ij}, \ i = 1, ..., n. \tag{76}$$

Here, g^{ij} stands for the amount of currency i exchanged into currency j and g^{ii} for the amount of currency i deposited with a bank account. Therefore the sum $\sum_j g^{ij}$ is subtracted from the ith position of the portfolio. As a result of the exchange (plus interest), one gets the amount $\sum_j \gamma_t^{ji} g^{ji}$ of currency i. Hence the sum $\sum_j \gamma_t^{ji} g^{ji}$ is added to the ith position of the portfolio. Thus a portfolio of currencies $b = (b^1, ..., b^n)$ can be obtained from a portfolio $a = (a^1, ..., a^n)$ if and only if $b - a \in M_t(\omega)$, where

$$M_t(\omega) := \{f \in \mathbb{R}^n : \ f^i \le \sum_{j=1}^{n} \gamma_t^{ji}(\omega) g^{ji} - \sum_{j=1}^{n} g^{ij} \text{ for some matrix } (g^{ij}) \ge 0\} \tag{77}$$

$(t = 0, ..., T)$.

Initial endowments of an investor trading on the market under consideration are portfolios of m_0 currencies and contingent claims are portfolios of m_T currencies, where $1 \le m_t \le n$ $(t = 0, T)$ are given numbers. For each $t = 0, ..., T - 1$, a polyhedral cone $Y_t(\omega) \subseteq \mathbb{R}^n$ is given. Those and only those currency portfolios are admissible which belong to $Y_t(\omega)$. To include the currency market model into our general framework, we define

$$V_0(\omega) := \{(a, b) \in \mathbb{R}^{m_0} \times \mathbb{R}^n : \ b - (a, 0) \in M_0(\omega)\}, \tag{78}$$

$$Z_t(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n : \ b - a \in M_t(\omega), \ b \in Y_t(\omega)\} \tag{79}$$

$(t = 1, ..., T - 1)$, and

$$V_T(\omega) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^{m_T} : \ a - (b, 0) \in M_T(\omega)\}. \tag{80}$$

The given cone $Y_0(\omega)$ specifies the set of admissible initial portfolios. In (78), we write $(a, 0)$ for the n-dimensional vector whose first m_0 coordinates coincide with the respective coordinates of the vector $a \in \mathbb{R}^{m_0}$ and all the other coordinates are equal to zero. The notation $(b, 0)$ in (80), where $b \in \mathbb{R}^{m_T}$, has the analogous meaning. The inclusion $(v_0, y_0) \in V_0(\omega)$ describes possibilities of constructing initial portfolios y_0 from initial endowments v_0 (which contain only the currencies $i = 1, ..., m_0 \le n$). Analogously, the inclusion $(y_{T-1}, v_T) \in V_T(\omega)$ describes portfolio pairs (y_{T-1}, v_T) such that

y_{T-1} can be converted into the contingent claim v_T—a portfolio of currencies $i = 1, ..., m_T$.

It is clear that the model under consideration is regular since all the cones in (78) – (80) are polyhedral. We will assume that the cones $Y_t(\omega)$ satisfy condition (B), i.e. contain the vector $e_1 = (1, 0, ..., 0)$. Then the model under consideration is strictly monotone, which follows from Lemma 12.1 below.

Define $Y_T(\omega)$ as the set of vectors $\{(b, 0) \in \mathbb{R}^n : b \in \mathbb{R}^{m_T}\}$.

Lemma 12.1. *For each $\omega \in \Omega$ and $t = 0, ..., T$, if $c \geq b \in Y_t(\omega)$, $c - a \in M_t(\omega)$, $a' \geq a$ and $a' \neq a$, then there is $b' \in Y_t(\omega)$ such that $b' \geq b$, $b' \neq b$ and $b' - a' \in M_t(\omega)$.*

Proof. The non-negative non-zero portfolio $d := a' - a$ can be exchanged into some portfolio $\kappa e_1 = (\kappa, 0, ..., 0)$ with $\kappa > 0$, i.e. $\kappa e_1 - d \in M_t(\omega)$ (formally, put in (76) $g^{j1} = d^j$ and $g^{ji} = 0$ if $i \neq 1$). Set $b' = b + \kappa e_1$. Then $b' \geq b$, $b' \neq b$ and $b' \in Y_t(\omega)$ because $b \in Y_t(\omega)$ and $e_1 \in Y_t(\omega)$. Finally, the vector $b' - a' = \kappa e_1 - d + b - a$ belongs to $M_t(\omega)$ because $b - a \leq c - a \in M_t(\omega)$ and $\kappa e_1 - d \in M_t(\omega)$. □

It follows from the strict monotonicity of the model that hypotheses (NA) and (NA_0) are equivalent (see Proposition 7.2). Condition (SH_T), which is a consequence of (77) and (80), implies the validity of hypothesis (SH). Thus the no-arbitrage criterion and the hedging criterion stated in Theorems 7.2 and 7.3 are valid. By applying these results to the model at hand, we arrive at Theorem 12.1 formulated below.

For any vector $a = (a^1, ..., a^n) \in \mathbb{R}^n$, we write

$$\check{a} := (a^1, ..., a^{m_0}),\ \hat{a} := (a^1, ..., a^{m_T}).$$

Theorem 12.1. *In the currency market model (78) - (80), consistent valuation systems are sequences $(q_0, p_0, ..., p_T, q_T)$ such that*

$$q_0 = \check{p}_0,\ q_T = \hat{p}_T, \tag{81}$$

and $p_0, ..., p_T$ is a strictly positive Y^*-martingale satisfying for each $t = 0, 1, ..., T$ the following inequalities:

$$\gamma_t^{ji} p_t^i \leq p_t^j,\ i, j = 1, ..., n, \tag{82}$$

where p_t^i is the ith coordinate of the vector p_t. The no-arbitrage hypothesis (NA) is equivalent to the existence of such a Y^*-martingale. A contingent claim $v_T \in \mathcal{L}_T(m_T)$ can be hedged starting from an initial endowment $v_0 \in \mathcal{L}_T(m_0)$ if and only if $E\check{p}_0 v_0 \geq E\hat{p}_T v_T$ for any Y^*-martingale $p_0, ..., p_T$ satisfying (82).

Proof. By using Proposition 8.2, we obtain that a sequence $(q_0, p_0, ..., p_T, q_T)$ is a consistent valuation system if and only if the following conditions hold:

$\mathcal{L}_0(m_0) \ni q_0 > 0$, $\mathcal{L}_T(m_T) \ni q_T > 0$; $p_0, ..., p_T$ is a non-negative Y^*-martingale;

$$q_0 a \geq p_0 b \text{ when } b - (a, 0) \in M_0(\omega); \tag{83}$$

$$p_t f \leq 0 \text{ for all } f \leq \sum_j \gamma_t^{ji}(\omega) g^{ji} - \sum_j g^{ij}; \tag{84}$$

where (g^{ij}) is any nonnegative matrix and $t = 1, ..., T-1$; and

$$p_T a \geq q_T b \text{ when } a - (b, 0) \in M_T(\omega). \tag{85}$$

Condition (84) is equivalent to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_t^i \gamma_t^{ji} g^{ji} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} p_t^j g^{ji}, \ (g^{ji}) \geq 0,$$

which, in turn, is equivalent to (82).

If property (83) holds, then by setting $b := (a, 0)$ with any $a \in \mathbb{R}^{m_0}$, we get $q_0 a = p_0(a, 0)$, which means $q_0 = \check{p}_0$. Further, by setting $a = 0$ in (83), we find that $-p_0 \in M_0^*$. This is equivalent to (82) with $t = 0$. Conversely, if $q_0 = \check{p}_0$ and $-p_0 \in M_0^*$, then, assuming that $b - (a, 0) \in M_0(\omega)$, we obtain $q_0 a - p_0 b = p_0(a, 0) - p_0 b \geq 0$. Analogously, we establish the equivalence of (85) and the relations $q_T = \hat{p}_T$ and $-p_T \in M_T^*$, where the latter holds if and only if $\gamma_T^{ji} p_T^i \leq p_T^j$.

The strict positivity of the Y^*-martingale $p_0, ..., p_T$ follows from condition (B) (which implies that the sequence $q_0^1 = p_0^1$, $p_1^1, ..., p_{T-1}^1$, $p_T^1 = q_T^1 > 0$ is a supermartingale) and the inequality $p_t^j \geq \gamma_t^{1j} p_t^1$, which is a consequence of (82). □

Consider the special case where $m_0 = m_T = 1$, so that initial endowments and contingent claims are measured in terms of currency 1. Further, assume that the structure of the exchange rates is as follows:

$$\gamma_t^{ij} = \frac{S_t^i(1 - \beta_t^i)}{S_t^j(1 + \alpha_t^j)}, \ i \neq j, \ i, j = 1, ..., n,$$

and we have $\gamma_t^{ii}(\omega) = 1$ for all i, t and ω. For one unit of currency i, one can get $S_t^i(1 - \beta_t^i)$ units of currency 1, playing the role of a *numeraire*, and then convert the amount obtained into $S_t^i(1 - \beta_t^i)/[S_t^j(1 + \alpha_t^j)]$ units of currency j. Here, S_t^i/S_t^j may be regarded as the "interbank exchange rate" (without transaction costs) of i into j. The numbers β_t^i and α_t^i $(i = 2, ..., n)$ are the transaction cost rates for exchanging currency i into the numeraire and backwards, respectively. We assume that $S_t^1 = 1$ and $\beta_t^1 = \alpha_t^1 = 0$, which corresponds to the role of currency 1 as a numeraire. The random variables $S_t^i > 0$, $\alpha_t^i > 0$ and $0 < \beta_t^i < 1$ are supposed to be $\mathcal{F}_t$-measurable. Under the assumptions imposed, conditions (81) and (82), describing consistent valuation systems in the model at hand, take on the following form: $q_0 = p_0^1$, $q_T = p_T^1$, and

$$p_t^i \frac{S_t^j(1-\beta_t^j)}{S_t^i(1+\alpha_t^i)} \leq p_t^j, \ i \neq j,$$

which can equivalently be written as

$$p_t^1 S_t^i(1-\beta_t^i) \leq p_t^i \leq p_t^1 S_t^i(1+\alpha_t^i), \ i = 1, ..., n,$$

because $S_t^1 = 1$ and $\beta_t^1 = \alpha_t^1 = 0$. Thus, in the case we consider, consistent valuation systems in the present model and in that examined in Section 9 coincide (see Theorem 9.1), and in this sense, these two models are equivalent.

13 Currency exchange without borrowing and short sales

We consider another currency market model—a version of that proposed in [23]. There are n currencies traded on the market at dates $t = 0, 1..., T$. As in the previous model (see Section 12), we are given a matrix $(\gamma_t^{ij}(\omega))$ $(i, j = 1, ..., n)$ whose elements are strictly positive $\mathcal{F}_t$-measurable random variables, specifying the exchange rates and the interest rates of the currencies traded. Initial endowments at date 0 and contingent claims at date T are portfolios of m_0 and m_T currencies, respectively $(1 \leq m_0,\ m_T \leq n)$.

A portfolio $a = (a^1, ..., a^k) \in \mathbb{R}^k$ of k currencies $i = 1, ..., k$ can be exchanged into portfolio $b = (b^1, ..., b^l) \in \mathbb{R}^l$ of l currencies $j = 1, ..., l$ at date $t = 0, ..., T$ if and only if there exists a nonnegative matrix (g^{ij}) $(i = 1, ..., k,$ $j = 1, ..., l)$ such that

$$a^i \geq \sum_{j=1}^{l} g^{ij}, \ i = 1, ..., k, \ b^j \leq \sum_{i=1}^{k} \gamma_t^{ij}(\omega) g^{ij}, \ j = 1, ..., l. \qquad (86)$$

Here, g^{ij} $(i \neq j)$ stands for the amount of currency i converted into currency j. The amount g^{ii} of currency $i \leq \min\{k, l\}$ is left unexchanged. It is deposited with a bank account, yielding $\gamma_t^{ii} g^{ii}$, where $\gamma_t^{ii} - 1$ is the interest rate.

We will assume that $\gamma_t^{ii} \geq 1$. The first inequality in (86) is a balance constraint for currency i: one cannot exchange more of it than is available at the beginning of date t (no borrowing is allowed). The second inequality in (86) says that, at the end of date t after the currency exchange, the jth position of the portfolio cannot be greater than the sum $\sum_{i=1}^{k} \gamma_t^{ij}(\omega) g^{ij}$ obtained as a result of the exchange (plus interest).

The set of all portfolio pairs (a, b) satisfying (86) for some $(g^{ij}) \geq 0$ will be denoted by $Z_t(\omega, k, l)$, $t = 0, ..., T$. The analogous set of (a, b) with the additional requirement $b \geq 0$ will be denoted by $Z_t^+(\omega, k, l)$. Clearly, $Z_t(\omega, k, l)$ and $Z_t^+(\omega, k, l)$ are polyhedral cones depending $\mathcal{F}_t$-measurably on ω. To include the model outlined above into our general framework, we define

$$V_0(\omega) := Z_0(\omega, n, m_0),\ Y_0(\omega) := \mathbb{R}_+^n\,, \tag{87}$$

$$Z_t(\omega) := Z_t^+(\omega, n, n),\ t = 1, ..., T-1,\ V_T(\omega) := Z_T(\omega, n, m_T). \tag{88}$$

It is clear that if a portfolio a can be exchanged into a portfolio b, i.e. $(a, b) \in Z_t(\omega, k, l)$, then any $a' \geq a$, $a \neq a$ can be exchanged into some $b' \geq b$, $b' \neq b$. This implies that the model defined by (87) and (88) is strictly monotone. Hypothesis (SH) is valid because of the monotonicity of the model and the property (SH$_T$) of the cone $V_T(\omega)$ (see (88)). Consequently, the hedging criterion established in Theorem 7.3 is applicable. It is easily seen that the model at hand does not have arbitrage opportunities: if $(v_0, y_0, ..., y_{T-1}, v_T)$ is a hedging strategy with $v_0 \leq 0$, then $v_0 = y_0 = ... = y_{T-1} = v_T = 0$. This guarantees, by virtue of Theorem 7.2, the existence of consistent valuation systems.

Theorem 13.1. *In the model defined by (87) and (88), consistent valuation systems are sequences $(q_0, p_0, ..., p_T, q_T)$ such that $p_0, ..., p_T$ $(p_t \in \mathcal{P}_t)$ is a strictly positive supermartingale, $0 < q_t \in \mathcal{L}_t(m_t)$, $t = 0, T$, and we have*

$$q_0^i \geq \gamma_t^{ij} p_0^j,\ i = 1, ..., m_0,\ j = 1, ..., n, \tag{89}$$

$$p_t^i \geq \gamma_t^{ij} \bar{p}_{t+1}^j,\ t = 1, 2, ..., T-1,\ i, j = 1, ..., n, \tag{90}$$

$$p_T^i \geq \gamma_T^{ij} q_T^j,\ i = 1, ..., n,\ j = 1, ..., m_T, \tag{91}$$

where $\bar{p}_{t+1}^i = E_t p_{t+1}^i$. The no-arbitrage hypothesis holds always. A contingent claim $v_T \in \mathcal{L}_T(m_T)$ can be hedged starting from an initial endowment $v_0 \in \mathcal{L}_0(m_0)$ if and only if $E q_0 v_0 \geq E q_T v_T$ for all consistent valuation systems $(q_0, p_0, ..., p_T, q_T)$ described above.

Proof. Observe that the set $Z_t(\omega, k, l)^\times$, as well as the set $[Z_t^+(\omega, k, l)]^\times$, consists of those $(c, d) \in \mathbb{R}_+^k \times \mathbb{R}_+^l$ for which

$$\sum_{j=1}^{l} d^j \sum_{i=1}^{k} \gamma_t^{ij} g^{ij} - \sum_{i=1}^{k} c^i \sum_{j=1}^{l} g^{ij} \leq 0, \tag{92}$$

where $(g^{ij}) \geq 0$. This inequality holds if and only if

$$d^j \gamma_t^{ij} \leq c^i, \; i = 1, ..., k, \; j = 1, ..., l. \tag{93}$$

By applying (93) in the cases $t = 0$, $1 \leq t \leq T - 1$, and $t = T$, we obtain (89), (90) and (91), respectively.

Since $Y_0(\omega) = \mathbb{R}_+^n$, condition (22) involved in the definition of a consistent price system, becomes $p_0 \geq \bar{p}_1$. From (90) it follows that $p_t \geq \bar{p}_{t+1}$ because $\gamma_t^{ii} \geq 1$. Thus a consistent price system $p_0, ..., p_T$ is a supermartingale. Its strict positivity follows from (91), as long as $q_T > 0$. □

Observe that the cone $Z_t^+(\omega, k, l)$ can be represented in the form

$$Z_t^+(\omega, k, l) = \{(a, b) \in \mathbb{R}_+^k \times \mathbb{R}_+^l :$$

$$a \geq Ag, \; 0 \leq b \leq B_t(\omega)g \text{ for some } g \in \mathbb{R}_+^{k \times l}\}. \tag{94}$$

Here $A : \mathbb{R}^{k \times l} \to \mathbb{R}^k$ and $B_t(\omega) : \mathbb{R}^{k \times l} \to \mathbb{R}^l$ are non-negative linear operators transforming a matrix $g = (g^{ij}) \in \mathbb{R}^{k \times l}$ into the vectors Ag and $B_t(\omega)g$ whose coordinates are defined by

$$(Ag)^i = \sum_{j=1}^{l} g^{ij} \text{ and } (B_t(\omega)g)^j = \sum_{i=1}^{k} \gamma_t^{ij}(\omega) g^{ij}.$$

Thus the model under consideration is a direct stochastic analogue of the von Neumann [63] model of economic growth (A and $B_t(\omega)$ being the counterparts of the "technology matrices").

14 On the von Neumann–Gale model. Concluding remarks

In this section, we briefly discuss links between the theory developed above and the von Neumann-Gale model [63], [26]. The latter describes an economy in which, at time $t = 0, 1, ...$, there are n commodities $i = 1, 2, ..., n$. The state of the economy at time t is characterized by a commodity vector $y_t =$

$(y_t^1, ..., y_t^n) \in \mathbb{R}_+^n$. Possible transitions from one state to another are specified by a sequence of cones $\mathcal{Z}_t \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^n$, $t = 1, 2, ...$ A *path (trajectory)* of the economic system is a finite or infinite sequence $y_0, y_1, y_2, ...$ such that $(y_{t-1}, y_t) \in \mathcal{Z}_t$, $t = 1, 2, ...$ Elements $(x, y) \in \mathcal{Z}_t$ are called *input-output* pairs or *technological processes.* The sets $\mathcal{Z}_t$ are termed *technology sets.* In the original [63] model, the cones $\mathcal{Z}_t$ were supposed to be polyhedral; they had a special structure motivated by economic considerations (cf. (94) above). Gale [26] generalized von Neumann's framework allowing general, not necessarily polyhedral, cones.

The main focus in the von Neumann-Gale model is on the analysis of paths that grow faster than others—e.g. maximize growth rates over each time period $(t-1, t]$. Such paths are called *efficient* or *rapid* (see Evstigneev and Schenk-Hoppé [21]). A precise definition is as follows. A path $y_0, y_1, y_2, ...$ is called efficient if there exists a sequence of price vectors $p_0, p_1, ...$ $(p_t \in \mathbb{R}_+^n)$ such that $p_t y_t = 1$ and

$$p_t y \leq p_{t-1} x \text{ for all } (x, y) \in \mathcal{Z}_t. \tag{95}$$

By virtue of this definition, the *growth rate* $p_t y_t / p_{t-1} y_{t-1}$ over the time period $(t-1, t]$ attains its maximum on the path $y_0, y_1, ...$ Since $p_t y_t = 1$, the growth rate on $y_0, y_1, ...$ is constant and equal to one. (In fact, what matters here is that $p_t y_t$ is a strictly positive constant.)

Sequences of price vectors $p_0, p_1, ...$ satisfying (95) are called *dual paths.* They may be regarded as paths in the *dual model* defined by the sequence of cross-dual cones $\mathcal{Z}_t^\times$ (see (27)). Thus *a trajectory $y_0, y_1, y_2, ...$ is efficient if there exists a dual trajectory $p_0, p_1, p_2, ...$ such that $p_t x_t = 1$.* Clearly, if $p_0, p_1, p_2, ...$ is a dual path, then $p_0 y_0 \geq p_1 y_1 \geq p_2 y_2 \geq ...$ for each path $y_0, y_1, y_2, ...$ of the original system.

In this work, we have used stochastic analogues of dual paths (consistent valuation systems) to characterize the set of those pairs y_0, y_T of states of the dynamical model at hand such that y_T can be reached from y_0. The main idea lies in that, under certain general assumptions, y_T is reachable from y_0 if and only if $p_0 y_0 \geq ... \geq p_T y_T$ for all dual paths $p_0, ..., p_T$. (In the stochastic case, these inequalities are supposed to hold in the sense of expectations.) Hedging problems considered in this paper are fully analogous to this reachability problem. The analogues of commodity vectors are contingent portfolios of assets, and the counterparts of paths in the von Neumann–Gale model are hedging strategies. The transition cones specifying self-financing and other portfolio constraints correspond in the financial context to the technology cones in the economic dynamics context.

In the conventional theory of the von Neumann–Gale model, the main theme is the modeling of economic growth. Dual paths play an important, but not the primary role; they are, rather, a key tool in the analysis. In this work, we did not consider questions of financial growth at all. However, the methodology of the von Neumann–Gale model can be applied in this field as well. Some steps in analyzing financial growth from this angle were made in [16] and [21, Section 6]. This is an interesting direction for further research, closely related to capital growth theory (Kelly [45], Breiman [9], Algoet and Cover [1], Hakansson and Ziemba [29], Iyengar and Cover [31]) and evolutionary finance (Blume and Easley [7], Evstigneev, Hens and Schenk-Hoppé [20]).

The study of stochastic versions of the von Neumann–Gale model was pioneered by Dynkin, Radner and their collaborators in the early 1970s. Radner [52] focused primarily on the issues of growth, typical for the von Neumann–Gale theory. Dynkin [17], [18, Chapter 9] analyzed stochastic analogues of dual paths and considered in the economics context what we call here consistent price systems.

There are some conceptual differences in the ways of "stochastization" of the von Neumann–Gale model in Dynkin's and Radner's approaches. In Dynkin's model, technology sets contain input-output pairs of contingent commodity vectors (x_t, y_t) associated with the same date t and influenced by the same random factors (mathematically, x_t and y_t are supposed to be measurable with respect to the same σ-algebra $\mathcal{F}_t$). In Radner's approach, inputs x_{t-1} and outputs y_t correspond to the beginning and the end of the production period $(t-1, t]$ and therefore they are influenced by different random factors (x_{t-1} is measurable with respect to the σ-algebra $\mathcal{F}_{t-1}$ smaller than $\mathcal{F}_t$). These distinctions in the frameworks—which might seem not very essential—lead in fact to substantial distinctions in the structure of the duality results underlying the construction of consistent price systems.

Radner's approach appeared to be more natural from the economic point of view and it became prevailing in the economic dynamics literature—see the survey in Arkin and Evstigneev [3]. However, it turned out that Dynkin's setting fits the financial context better. First of all, it is quite natural to assume that portfolio rebalancing is performed with full information about the current asset prices and transaction costs (and so both contingent portfolios x_t and y_t are measurable with respect to the σ-algebra $\mathcal{F}_t$). Moreover, the duality results, such as Theorem 7.1 in this paper, obtained along Dynkin's lines (cf. [18, Section 9.5]) fit exactly the objectives of the present work. They allow one to obtain general hedging criteria that unify the whole range of available examples and applications.

To conclude, we would like to list several directions of further research, which, in our opinion, are of interest. These are: (i) as already mentioned above—the analysis of questions of financial growth within the framework proposed; (ii) portfolio optimization within this framework; (iii) developing numerical algorithms based on duality considerations (involving consistent valuation systems) for computing approximate solutions to hedging and optimization problems; (iv) finding explicit analytical solutions to these problems in specialized models[6]; and (v) mathematical generalizations of the results obtained (in particular, their extension of to the case of a general, not necessarily finite, probability space).

Appendix

For a cone K in $\mathbb{R}^n$, we denote by K^* the *dual* to the cone K, i.e. the set of all those linear functionals q on $\mathbb{R}^n$ for which $qx \geq 0$, $x \in K$. We write K^+ for the set of those q in K^* that satisfy $qx > 0$ for each non-zero element $x \in K$. A cone K is called *proper* if $K \cap (-K) = \{0\}$.

Theorem A.1. *Let H and M be closed cones in $\mathbb{R}^n$ such that $H \cap (-M) = \{0\}$. Then $H + M$ is a closed cone. If, additionally, H and M are proper, then $H + M$ is proper.*

Theorem A.1 is a consequence of [53, Corollary 9.1.3].

Let H and K be a closed cones in $\mathbb{R}^n$, K being proper. These cones will be fixed throughout the rest of the Appendix. The next result can be deduced from [53, Corollary 11.4.2].

Theorem A.2. *The following two assertions are equivalent:*

(a) $H \cap K = \{0\}$.

(b) There exists $l \in K^+$ such that $lh \leq 0$ for all $h \in H$.

Define

$$N = \{l \in K^+ : \ lh \leq 0, \ h \in H\}.$$

Theorem A.2 says that

$$H \cap K = \{0\} \Leftrightarrow N \neq \emptyset.$$

When H is a linear space, we have

$$N = \{l \in K^+ : \ lh = 0, \ h \in H\}. \tag{96}$$

Thus, by using Theorem A.2, we obtain the following result.

[6]In this connection, see Stettner [61] and Sass [56].

Theorem A.3. *Let H be a linear space. Then the following assertions are equivalent.*

(a) $H \cap K = \{0\}$.

(b) There exists $l \in K^+$ such that $lh = 0$ for all $h \in H$.

Theorem A.4 below characterizes elements u in the cone $H - K$ in terms of *dual variables*—linear functionals l in N.

Theorem A.4. *Let $H \cap K = \{0\}$. Then, for any $u \in \mathbb{R}^n$, the following assertions are equivalent.*

(i) $u \in H - K$.

(ii) We have $lu \leq 0$ for all $l \in N$.

Proof. Clearly (ii) is a consequence of (i). Let (ii) hold. Suppose $u \notin H - K$. Put $U = \{\lambda u : \lambda \geq 0\}$. Since $u \notin (-K)$, the cone $U + K$ is proper and closed—see Theorem A.1. Furthermore $(U + K) \cap H = \{0\}$ because $u \notin H - K$. By virtue of Theorem A.2, there exists $l \in (U + K)^+$ such that $lh \leq 0$ for all $h \in H$. Then $l \in N$ because $l \in (U + K)^+ \subseteq K^+$. At the same time $lu > 0$ since $l \in (U + K)^+ \subseteq U^+$. A contradiction. □

If H is a linear space, then we can characterize not only $H - K$, but also H, in terms of dual variables l in N.

Theorem A.5. *Let $H \cap K = \{0\}$. If H is a linear space, then the following assertions are equivalent.*

(i) $u \in H$.

(ii) We have $lu = 0$ for all $l \in N$.

Proof. We can see that (i) implies (ii) by virtue of (96). Suppose (ii) holds. Then $u \in H - K$ by virtue of Theorem A.4. Let us apply Theorem A.4 to the cone $-K$ and the linear space H. This is possible since $-K$ is proper and closed, and $(-K) \cap H = (-K) \cap (-H) = \{0\}$. Define $\bar{N} = \{l \in (-K)^+ : lh = 0,\ h \in H\}$. Clearly $\bar{N} = -N$, and so $lu = 0$ for all $l \in \bar{N}$ in view of (ii). By virtue of Theorem A.4 applied to $-K$ and H, we find that $u \in H + K$. On the other hand $u \in H - K$, which implies $u \in H$ because $H \cap K = K \cap (-K) = \{0\}$. □

Let X be a convex subset in $\mathbb{R}^n$ and $f(x)$, $x \in X$, a concave real-valued function defined on X. Let C be a cone in $\mathbb{R}^k$ and $G(x)$ a vector function on X with values in $\mathbb{R}^k$. Assume that G is concave in the following sense

$$G(\lambda x_1 + (1 - \lambda)x_2) - \lambda G(x_1) - (1 - \lambda)G(x_2) \in C \tag{97}$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. (Clearly (97) holds if G is affine, i.e., the expression in (97) is equal to zero.) Consider the following optimization problem.

(M) Maximize $f(x)$ on the set X subject to the constraint

$$G(x) \in C. \tag{98}$$

Suppose that one of the conditions (SL) or (LP) below holds.

(SL) (Slater's constraint qualification.) There is a vector $x \in X$ such that $G(x) \in \mathrm{int}\, C$.

(LP) The sets X and C are polyhedral and f, G are affine.

If (LP) holds, then (M) is a linear programming problem.

Theorem A.6. *Let $\bar{x}$ be an element of X satisfying constraint (98). Then the following assertions are equivalent.*

(i) The vector $\bar{x}$ is a solution to optimization problem (M).

(ii) There exists a linear functional $p \in C^*$ such that

$$f(x) + p(G(x)) \leq f(\bar{x}) + p(G(\bar{x})),\ x \in X, \tag{99}$$

and

$$p(G(\bar{x})) = 0. \tag{100}$$

Condition (99) states that $\bar{x}$ maximizes the Lagrangian $L(p, x) = f(x) + p(G(x))$ over all $x \in X$ (not necessarily satisfying (98)). In this sense, p is said to *relax* the constraint (98). Equality (100) is the *complementary slackness condition.* If $p \in C^*$, then two relations (99) and (100) are equivalent to one: $f(x)+\ pG(x) \leq f(\bar{x})$, $x \in X$. For a proof of Theorem A.6 see, e.g., [47, Theorem 8.3.1].

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space. For each $\omega \in \Omega$, let $A(\omega) \subseteq \mathbb{R}^n$ be a set, possibly empty for some ω. Let $\mathcal{G} \subseteq \mathcal{F}$ be an algebra of subsets of Ω. We say that the set $A(\omega)$ depends $\mathcal{G}$*-measurably* on ω if there exists a partition $\Omega = \Omega_1 \cup ... \cup \Omega_k$ of Ω into disjoint sets $\Omega_1 \in \mathcal{G}, ..., \Omega_k \in \mathcal{G}$ such that $A(\omega)$ is constant on each Ω_i, $i = 1, ..., k$. The proof of the following fact is straightforward.

Theorem A.7. *If a set $A(\omega)$ depends $\mathcal{G}$-measurably on ω, then there is a $\mathcal{G}$-measurable mapping $\alpha : \Omega \to R^n$ such that $\alpha(\omega) \in A(\omega)$ for all those ω for which $A(\omega) \neq \emptyset$.*

The mapping α described in the above theorem is called a $\mathcal{G}$-measurable *selector* of the set-valued mapping $\omega \mapsto A(\omega)$. By using Theorem A.7, one can easily obtain the following result.

Theorem A.8. *Let $A(\omega) \subseteq \mathbb{R}^n$ be a non-empty set depending $\mathcal{G}$-measurably on ω and let $F(\omega, x)$ ($\omega \in \Omega$, $x \in \mathbb{R}^n$) be a real valued function $\mathcal{G}$-measurable in ω for each $x \in \mathbb{R}^n$ and continuous in x for each ω. Let $\bar{\alpha}(\omega)$ be a $\mathcal{G}$-measurable selector of $A(\omega)$. Then the following two assertions are equivalent:*

(a) The inequality $EF(\omega, \alpha(\omega)) \leq EF(\omega, \bar{\alpha}(\omega))$ is valid for each $\mathcal{G}$-measurable selector α of the set-valued mapping $A(\omega)$.

(b) With probability one, $F(\omega, a) \leq F(\omega, \bar{\alpha}(\omega))$ for all $a \in A(\omega)$.

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