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Airline Network Revenue Management with Buy-up

Houyuan Jiang* Giovanna Miglionico†

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Abstract

The emergence of budget airlines, the advent of new sales channels, the disappearance of traditional fare-class fences, and other on-going changes in the airline industry pose serious challenges to traditional revenue management models, which make some rigid and unrealistic assumptions. One of these is that passengers who do not get the fare they want book and travel on other airlines or do not travel at all. In reality many are not necessarily lost to the airline but buy up, i.e., buy a more expensive ticket. In this paper, we model network revenue management which incorporates buy-up using dynamic programming (DP). The resulting DP model is unlikely to be solved optimally due to the curse of dimensionality and hence is solved approximately by various simpler models such as deterministic linear programming (DLP), probabilistic nonlinear programming (PNLP), randomized linear programming (RLP), and approximate dynamic programming. Policies based on partitioned booking limits and bid prices in conjunction with some of the above approximate models are proposed to control capacity. We show that the partitioned booking limit policies for several models are asymptotically optimal. The bid-price policy is also shown to be asymptotically optimal provided that correct bid prices are used. We establish some analytical results to estimate expected network revenue gradients based on the RLP. Numerical results show that a significant increase in revenue is obtainable for all five booking schemes on four test examples even when the buy-up probability is relatively small.

Key words. Revenue management, capacity control, buy-up, dynamic programming, linear programming, nonlinear programming.

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1 Introduction

Revenue management (RM) has experienced a boom period in both theory and practice in the last two decades. The state of the art is presented in the recent book by Talluri and van Ryzin [21]. The emergence of budget airlines, the advent of new sales channels, the disappearance of traditional fare-class fences, and other on-going changes in the airline industry pose serious challenges to traditional RM models.

It has long been recognized that many traditional RM models are based on rigid and unrealistic assumptions. Hornick [12] describes five basic assumptions in the context of airlines that need to be challenged:

(1) Fares themselves are static and the thing that changes is the availability of fare classes;

(2) Demand is distinct for specific products or groups that are "fenced off" from one another to prevent less price-sensitive segments from buying lower-fare products;

(3) Passengers who do not get the fare they want book travel on another carrier or do not travel at all;

(4) Demand forecasts are based on historical patterns, not up-to-the-minute market conditions;

(5) Competitive actions from other carriers or other unexpected events are captured by the manual adjustment of forecasts.

In this paper we make one small step forward by weakening assumptions (2) and (3) above, which have often been discussed in the literature, see, for example, Talluri and van Ryzin [21]. In reality, many initially denied customers are not necessarily lost to the airline. According to Belobaba [3], the unavailability of a desired flight and fare class can lead to:

(a) A vertical shift to a higher fare class, same flight;

(b) A horizontal shift to a different flight, same fare class and airline;

(c) A booking loss to the refusing airline.

In [15], purchase behavior (a) is called buy-up or upgrade, and purchase behavior (b) is called recapture. We are particularly interested in customers' choices related to (a) and (b), and shall consider a buy-up model which captures both behaviors.

In the traditional RM models based on assumptions (2) and (3), each customer requests a unique product. The customer's request is either accepted or the customer is lost to the airline.

Our buy-up model can be interpreted in the following way in practice. When customers arrive, they have a product in mind. They either directly requests a product from an agent or checks seat availability over the internet. If their requested product is available, the request will be accepted. If not, they may ask for another product, but it is more likely that the agent will recommend a new set of products which could be an upgraded product with the same travel time and route or similar products with different travel time(s) and route(s). If they accept the agent's recommendation, they purchase a new product. If not, they leave the airline. In the case of internet bookings, the customers can act as an agent for themselves by repeatedly sending enquires to the booking system.

Several models without assumptions (2) and (3) have been proposed in the literature. In a single-leg setting, Belobaba [2, 3] proposes a heuristic algorithm based on the well-known expected marginal seat revenue (EMSR) for static capacity management which considers buy-up. You [27, 28] studies the optimal dynamic programming model and analyzes the booking policy. You obtains the following results: (1) The rejection decision can be optimally controlled using either a set of critical booking capacities or a set of critical decision periods; (2) The offering decision of possible buy-up can also be optimally controlled using a set of critical values. In the context of RM in a single-leg setting, Shumsky and Zhang [17] consider a problem similar to the one studied by You [27, 28] and examine a multiperiod capacity allocation model with upgrading. Buy-up models with only two classes of customers are also considered in [5, 16, 25].

A somewhat different approach incorporating buy-up is the so-called choice model. More precisely, at any point of time, possibly more than one product is made available to customers. Purchase behavior of customers may depend on which products are available and is characterized by the probability distribution as a function of the set of products offered. A general analysis based on dynamic programming is provided by Talluri and van Ryzin [20] for a single-leg setting. The same model has been extended to a network setting in [9, 22, 24].

In this paper, we propose a buy-up model for network RM using dynamic programming (DP). This new model reduces to the one proposed by You [27] in a single-leg setting and to the one proposed by Bertsimas and Popescu [4] without buy-up in a network setting. The resulting DP model is unlikely to be solved optimally due to the curse of dimensionality and hence is solved approximately by various simpler models such as deterministic linear programming (DLP), probabilistic nonlinear programming (PNLP), randomized linear programming (RLP), and approximate dynamic programming. Policies based on partitioned booking limits and bid prices in conjunction with some of the above approximate models are proposed to control capacity. We show that the partitioned booking limit policy based on both the DLP and the PNLP are asymptotically optimal. We further show that the bid-price policy is also asymptotically optimal provided that bid prices are obtained from the appropriate approximate model. It is known that the RLP is used to estimate the gradient of the expected perfect information network revenue, which in turn is used to approximate the gradient of the optimal expected network revenue. The resulting gradient can be used in the bid-price policy. We establish some analytical results to estimate expected network revenue gradients based on the RLP.

The rest of this paper is organized as follows. In the next section, we define the buy-up model more precisely and present a dynamic programming formulation. We then show that the airline is always better off using the buy-up model if some customers do buy up. In section 3, we present several approximate formulations to the dynamic programming formulations including the DLP, the PNLP, the RLP and the opportunity cost-based approximation using the DLP. It turns out that those approximations are direct extensions of the same approximations from the traditional network revenue management. Policies based on partitioned booking limits and bid prices in conjunction with some of the above approximate models are proposed to control capacity. In section 4, we show that the partitioned booking limit policy based on either the DLP or the PNLP is asymptotically optimal. The bid-price policy is also shown to be asymptotically optimal in section 5 provided that correct bid prices are used. In section 6, we establish some analytical results to estimate expected network revenue gradients based on the RLP, which provides some basis for using the RLP as an approximation for solving the DP. Some further analytical properties for the DP, the PNLP and the opportunity cost-based approximation are discussed in section 7. We report our numerical experiments on some test examples in section 8. The numerical results confirms that the buy-up model is indeed a better model than the traditional network revenue management models when some customers do elect to buy up. In section 9, we make some concluding remarks.

2 The Dynamic Programming Model (DP)

Assume an airline company sells n products over a network that has m resources/legs. Each product j is a combination of an itinerary and a fare and has an associated unit revenue r_j. Define the incidence matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if resource i is used by product j, $a_{ij} = 0$ otherwise. Let A_j denote the j-th column of matrix A. The initial capacity of the network is $c = (c_1, \dots, c_m)^\top \in \mathbb{R}^m$. The state of the network is described by a vector $x = (x_1, \dots, x_m)^\top \in \mathbb{R}^m$ of resource capacities. The booking horizon is divided into T periods indexed by t, which runs backwards so that $t = 0$ is the departure time.

Let us formally make some assumptions for our buy-up model.

(A1) In each time-period t no more than one request for one product can arrive with probability λ_{tj} for product j. It holds that $\lambda_{t0} + \sum_{j=1}^{n} \lambda_{tj} = 1$, where λ_{t0} is the probability that no booking request arrives at time t.

(A2) In any time period, a customer who is denied for their initial booking request can ask for any other product if they wish to do so. If a customer with a booking request of product j arriving at time t is rejected, then let $q^j_{t\ell}$ denote the probability that this customer will express to buy product ℓ. General conditions imposed on $q^j_{t\ell}$ are $q^j_{t\ell} \geq 0$, $\forall j, \ell, t$ and $q^j_{t0} + \sum_{\ell=1}^{n} q^j_{t\ell} = 1\ \forall j, t$, where $q^j_{t0} \geq 0$ denotes the probability of having no further booking requests for the denied booking of product j at time t.

(A3) No customer will ask for another product if they are denied a second time, i.e., customers who are denied twice are lost forever.

Assumption A1 is standard for a dynamic network RM model in the literature. Under Assumption A2, a booking request for any other product is possible for any denied booking in any time period. In other words, this model allows the customers to upgrade or downgrade to a product on the same itinerary or on different itineraries. In reality, many of $q^j_{t\ell} = 0$ for any fixed t and j as many customers are only able to buy up (or upgrade) to closely related products. Assumption A3 is still restrictive in reality. Nevertheless, our model is one step closer to reality than the traditional network RM models which do not consider buy-up.

At any time t, the airline has to make the following decisions when a booking arrives:

(**D1**) Accept or reject a booking request for product j.

(**D2**) In the case of rejection in D1 and if the customer wants to buy up to product ℓ, accept or reject the customer's new booking request.

Let us introduce the boolean variables u_{tj} and $v^j_{t\ell}$ to represent decisions D1 and D2 for any t, j and ℓ. In particular, $u_{tj} = 1$ if and only if a booking request for product j at time t is accepted. When $u_{tj} = 1$, $v^j_{t\ell} = 0$ for all ℓ, and when $u_{tj} = 0$, $v^j_{t\ell} = 1$ if and only if an upgrade to product ℓ from j is accepted.

The problem can be formulated as a dynamic program by letting $V_t(x)$ be the maximum expected revenue obtainable from periods $t, t-1, \ldots, 1$ given that there are x units of capacity at time t. Then the Bellman equation for $V_t(x)$ is:

$$
\begin{aligned}
V_t(x) &= \sum_{j=1}^{n} \lambda_{tj} \max_{\substack{u_{tj} + v^j_{t\ell} \le 1 \\ u_{tj} \in \{0,1\} \\ v^j_{t\ell} \in \{0,1\}}} \Big[r_j u_{tj} + u_{tj} V_{t-1}(x - A_j u_{tj}) \\
&\quad + (1-u_{tj}) \sum_{\ell=1}^{n} q^j_{t\ell} (r_\ell v^j_{t\ell} + V_{t-1}(x - A_\ell v^j_{t\ell})) \\
&\quad + (1-u_{tj}) q^j_{t0} V_{t-1}(x) \Big] \\
&\quad + \lambda_{t0} V_{t-1}(x)
\end{aligned} \tag{1}
$$

with boundary conditions

$$
\begin{aligned}
&V_t(0) = 0, \qquad \forall t; \\
&V_0(x) = 0, \qquad \forall x \ge 0; \\
&V_0(x) = -\infty, \qquad \forall x \text{ such that } x_j < 0 \text{ for some } j.
\end{aligned}
$$

According to definitions of λ_{t0} and q^j_{t0}, the Bellman equation can be rewritten as:

$$
\begin{aligned}
V_t(x) &= \sum_{j=1}^{n} \lambda_{tj} \max \quad \{ r_j + V_{t-1}(x - A_j), \\
&\qquad \sum_{\ell=1}^{n} q^j_{t\ell} \max[r_\ell + V_{t-1}(x - A_\ell), V_{t-1}(x)] + q^j_{t0} V_{t-1}(x) \} \\
&\qquad + \lambda_{t0} V_{t-1}(x) \\
&= \sum_{j=1}^{n} \lambda_{tj} \max \quad \{ r_j + V_{t-1}(x - A_j), \sum_{\ell=1}^{n} q^j_{t\ell} \max[r_\ell - \Delta V_{t\ell}(x), 0] + V_{t-1}(x) \} \\
&\qquad + \lambda_{t0} V_{t-1}(x) \\
&= \sum_{j=1}^{n} \lambda_{tj} \max \quad \{ r_j - \Delta V_{tj}(x), \sum_{\ell=1}^{n} q^j_{t\ell} \max[r_\ell - \Delta V_{t\ell}(x), 0] \} \\
&\qquad + V_{t-1}(x)
\end{aligned} \tag{2}
$$

where $\Delta V_{tj}(x) = V_{t-1}(x) - V_{t-1}(x - A_j)$ represents the opportunity cost/profit of holding a unit of product j at time period $t-1$.

The representation of $V_t(x)$ in the last equation above is quite intuitive and can be explained as follows. An initially denied booking request of product j is accepted ($v^j_{t\ell} = 1$) to be upgraded to product ℓ if and only if $r_\ell - \Delta V_{t\ell}(x) > 0$, i.e., when the revenue from a unit of product ℓ is higher than the opportunity cost of holding it in time period $t-1$. Let $\Delta W_{tj}(x) = \sum_{\ell=1}^{n} q^j_{t\ell} \max[r_\ell - \Delta V_{t\ell}(x), 0]$. Then $\Delta W_{tj}(x)$ is nonnegative and can be interpreted as the benefit of denying an initial booking request of product j. The above equation also states that a booking request of product j is accepted ($u_{tj} = 1$) if and only if $r_j - \Delta V_{tj}(x) > \Delta W_{tj}(x)$, i.e., the booking request of product j at time t is accepted if and only if the benefit of accepting it is more than the benefit of rejecting it. Notice that $\max\{r_j - \Delta V_{tj}(x), \Delta W_{tj}(x)\} \geq 0$ because $\Delta W_{tj}(x) \geq 0$ holds in any case.

We point out that the dynamic programming formulation (1) or (2) reduces to that presented in [27] in a single-leg setting. Also the DP reduces to the dynamic programming formulation without buy-up presented by Bertsimas and Popescu [4].

A natural question arises as to whether or not the dynamic programming formulation (2) is better than the traditional capacity management model without buy-up, which is defined in [4] by

$$G_t(x) = \sum_{j=1}^{n} \lambda_{tj} \max\{r_j + G_{t-1}(x - A_j), G_{t-1}(x)\} + \lambda_{t0} G_{t-1}(x). \tag{3}$$

For all x and t, is $V_t(x) \geq G_t(x)$? The answer is positive and the claim can be proven by induction. Clearly, $V_0(x) = G_0(x) = 0$ for any $x \geq 0$ and $V_0(x) = G_0(x) = -\infty$ for any x such that for some j, $x_j < 0$. Hence for any x, $V_0(x) \geq G_0(x)$. Now assuming that $V_{t-1}(x) \geq G_{t-1}(x)$ for any x, we aim to prove that $V_t(x) \geq G_t(x)$ for any x. This can be shown using (2):

$$\begin{aligned} V_t(x) &= \sum_{j=1}^{n} \lambda_{tj} \max \quad \{r_j + V_{t-1}(x - A_j), \\ &\qquad \sum_{\ell=1}^{n} q_{t\ell}^{j} \max[r_\ell + V_{t-1}(x - A_\ell), V_{t-1}(x)] + q_{t0}^{j} V_{t-1}(x)\} \\ &\qquad + \lambda_{t0} V_{t-1}(x) \\ &\geq \sum_{j=1}^{n} \lambda_{tj} \max \quad \{r_j + G_{t-1}(x - A_j), \\ &\qquad \sum_{\ell=1}^{n} q_{t\ell}^{j} \max[r_\ell + G_{t-1}(x - A_\ell), G_{t-1}(x)] + q_{t0}^{j} G_{t-1}(x)\} \\ &\qquad + \lambda_{t0} G_{t-1}(x) \\ &\geq \sum_{j=1}^{n} \lambda_{tj} \max \quad \{r_j + G_{t-1}(x - A_j), G_{t-1}(x)\} + \lambda_{t0} G_{t-1}(x) \\ &= G_t(x). \end{aligned} \tag{4}$$

The above result is summarized below.

Proposition 2.1 *The buy-up model performs no worse than the non buy-up model, i.e., for any t and x,*

$$V_t(x) \geq G_t(x).$$

3 Deterministic and Probabilistic Approximations

The DP model proposed in the last section is unlikely to be solved optimally due to the curse of dimensionality. In this section, we propose several approximations to solutions of the DP, which are extensions of well-known approximations for the DP of traditional network capacity management. In particular, we are interested in approximations by deterministic linear programming (DLP) [8, 26], probabilistic nonlinear programming (PNLP) [21, 26], randomized linear programming (RLP) [19], and opportunity cost-based methods [4].

3.1 The Deterministic Linear Programming Model (DLP)

In the DLP, we replace stochastic demand quantities by their mean values and assume that capacity and demand are continuous.

Let $d \in \mathbb{R}^n$ be the random vector of cumulative future demand at time t, and $\bar{d} \in \mathbb{R}^n$ its mean vector. In particular, d_j represents the aggregate demand for product j from period t to the departure time. Let y_j be the initial booking limit of capacity for product j and z_j^ℓ the upgraded booking limit of capacity for product j when a customer whose initial request to buy product ℓ is rejected wishes to buy a unit of product j with probability Q_j^ℓ. If time index t is ignored, Q_j^ℓ can be thought as an average of q_{tj}^ℓ across the entire booking horizon.

For any given capacity x at time t, the deterministic linear program can be formulated as:

$$\begin{aligned} V_t^{\mathrm{DLP}}(x) = \quad & \max \quad r^\top (y + \sum_{\ell=1}^n z^\ell) \\ & \text{s.t.} \quad A(y + \sum_{\ell=1}^n z^\ell) \le x \\ & \qquad\;\; y \le \bar{d} \\ & \qquad\;\; z^\ell \le (\bar{d}_\ell - y_\ell) Q^\ell, \qquad \forall \ell = 1, \ldots, n \\ & \qquad\;\; y \ge 0, z^\ell \ge 0, \qquad \forall \ell = 1, \ldots, n. \end{aligned} \tag{5}$$

It is well known [21] that by solving the DLP model we can use either the primal variables to construct a partitioned booking limit control directly or the dual variables to define a bid-price control. In the partitioned booking limit control, a fixed amount of capacity of each resource is allocated to every product offered. The demand for each product has access only to its allocated capacity and no other product may use this capacity. In contrast, a bid-price control policy sets a threshold price or bid price for each resource in the network. Roughly this bid price is an estimate of the marginal cost of consuming the next incremental unit of the resource's capacity. When a booking request for a product arrives, the revenue of the request is compared to the sum of the bid prices of all the resources required by the product. If the revenue exceeds the sum of the bid prices, the request is accepted provided that all the resources associated with the requested product are still available; if not, the request is rejected.

In the context of the DLP, optimal solutions y^* and z^* give partitioned booking limits while bid prices are formed from optimal dual variables of constraint $A(y + \sum_{\ell=1}^{n} z^{\ell}) \leq x$. Assume α^* is the optimal dual solution of the DLP. Next we formally define the partitioned booking limit policy and the bid-price policy based on the DLP.

Partitioned booking limit policy (BLP).

Step BL1. A new booking request of product j arrives. Go to Step BL2.

Step BL2. If the number of bookings of product j accepted from initial requests is less than or equal to $y_j^* - 1$ and there is enough resource on each leg covered by product j, then accept the booking request and go to Step BL1. Otherwise, reject the booking request and go to Step BL3.

Step BL3. If the rejected customer does not wish to upgrade to any other product, go to Step BL1. Otherwise, the rejected customer wishes to upgrade to another product ℓ. If the number of accepted bookings of product ℓ which have been upgraded from product j to product ℓ is less than or equal to $(z^*)_{\ell}^{j} - 1$, and there is enough resource on each leg covered by product ℓ, then accept the upgraded booking request and go to Step BL1. Otherwise, reject the upgraded booking request and go to Step BL1.

Bid-price policy (BPP).

Step BP1. A new booking request of product j arrives. Go to Step BP2.

Step BP2. If $r_j - A_j^{\top}\alpha^* > \sum_{\ell=1}^{n} \max\{r_{\ell} - A_{\ell}^{\top}\alpha^*, 0\} Q_{\ell}^{j}$, and there is enough resource on each leg covered by product j, then accept it and go to Step BP1. Otherwise, reject the booking request and go to Step BP3.

Step BP3. If the rejected customer does not wish to upgrade to any other product, go to Step BP1. Otherwise, the rejected customer wishes to upgrade to another product ℓ. If $r_{\ell} > A_{\ell}^{\top}\alpha^*$ and there is enough resource on each leg covered by product ℓ, then accept the upgraded booking request and go to Step BP1. Otherwise, reject the upgraded booking request and go to Step BP1.

Remarks. (1) The DLP reduces to the traditional DLP [8, 26] when upgrades are not considered, i.e., $Q^j_\ell \equiv 0$ for all j, ℓ. (2) The partitioned booking limit policy also reduces to the corresponding policy for the traditional DLP when upgrades are not considered. (3) In the BPP, if $r_j \leq A_j^\top \alpha^*$, then $r_j - A_j^\top \alpha^* > \sum_{\ell=1}^n \max\{r_\ell - A_\ell^\top \alpha^*, 0\} Q^j_\ell$ does not hold and hence the initial booking request should be rejected. Once again, the traditional bid-price policy is recovered if the upgrade probabilities are assumed to be zero for all products.

3.2 The probabilistic nonlinear programming model (PNLP)

The probabilistic nonlinear program can be formulated as:

$$
\begin{aligned}
V_t^{\mathrm{PNLP}}(x) = \quad & \max \quad \mathbb{E}\left[\sum_{j=1}^n r_j \min(y_j, d_j) + \sum_{\ell=1}^n \sum_{j=1}^n r_j \min\{z^\ell_j, Q^\ell_j(d_\ell - \min(y_\ell, d_\ell))\}\right] \\
& \text{s.t.} \quad A(y + \sum_{\ell=1}^n z^\ell) \leq x \\
& \qquad\quad y \geq 0, z^\ell \geq 0, \qquad \forall \ell = 1, \dots, n.
\end{aligned} \tag{6}
$$

When upgrades are not considered, the PNLP reduces to the traditional PNLP [21, 26]. Similar to the DLP, one can propose both the partitioned booking limit and the bid-price policies based on the optimal primal and dual solutions of the PNLP, which reduce to the corresponding policies for the traditional PNLP when upgrades are not considered.

3.3 The randomized linear programming model (RLP)

The randomized linear program is based on replacing $\bar{d}$, the expected value of d, with the random vector d itself into the constraints of the DLP.

$$
\begin{aligned}
V_t^{\mathrm{RLP}}(x, d) = \quad & \max \quad r^\top (y + \textstyle\sum_{\ell=1}^n z^\ell) \\
& \text{s.t.} \quad A(y + \sum_{\ell=1}^n z^\ell) \leq x \\
& \qquad\quad y \leq d \\
& \qquad\quad z^\ell \leq (d_\ell - y_\ell) Q^\ell, \qquad \forall \ell = 1, \dots, n \\
& \qquad\quad y \geq 0, z^\ell \geq 0, \qquad \forall \ell = 1, \dots, n.
\end{aligned} \tag{7}
$$

Note that the optimal value $V_t^{\mathrm{RLP}}(x, d)$ is a random variable. Indeed, $V_t(x)$ can be approximated by $V_t^{\mathrm{RLP}}(x) = \mathbb{E}[V_t^{\mathrm{RLP}}(x, d)]$. If $V_t^{\mathrm{RLP}}(x, d)$ is continuously differentiable with respect to x for all d, then so is $\mathbb{E}[V_t^{\mathrm{RLP}}(x, d)]$ and $\mathbb{E}[\frac{\partial}{\partial x} V_t^{\mathrm{RLP}}(x, d)] = \frac{\partial}{\partial x}\mathbb{E}[V_t^{\mathrm{RLP}}(x, d)]$. Therefore the opportunity cost of $\mathbb{E}[V_t^{\mathrm{RLP}}(x, d)]$ is equal to the expected value of the opportunity cost of

$V_t^{\mathrm{RLP}}(x, d)$, which in turn can be approximated by sample averages of the optimal dual solution of the RLP.

When upgrades are not considered, the RLP reduces to the traditional RLP [19] and one can propose the bid-price policy based on the RLP, which matches the same policy for the traditional RLP. Note that there is no partitioned booking limit policy for the RLP similar to the traditional RLP.

3.4 Opportunity cost-based approximations

Bertsimas and Popescu [4] point out that two main disadvantages of the bid-price policy based on the DLP are: (a) the bid price may not be uniquely defined; (b) the bid price provides an additive approximation of the opportunity costs, which are not necessarily additive due to bundle effects (group or multileg itinerary requests may determine basis changes in the dual linear program). To rectify these shortcomings of the bid-price policy, Bertsimas and Popescu propose a different approximation scheme for the opportunity cost of product j by

$$\mathrm{OC}_{tj}^{\mathrm{DLP}}(x) \equiv V_t^{\mathrm{DLP}}(x, \bar{d}(t-1)) - V_t^{\mathrm{DLP}}(x - A_j, \bar{d}(t-1))$$

where $V_t^{\mathrm{DLP}}(x, \bar{d}(t-1))$ denotes the optimal objective function value of the DLP at time t when the expected future demand is $\bar{d}(t-1)$. In other words, the opportunity cost of a seat for product j is equal to the difference between the approximate total future revenue obtainable by keeping this seat and that obtainable by selling it.

With the same notation, this is exactly the opportunity cost for the buy-up model. Then an initial booking request of product j is accepted if and only if $r_j - \mathrm{OC}_{tj}^{\mathrm{DLP}}(x) > \max\{\sum_{\ell=1}^{n} Q_\ell^j [r_\ell - \mathrm{OC}_{t\ell}^{\mathrm{DLP}}(x)], 0\}$, and the upgraded booking of product ℓ is accepted if and only if $r_\ell > \mathrm{OC}_{t\ell}^{\mathrm{DLP}}(x)$. When upgrades are not considered, the above opportunity based booking policy reduces to the one proposed in [4].

4 Asymptotic Optimality of the Booking Limit Policies

The boundedness of the feasible region of the DLP shows the existence of its dual variables. Let $\alpha \in \mathbb{R}^m$, $\beta \in \mathbb{R}^n$, and $\gamma^\ell \in \mathbb{R}^n$ for any ℓ be dual variables of the first three inequalities of the

DLP. Then the dual problem of the DLP is the following linear program:

$$\begin{aligned}
\min \quad & x^\top \alpha + (\bar{d})^\top \beta + \sum_{\ell=1}^{n} (\bar{d}_\ell Q^\ell)^\top \gamma^\ell \\
\text{s.t.} \quad & A^\top \alpha + \beta + ((Q^1)^\top \gamma^1 \dots (Q^n)^\top \gamma^n)^\top \geq r \\
& A^\top \alpha + \gamma^\ell \geq r, \qquad \forall \ell = 1, \dots, n \\
& \alpha \geq 0, \beta \geq 0, \gamma^\ell \geq 0, \qquad \forall \ell = 1, \dots, n.
\end{aligned} \tag{8}$$

Using the DP, the DLP and its dual, we are able to establish the relationship of the optimal objective function values between the DP and the DLP and also the so-called asymptotic optimality property which is well known in the traditional DLP [7, 21]. The asymptotic optimality property basically states that the expected revenue generated from the partitioned booking limit policy based on the optimal primal solution of the DLP is asymptotically convergent to the optimal expected revenue $V_t(x)$ when both the available capacity x and the demand are scaled up proportionally.

More precisely, consider a sequence of the network buy-up problems indexed by θ in which the available capacity is equal to θx, the expected demand $\theta \bar{d}$, and the demand random variable $d(\theta)$. When $\theta = 1$, the θ-scaled problem corresponds to the original problem defined in Section 2. Let $V_t(x, \theta)$ and $V_t^{\text{DLP}}(x, \theta)$ be the optimal objective function values of the DP and the DLP of the θ-scaled problem respectively. With a given demand scenario $d(\theta)$, define $U_t^{\text{DLP}}(x, d(\theta))$ to be the revenue generated from the partitioned booking limit policy based on the optimal primal solution of the DLP corresponding to the θ-scaled problem. Then the asymptotic optimality property states that under appropriate demand conditions, $V_t(x, \theta)$ scaled by $1/\theta$ converges to $V_t^{\text{DLP}}(x)$ as θ approaches infinity and the revenue generated from the optimal solution of the DLP is also asymptotically optimal.

Theorem 4.1 *(a) For any fixed t, $V_t^{\text{DLP}}(x)$ is a piecewise linear, nonincreasing and concave function of the available capacity x and the expected future demand $\bar{d}$.*

(b) For any t and x, the optimal revenue of the DP is bounded above by the optimal objective function value of the DLP, i.e.,

$$V_t(x) \leq V_t^{\text{DLP}}(x).$$

(c) The vector $(y^*, (z^*)^1, \dots, (z^*)^n)$ is an optimal solution of the DLP if and only if $(\theta y^*, \theta (z^*)^1, \dots, \theta (z^*)^n)$ is an optimal solution of the DLP of the θ-scaled problem. Moreover,

$$V_t^{\text{DLP}}(x, \theta) = \theta V_t^{\text{DLP}}(x).$$

(d) *For any fixed available capacity x, if the expected demand of the θ-scaled problem is $\theta\bar{d}$ and $\dfrac{d(\theta)}{\theta}$ converges to $\bar{d}$ in distribution, then, in distribution,*

$$\frac{U_t^{\mathrm{DLP}}(x, d(\theta))}{\theta} \to V_t^{\mathrm{DLP}}(x) \tag{9}$$

and

$$\lim_{\theta\to\infty} \frac{V_t(x,\theta)}{\theta} = \lim_{\theta\to\infty} \frac{V_t^{\mathrm{DLP}}(x,\theta)}{\theta} = V_t^{\mathrm{DLP}}(x).$$

Proof. (a) Let Ω be the set of vertices of the feasible region of (8). Clearly, the cardinality of Ω is finite. Recall that both the DLP and its dual (8) have optimal solutions since the feasible region of the DLP is bounded. Furthermore, both the DLP and its dual have the same optimal objective function value $V_t^{\mathrm{DLP}}(x)$ satisfying

$$V_t^{\mathrm{DLP}}(x) = \min_{(\alpha,\beta,\gamma^1,\ldots,\gamma^n)\in\Omega} x^\top \alpha + (\bar{d})^\top \beta + \sum_{\ell=1}^{n} (\bar{d}_\ell Q^\ell)^\top \gamma^\ell.$$

That is, $V_t^{\mathrm{DLP}}(x)$ is the minimum of finitely many linear functions of x and $\bar{d}$, and hence it is piecewise linear and concave. Because Q^ℓ and all α, β and γ^ℓ in Ω are nonnegative, all linear functions defining $V_t^{\mathrm{DLP}}(x)$ in the above equation have nonnegative coefficients. Therefore, $V_t^{\mathrm{DLP}}(x)$ is a nondecreasing function of both x and $\bar{d}$.

(b) For any demand realization d, the revenue generated from the optimal policy of the DP is bounded above by $V_t^{\mathrm{RLP}}(x,d)$ (see the RLP) as the latter is obtained from the optimal policy that is based on the perfect demand information. It follows that

$$V_t(x) \le \mathbb{E}[V_t^{\mathrm{RLP}}(x,d)].$$

Applying the result in (a) to the RLP, we obtain that $V_t^{\mathrm{RLP}}(x,d)$ is a piecewise linear, nonincreasing and concave function of d. Jensen's inequality shows that

$$\mathbb{E}[V_t^{\mathrm{RLP}}(x,d)] \le V_t^{\mathrm{RLP}}(x, \mathbb{E}[d]) = V_t^{\mathrm{DLP}}(x)$$

which implies the result.

(c) It is easy to verify that $(\theta y^*, \theta(z^*)^1, \ldots, \theta(z^*)^n)$ is a feasible solution and an optimal solution of the DLP of the θ-scaled problem if $(y^*, (z^*)^1, \ldots, (z^*)^n)$ is an optimal solution of the DLP, and vice versa. The result follows easily.

(d) By definition,

$$U_t^{\mathrm{DLP}}(x, d(\theta)) = \sum_{j=1}^{n} r_j \min[\theta y_j^*, d_j(\theta)] + \sum_{\ell=1}^{n}\sum_{j=1}^{n} r_j \min\left\{\theta(z^*)_j^\ell, Q_j^\ell(d_j(\theta) - \min[\theta y_j^*, d_j(\theta)]\right\}$$

and since the min-operator is continuous, it follows from Lemma 1 of [7] that when θ approaches infinity, in distribution,

$$\begin{aligned}\frac{U_t^{\mathrm{DLP}}(x,d(\theta))}{\theta} \quad &\to \quad \sum_{j=1}^{n} r_j \min[y_j^*, \bar{d}_j] + \sum_{\ell=1}^{n}\sum_{j=1}^{n} r_j \min\left\{(z^*)_j^\ell, Q_j^\ell(\bar{d}_j - \min[y_j^*, \bar{d}_j]\right\} \\ &= \sum_{j=1}^{n} r_j y_j^* + \sum_{\ell=1}^{n}\sum_{j=1}^{n} r_j (z^*)_j^\ell,\end{aligned}$$

i.e., in distribution, (9) holds, where the last equality follows from (5). Since $V_t(x,\theta)$ is the optimal value of the θ-scaled problem, it yields

$$\mathbb{E}[U_t^{\mathrm{DLP}}(x,d(\theta))] \le V_t(x,\theta) \le V_t^{\mathrm{DLP}}(x,\theta) = \theta V_t^{\mathrm{DLP}}(x),$$

where the second inequality follows from an application of (b) to the θ-scaled problem and the last equality follows from (c). This shows that

$$\frac{\mathbb{E}[U_t^{\mathrm{DLP}}(x,d(\theta))]}{\theta} \le \frac{V_t(x,\theta)}{\theta} \le \frac{V_t^{\mathrm{DLP}}(x,\theta)}{\theta} = V_t^{\mathrm{DLP}}(x).$$

The results follow from (9). ■

The next proposition shows that we can obtain the similar asymptotic property for the PNLP and some analytical property for the RLP. To our knowledge, the results in Theorem 4.2 have not been addressed in the literature for the traditional network capacity management problem.

Theorem 4.2 *(a) $V_t^{\mathrm{PNLP}}(x)$ is a lower bound of the optimal value function $V_t(x)$ and on average, the PNLP performs no worse than the DLP, i.e.,*

$$\mathbb{E}[U_t^{\mathrm{DLP}}(x)] \le V_t^{\mathrm{PNLP}}(x) \le V_t(x).$$

(b) For any fixed available capacity x, if the expected demand of the θ-scaled problem is $\theta\bar{d}$ and $\frac{d(\theta)}{\theta}$ converges to $\bar{d}$ in distribution, then

$$\frac{V_t^{\mathrm{PNLP}}(x,\theta)}{\theta} \to V_t^{\mathrm{DLP}}(x), \tag{10}$$

i.e., asymptotically, the optimal solution of the PNLP provides an optimal inventory control policy.

(c) $\mathbb{E}[V_t^{\mathrm{RLP}}(x,d)]$ also provides an upper bound for $V_t(x)$, i.e.,

$$V_t(x) \le \mathbb{E}[V_t^{\mathrm{RLP}}(x,d)] \le V_t^{\mathrm{DLP}}(x)$$

and moreover

$$\frac{\mathbb{E}\left[V_t^{\mathrm{RLP}}(x,d(\theta))\right]}{\theta} \to V_t^{\mathrm{DLP}}(x).$$

Proof. (a) It is trivial to prove that $V_t^{\mathrm{PNLP}}(x)$ is a lower bound of $V_t(x)$ as the former is exactly equal to the expected revenue generated from the partitioned booking limit policy of the PNLP, which must not exceed the optimal expected revenue $V_t(x)$ generated from the optimal policy. Note that the optimal solution of the DLP is a feasible solution of the PNLP. Therefore the expected revenue generated from the optimal booking limits $(y^*, (z^1)^*, \ldots, (z^n)^*)$ of the DLP is equal to the objective function value of the PNLP evaluated at $(y^*, (z^1)^*, \ldots, (z^n)^*)$, which must not exceed the optimal objective function value of the PNLP, which in turn is equal to the expected revenue generated from its optimal booking limits.

(b) By (a) and Theorem 4.1 (d),

$$\limsup_{\theta\to\infty} \frac{V_t^{\mathrm{PNLP}}(x,\theta)}{\theta} \leq \lim_{\theta\to\infty} \frac{V_t(x,\theta)}{\theta} = V_t^{\mathrm{DLP}}(x).$$

On the other hand, by (a) again and Theorem 4.1 (d),

$$\frac{V_t^{\mathrm{PNLP}}(x,\theta)}{\theta} \geq \frac{\mathbb{E}[U_t^{\mathrm{DLP}}(x,d(\theta))]}{\theta} \to V_t^{\mathrm{DLP}}(x)$$

which shows that

$$\liminf_{\theta\to\infty} \frac{V_t^{\mathrm{PNLP}}(x,\theta)}{\theta} \geq V_t^{\mathrm{DLP}}(x).$$

This proves (10).

(c) The two inequalities are proved in the proof of Theorem 4.1 (b). The last result follows from the two inequalities and Theorem 4.1 (d). ■

5 Asymptotic Optimality of the Bid-Price Policy

Talluri and van Ryzin [18] prove that the bid-price policy is also asymptotically optimal when leg capacities and demand are large provided that the right bid prices are used. Our next result is to extend the asymptotic optimality of Talluri and van Ryzin to the buy-up model.

As in [18], we replace price r_j of product j by a continuous random variable $\mathcal{R}_j$ defined over $(0, +\infty)$ with mean r_j. It is assumed that $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are mutually and statistically independent. It is also assumed that $\mathcal{R}_j$ has a bounded support: there exists a positive constant $\bar{r}_j$ such that $P(\mathcal{R}_j \leq \bar{r}_j) = 1$, i.e., the probability that the unit revenue of product j does not exceed $\bar{r}_j$ is one. A price range is preferable to a single value for product j since the price changes with time with some small perturbations. More discussions on this rationale can be found in [18]. We shall call this problem the DP' which is more general than the DP (1).

Based on the revenue random variables $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$, we define a probabilistic mathematical program (PMP):

$$
\begin{aligned}
V_t^{\mathrm{PMP}}(x) = \quad \max \quad & \mathbb{E}\left[R^\top (y + \sum_{\ell=1}^{n} z^\ell)\right] \\
\text{s.t.} \quad & A(y + \sum_{\ell=1}^{n} z^\ell) \le x \\
& y \le \bar{d} \\
& z^\ell \le (\bar{d}_\ell - y_\ell) Q^\ell, \qquad \forall \ell = 1, \ldots, n \\
& y \ge 0, z^j \ge 0, \qquad \forall j = 1, \ldots, n.
\end{aligned}
\tag{11}
$$

where the expectation $\mathbb{E}$ is defined over the joint probability space of $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$ with $(R_1, \ldots, R_n) \in \mathbb{R}^n$ representing a particular price scenario, and for each scenario $(R_1, \ldots, R_n)$, there is a corresponding optimal solution $(y, z^1, \ldots, z^n)$. The PMP is more general than the DLP, but it reduces to the DLP when $\mathcal{R}_j$ takes only a single value r_j for all j. The PMP is also quite different from the PNLP as the expectation in the objective of the former is taken over the prices of products but not over demand as in the latter.

Let $\Phi_t(x)$ denote the optimal expected revenue for the DP'. Similar to Theorem 4.1 (b) and (c), we can prove the following results.

Proposition 5.1 *For any t and x, the optimal expected revenue of the DP' is bounded above by the optimal objective function value of the PMP, i.e.,*

$$\Phi_t(x) \le V_t^{\mathrm{PMP}}(x).$$

Moreover,

$$V_t^{\mathrm{PMP}}(x, \theta) = \theta V_t^{\mathrm{PMP}}(x),$$

where $V_t^{\mathrm{PMP}}(x, \theta)$ is the optimal objective function value of the θ-scaled problem of the PMP.

Consider an Lagrangian relaxation (LR) problem of the PMP by dualizing the network leg capacity constraints with a common dual variable $\alpha \in \mathbb{R}^m$ for all different price scenarios.

$$
\begin{aligned}
V_t^{\mathrm{LR}}(x, \alpha) = \quad \max \quad & \mathbb{E}\left[\sum_{j=1}^{n} (R_j - A_j^\top \alpha) y_j + \sum_{j=1}^{n} \sum_{\ell=1}^{n} (R_j - A_j^\top \alpha) z_j^\ell)\right] + \alpha^\top x \\
\text{s.t.} \quad & y \le \bar{d} \\
& z^\ell \le (\bar{d}_\ell - y_\ell) Q^\ell, \qquad \forall \ell = 1, \ldots, n \\
& y \ge 0, z^\ell \ge 0, \qquad \forall \ell = 1, \ldots, n
\end{aligned}
\tag{12}
$$

where $\alpha \ge 0$ is the Lagrangian multiplier corresponding to the network leg capacity constraints.

The Lagrangian dual (LD) of the LR (12) is the following minimization problem:

$$V_t^{\mathrm{LD}}(x) = \begin{array}{ll} \min & V_t^{\mathrm{LR}}(x,\alpha) \\ \text{s.t.} & \alpha \geq 0. \end{array} \tag{13}$$

Remark. It is easy to see that for any nonnegative α, the optimal value $V_t^{\mathrm{LR}}(x,\alpha)$ of the LR provides an upper bound to the optimal value $V_t^{\mathrm{PMP}}(x)$ of the PMP. Together with Proposition 5.1, the last result implies

$$V_t^{\mathrm{LD}}(x) \geq V_t^{\mathrm{PMP}}(x) \geq \Phi_t(x). \tag{14}$$

For each fixed scenario $(R_1,\ldots,R_n)$ and a given $\alpha \geq 0$, define several index sets:

$$\begin{aligned}
&\mathcal{J}^{>} = \{j : R_j > A_j^\top \alpha\} \\
&\mathcal{J}^{=} = \{j : R_j = A_j^\top \alpha\} \\
&\mathcal{J}^{<} = \{j : R_j < A_j^\top \alpha\} \\
&\mathcal{K} = \{j : R_j - A_j^\top \alpha > \sum_{\ell:\ell\in\mathcal{J}^{>}} (R_\ell - A_\ell^\top \alpha) Q_\ell^j\} \\
&\mathcal{M} = \{1,\ldots,n\} \setminus \mathcal{K}.
\end{aligned}$$

Proposition 5.2 *For any $\alpha \geq 0$, we can find an optimal solution $(y^*,(z^*)^1,\ldots,(z^*)^n)$ of the LR (12) for scenario $(R_1,\ldots,R_n)$ satisfying the conditions specified in Table 1.*

		$j \in \mathcal{J}^{>} \cap \mathcal{K}$	$j \in \mathcal{J}^{>} \cap \mathcal{M}$	$j \in \mathcal{J}^{=}$	$j \in \mathcal{J}^{<}$
y_j^*		$\bar{d}_j$	0	0	0
$(z^*)_\ell^j$	$\ell \in \mathcal{J}^{>}$	0	$\bar{d}_j Q_\ell^j$	$\bar{d}_j Q_\ell^j$	$\bar{d}_j Q_\ell^j$
	$\ell \in \mathcal{J}^{=}$	0	0	0	0
	$\ell \in \mathcal{J}^{<}$	0	0	0	0

Table 1: Optimal values for y^* and z^*.

Furthermore, the optimal objective function value of the LR can be evaluated exactly

$$V_t^{\mathrm{LR}}(x,\alpha) = \sum_{j=1}^n \bar{d}_j \mathbb{E}\left[\max\{(R_j - A_j^\top \alpha)^+, \sum_{\ell=1}^n Q_\ell^j (R_\ell - A_\ell^\top \alpha)^+\}\right] + \alpha^\top x \tag{15}$$

where $f^+ = \max\{0,f\}$.

Proof. Observe that for each scenario $(R_1,\ldots,R_n)$, the LR can be decomposed into n simple linear programs with simple constraints. By checking index sets of j and ℓ, it is easy to establish the result by pushing optimal values of y and z^j to either their lower or upper bounds appropriately. We omit the detail. ∎

Under some conditions, $V_t^{\mathrm{LR}}(x,\alpha)$ can be proven to be continuously differentiable with respect to α. We need a result from [10].

Lemma 5.1 *Suppose $\mathcal{R}(\theta)$ is a random function of θ satisfying the global Lipschitz condition: there exists a positive constant L such that for any $\Delta\theta \geq 0$, it holds that*

$$\|\mathcal{R}(\theta+\Delta\theta)-\mathcal{R}(\theta)\| \leq L\Delta\theta.$$

If $\mathcal{R}(\theta)$ is continuously differentiable with respect to θ with probability one, then $\mathbb{E}[\mathcal{R}(\theta)]$ is continuously differentiable with respect to θ and

$$\frac{d}{d\theta}\mathbb{E}[\mathcal{R}(\theta)] = \mathbb{E}[\frac{d}{d\theta}\mathcal{R}(\theta)].$$

Proposition 5.3 *If $\mathcal{R}_1,\ldots,\mathcal{R}_n$ are continuous random variables over $(0,+\infty)$ respectively and mutually statistically independent, then for any fixed $x \geq 0$,*

(a) $V_t^{\mathrm{LR}}(x,\alpha)$ is continuously differentiable with respect to α,

(b) $V_t^{\mathrm{LR}}(x,\alpha)$ is convex with respect to α.

Proof. (a) Note that if f is globally Lipschitz continuous, then so is $\max\{0,f\}$, and a linear combination of two globally Lipschitz continuous functions is still globally Lipschitz continuous. Since a linear function is globally Lipschitz continuous and

$$\max\{(R_j-A_j^\top\alpha)^+,\sum_{\ell=1}^n Q_\ell^j(R_\ell-A_\ell^\top\alpha)^+\} = (R_j-A_j^\top\alpha)^+ + \max\{0,\sum_{\ell=1}^n Q_\ell^j(R_\ell-A_\ell^\top\alpha)^+ - (R_j-A_j^\top\alpha)^+\},$$

$F_j = \max\{(R_j-A_j^\top\alpha)^+,\sum_{\ell=1}^n Q_\ell^j(R_\ell-A_\ell^\top\alpha)^+\}$ is globally Lipschitz continuous.

On the other hand, F_j is continuously differentiable with respect to α except at the points where one of the following conditions holds:

$$R_j - A_j^\top\alpha = 0, R_\ell - A_\ell^\top\alpha = 0,$$

$$R_j - A_j^\top\alpha = \sum_{\ell:(R_\ell-A_\ell^\top\alpha)^+>0} Q_\ell^j(R_\ell - A_\ell^\top).$$

That is, F_j is not continuously differentiable with respect to α at some points that belong to finitely many hyperplanes of the n-dimensional probability space. In other words, F_j is continuously differentiable with probability one. Continuous differentiability of $V_t^{\mathrm{LR}}(x,\alpha)$ follows from Lemma 5.1.

(b) It is easy to prove convexity of $V_t^{\mathrm{LR}}(x,\alpha)$ from (15) by noticing that the max of finitely many convex functions is still convex, and the integral operator retains the convexity property. ■

Proposition 5.3 states that the LD (13) is a smooth and convex constrained optimization problem with simple bound constraints, for which the KKT conditions are both necessary and sufficient for optimality. We introduce some notation before presenting this result. Suppose $\alpha^* \in \mathbb{R}^m$ is an optimal solution of the LD. Define several event sets of $(\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n)$:

$$\begin{aligned}
\mathcal{X}_j^{>} &= \{\mathcal{R}_j > A_j^\top \alpha^*\}\\
\mathcal{X}_j^{=} &= \{\mathcal{R}_j = A_j^\top \alpha^*\}\\
\mathcal{X}_j^{<} &= \{\mathcal{R}_j < A_j^\top \alpha^*\}\\
\mathcal{Y}_j &= \{\mathcal{R}_j - A_j^\top \alpha^* > \sum_{\ell=1}^n (\mathcal{R}_\ell - A_\ell^\top \alpha^*)^+ Q_\ell^j\}\\
\mathcal{Z}_j &= \{\mathcal{R}_j - A_j^\top \alpha^* \le \sum_{\ell=1}^n (\mathcal{R}_\ell - A_\ell^\top \alpha^*)^+ Q_\ell^j\}.
\end{aligned}$$

Proposition 5.4 *Suppose $\mathcal{R}_1, \dots, \mathcal{R}_n$ are continuous random variables over $(0, +\infty)$ with a finite support, and mutually and statistically independent, and α^* is an optimal solution of the LD (13). Then the Karush-Kuhn-Tucker (KKT) conditions of the LD at α^* hold:*

$$\begin{aligned}
\sum_{j=1}^n \bar{d}_j P(\mathcal{X}_j^{>} \cap \mathcal{Y}_j) A_j + \sum_{j=1}^n \bar{d}_j P(\mathcal{X}_j^{>} \cap \mathcal{Z}_j) \sum_{\ell=1}^n Q_\ell^j P(\mathcal{X}_\ell^{>}) A_\ell & \\
+ \sum_{j=1}^n \bar{d}_j P(\mathcal{X}_j^{=} \cup \mathcal{X}_j^{<}) \sum_{\ell=1}^n Q_\ell^j P(\mathcal{X}_\ell^{>}) A_\ell + \lambda &= x,\\
\lambda^\top \alpha^* &= 0,\\
\lambda &\ge 0.
\end{aligned}$$

Proof. The result follows from the standard KKT conditions for constrained optimization problems and calculations of the gradients of $V_t^{\mathrm{LR}}(x,\alpha)$. ■

We are now ready to present the asymptotic optimality property of the bid-price policy based on the LD.

Theorem 5.1 *For any fixed available capacity x, suppose the expected demand of the θ-scaled problem is $\theta\bar{d}$ and $\dfrac{d(\theta)}{\theta}$ converges to $\bar{d}$ in distribution. Then the bid-price policy based on the LD (13) is asymptotically optimal, i.e., in distribution,*

$$\frac{U_t^{\mathrm{LD}}(x, d(\theta))}{\theta} \to V_t^{\mathrm{PMP}}(x) \tag{16}$$

and

$$\lim_{\theta \to \infty} \frac{\Phi_t(x,\theta)}{\theta} = V_t^{\mathrm{PMP}}(x) \tag{17}$$

where $U_t^{\mathrm{LD}}(x, d(\theta))$ is the total revenue generated from the bid price based on the LD and $\Phi_t(x,\theta)$ is the optimal expected revenue of the θ-scaled problem of the DP'.

6 Unbiased Estimate of the Gradient of the RLP

Recall that $V_t(x)$ can be approximated by $V_t^{\mathrm{RLP}}(x) = \mathbb{E}[V_t^{\mathrm{RLP}}(x,d)]$. It is proved in [19] that under certain conditions, for the traditional capacity management problem without buy-up, $\mathbb{E}[V_t^{\mathrm{RLP}}(x,d)]$ is continuously differentiable with respect to x and $\mathbb{E}[\frac{\partial}{\partial x}V_t^{\mathrm{RLP}}(x,d)] = \frac{\partial}{\partial x}\mathbb{E}[V_t^{\mathrm{RLP}}(x,d)]$. Therefore the opportunity cost of $\mathbb{E}[V_t^{\mathrm{RLP}}(x,d)]$ can be approximated by sample averages of the optimal dual solution of the RLP:

$$\frac{\partial}{\partial x}\mathbb{E}[V_t^{\mathrm{RLP}}(x,d)] \approx \frac{1}{N}\sum_{i=1}^{N}\alpha(x,d_i)$$

where $\alpha(x,d_i)$ is an optimal dual solution of the RLP (x,d_i) and N is the sample size.

The above result can be extended to the buy-up model. Similar to [19], the proof is based on Lemma 5.1. To this end, define a Lagrangian relaxation problem of the RLP, which we call the RLR, by dualizing the capacity constraints $A(y+\sum_{\ell=1}^{n} z^{\ell}) \le x$ of the RLP. Suppose the objective function value of the RLR is denoted by $V_t^{\mathrm{RLR}}(x,d,\alpha)$ for any given demand scenario d and a nonnegative Lagrangian multiplier $\alpha \in \mathbb{R}^m$. For any fixed $x \in \mathbb{R}^m$ and $d \in \mathbb{R}^n$, a Lagrangian dual of the RLR is the so-called RLD, which is

$$\begin{aligned} \min \quad & V_t^{\mathrm{RLR}}(x,d,\alpha) \\ \text{s.t.} \quad & \alpha \ge 0. \end{aligned} \tag{18}$$

By the strong duality theorem, if α^* is an optimal solution of the RLD, then $V_t^{\mathrm{RLR}}(x,d,\alpha^*) = V_t^{\mathrm{RLP}}(x,d)$ and α^* is always a subgradient of $V_t^{\mathrm{RLP}}(x,d)$ with respect to x. Furthermore, if α^* is a unique solution of the RLD (18), then, by Theorem 6.3.3 in [1], $V_t^{\mathrm{RLP}}(x,d)$ is continuously differentiable with respect to x, which is what we need according to Lemma 5.1. Another condition required in Lemma 5.1 is the Lipschitz continuous property of $V_t^{\mathrm{RLP}}(x,d)$ with respect to x. This can be proven by simply showing that for any integer $\Delta\theta \ge 0$

$$V_t^{\mathrm{RLP}}(x,d) \le V_t^{\mathrm{RLP}}(x+\Delta\theta e^i, d) \le V_t^{\mathrm{RLP}}(x,d) + \Delta\theta \max\{r_j\}$$

where $e^i \in \mathbb{R}^m$ is a vector with all zero elements except for its i-th element which is equal to one. The left hand side inequality is obviously true and the right-hand side inequality can be proven by contradiction by noting that a unit increase on x_i results in at most an additional increase of revenue, $\max\{r_j\}$, to the objective function value.

By Lemma 5.1, we only need to prove that the RLD (18) has a unique solution with probability one with respect to demand d. Analogous to Proposition 5.2, we have

$$V_t^{\mathrm{RLR}}(x,d,\alpha)=\sum_{j=1}^{n} d_j \max\{(r_j-A_j^\top\alpha)^+, \sum_{\ell=1}^{n} Q_\ell^j (r_\ell - A_\ell^\top \alpha)^+\}+\alpha^\top x. \tag{19}$$

Let $\alpha^*(x,d)$ (or α^* for ease of notation) be an optimal solution of the RLD for any given x and d. With a given feasible direction $\Delta\alpha$ of α^* and sufficiently small positive t, define several index subsets of $\{1,2,\dots,n\}$:

$$\begin{aligned}
\mathcal{E}^{>} &= \{j : r_j - A_j^\top\alpha^* > \sum_{\ell=1}^{n} Q_\ell^j (r_\ell - A_\ell^\top\alpha^*)^+\}\\
\mathcal{E}^{=} &= \{j : r_j - A_j^\top\alpha^* = \sum_{\ell=1}^{n} Q_\ell^j (r_\ell - A_\ell^\top\alpha^*)^+\}\\
\mathcal{E}^{<} &= \{j : r_j - A_j^\top\alpha^* < \sum_{\ell=1}^{n} Q_\ell^j (r_\ell - A_\ell^\top\alpha^*)^+\}\\
\mathcal{H}^0 &= \{j : r_j > A_j^\top\alpha^*\}\\
\mathcal{H}^1 &= \{j : r_j = A_j^\top\alpha^*, r_j - A_j^\top\alpha^* - tA_j^\top\Delta\alpha > 0\}\\
\mathcal{H}^2 &= \{j : r_j = A_j^\top\alpha^*, r_j - A_j^\top\alpha^* - tA_j^\top\Delta\alpha \le 0\}\\
\mathcal{F}^1 &= \{j \in \mathcal{E}^{=}\cap(\mathcal{H}^0\cup\mathcal{H}^1) : -(\Delta\alpha)^\top A_j > -\sum_{\ell\in\mathcal{H}^0\cup\mathcal{H}^1} Q_\ell^j (\Delta\alpha)^\top A_\ell\}\\
\mathcal{F}^2 &= \mathcal{E}^{=}\cap(\mathcal{H}^0\cup\mathcal{H}^1)\setminus\mathcal{F}^1.
\end{aligned}$$

By (19) and careful calculations, it is easy to obtain the following

$$\begin{aligned}
& V_t^{\mathrm{RLR}}(x,d,\alpha^*+t\Delta\alpha) - V_t^{\mathrm{RLR}}(x,d,\alpha^*)\\
= \;& t(\Delta\alpha)^\top x - \sum_{j\in\mathcal{E}^{>}} t(\Delta\alpha)^\top d_j A_j \; - \sum_{j\in\mathcal{E}^{<}}\sum_{\ell\in\mathcal{H}^0\cup\mathcal{H}^1} Q_\ell^j t(\Delta\alpha)^\top d_j A_\ell\\
& + \sum_{j\in\mathcal{E}^{=}\cap(\mathcal{H}^0\cup\mathcal{H}^1)} d_j \max\{-t(\Delta\alpha)^\top A_j, -\sum_{\ell\in\mathcal{H}^0\cup\mathcal{H}^1} Q_\ell^j t(\Delta\alpha)^\top A_\ell\}.
\end{aligned}$$

Rewriting the above equation, we arrive at

$$\begin{aligned}
& V_t^{\mathrm{RLR}}(x,d,\alpha^*+t\Delta\alpha) - V_t^{\mathrm{RLR}}(x,d,\alpha^*)\\
= \;& t(\Delta\alpha)^\top \Bigg[x - \sum_{j\in\mathcal{E}^{>}} A_j d_j - \sum_{j\in\mathcal{E}^{=}\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^1} A_j d_j\\
& - \sum_{j\in\mathcal{E}^{=}\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^2} (\sum_{\ell\in\mathcal{H}^0\cup\mathcal{H}^1} Q_\ell^j A_\ell) d_j - \sum_{j\in\mathcal{E}^{<}} (\sum_{\ell\in\mathcal{H}^0\cup\mathcal{H}^1} Q_\ell^j A_\ell) d_j\Bigg]\\
= \;& t(\Delta\alpha)^\top [x - \sum_{j=1}^{n} \bar{A}_j d_j]
\end{aligned} \tag{20}$$

where $\bar{A}_j$ is a vector of n-dimension with nonnegative elements, and

$$
\begin{aligned}
& V_t^{\mathrm{RLR}}(x,d,\alpha^*+t\Delta\alpha) - V_t^{\mathrm{RLR}}(x,d,\alpha^*) \\
= \; & t(\Delta\alpha)^\top \Bigg[x - \sum_{j\in\mathcal{E}^>\cap\mathcal{H}^2} d_j A_j \\
& - \sum_{j\in\mathcal{E}^>\cap(\mathcal{H}^0\cup\mathcal{H}^1)} (d_j + \sum_{\ell\in\mathcal{E}^=\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^2} Q_j^\ell d_\ell + \sum_{\ell\in\mathcal{E}^<} Q_j^\ell d_\ell) A_j \\
& - \sum_{j\in\mathcal{E}^=\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^1} (d_j + \sum_{\ell\in\mathcal{E}^=\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^2} Q_j^\ell d_\ell + \sum_{\ell\in\mathcal{E}^<} Q_j^\ell d_\ell) A_j \\
& - \sum_{j\in\mathcal{E}^=\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^2} (\sum_{\ell\in\mathcal{E}^=\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^2} Q_j^\ell d_\ell + \sum_{\ell\in\mathcal{E}^<} Q_j^\ell d_\ell) A_j \\
& - \sum_{j\in\mathcal{E}^<\cap(\mathcal{H}^0\cup\mathcal{H}^1)} (\sum_{\ell\in\mathcal{E}^=\cap(\mathcal{H}^0\cup\mathcal{H}^1)\cap\mathcal{F}^2} Q_j^\ell d_\ell + \sum_{\ell\in\mathcal{E}^<} Q_j^\ell d_\ell) A_j \Bigg] \\
= \; & t(\Delta\alpha)^\top [x - \sum_{j=1}^n D_j A_j]
\end{aligned}
\tag{21}
$$

where D_j is a random variable which is a linear combination of random variables $d_1, d_2, \ldots, d_n$ with nonnegative coefficients.

Proposition 6.1 *Suppose (a) If $x = \sum_{j\in\mathcal{S}} \delta_j A_j$, then $\{A_j : j \in \mathcal{S}\}$ is of rank m; (b) The demand random variable d_j is continuous for all products j. Then $V_t^{\mathbf{RLP}}(x,d)$ is continuously differentiable with respect to x and*

$$\frac{\partial}{\partial x}\mathbb{E}[V_t^{\mathrm{RLP}}(x,d)] = \mathbb{E}[\frac{\partial}{\partial x} V_t^{\mathrm{RLP}}(x,d)] = \mathbb{E}[\alpha(x,d)]$$

where $\alpha(x,d)$ is the optimal solution of the RLD (18).

Proof. This proof should be the same as that of [19] and is done by contradiction. By the comments in the third paragraph of this section and Lemma 5.1, we need only show that the RLD (18) has a unique solution with probability one over the random variable d for any given x. Assume this is not true. Then there exist $\Delta\alpha \neq 0$ such that for all sufficiently small $t > 0$ and with a positive probability over d

$$V_t^{\mathrm{RLR}}(x,d,\alpha^*+t\Delta\alpha) - V_t^{\mathrm{RLR}}(x,d,\alpha^*) = 0.$$

Condition (21) shows that with a positive probability over d, either $x = \sum_{j=1}^n D_j A_j$ or $(\Delta\alpha)^\top x = 0$, $(\Delta\alpha)^\top A_j = 0$ for all j. According to Condition (b) of this proposition, $P(x = \sum_{j=1}^n D_j A_j) = 0$. So the above expression can hold only if $(\Delta\alpha)^\top x = 0$ and $(\Delta\alpha)^\top A_j = 0$ for all j. For Lemma 3 of [19] and Condition (a) of this proposition, the latter implies $\Delta\alpha = 0$ and we obtain a contradiction. ∎

Some simple examples are given by Talluri and van Ryzin [19] to illustrate that $\frac{\partial}{\partial x}\mathbb{E}[V_t^{\text{RLP}}(x,d)]$ does not exist there there is no buy-up. In the nondifferentiable case, the authors further discuss calculations of subgradients and directional derivatives of $\mathbb{E}[V_t^{\text{RLP}}(x,d)]$.

7 Additional Structural Properties

In this section, we establish some analytical results for the DP, the PNLP and the opportunity cost-based approximation. The first result below states that for any fixed capacity, the closer one is to the departure time the less revenue can be generated, and for any fixed time period, a smaller capacity results in a smaller value of revenue.

Proposition 7.1 *In a network setting, it holds that*

(a) $V_t(x)$ is non-decreasing in t for any fixed x.

(b) $V_t(x)$ is non-decreasing in x for any fixed t.

Proof. (a) This has been proven in (2).

(b) This can be proven by induction. The result clearly holds when $t = 0$. Assuming that $V_{t-1}(x) \geq V_{t-1}(x - A_k)$ for all x and $k = 1, \ldots, n$, we want to prove that $V_t(x) \geq V_t(x - A_k)$. Again by (2), for any k, we obtain

$$\begin{aligned}
V_t(x) &= \sum_{j=1}^{n} \lambda_{tj} \max\{r_j + V_{t-1}(x - A_j), \\
&\quad \sum_{\ell=1}^{n} q_{t\ell}^{j} \max[r_\ell + V_{t-1}(x - A_\ell), V_{t-1}(x)] + q_{t0}^{j} V_{t-1}(x)\} + \lambda_{tj}^{0} V_{t-1}(x) \\
&\geq \sum_{j=1}^{n} \lambda_{tj} \max\{r_j + V_{t-1}(x - A_j - A_k), \\
&\quad \sum_{\ell=1}^{n} q_{t\ell}^{j} \max[r_\ell + V_{t-1}(x - A_\ell - A_k), V_{t-1}(x - A_k)] + q_{t0}^{j} V_{t-1}(x - A_k)\} + \lambda_{tj}^{0} V_{t-1}(x - A_k) \\
&= V_t(x - A_k)
\end{aligned}$$

which implies the result. ■

In a single-leg setting, some stronger results have been established on the marginal cost of $V_t(x)$. In particular, it is proved in [27] that $\Delta V_t(x)$ is non-decreasing in t for any fixed x, and $\Delta V_t(x)$ is non-increasing in x for any fixed t. One would hope to extend those fine analytical properties to the network setting with buy-up. However, this is not the case because extensions are invalid even without buy-up. See [4] for counterexamples.

Our next result studies the relationship between the OC based approximations and the DLP. Consider two versions of the DLP at time t denoted by P_1 and P_2, where the remaining capacities are x and $x - A_j$ for P_1 and P_2 respectively. Assume α, β and $\gamma^1, \ldots, \gamma^n$ is an optimal dual solution of P_1, and $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}^1, \ldots, \tilde{\gamma}^n$ is an optimal dual solution of P_2. Let $\mathrm{BP}_j(x,t) = A_j^\top \alpha$ and $\mathrm{BP}_j(x - A_j, t) = A_j^\top \tilde{\alpha}$ be the bid prices of product j for P_1 and P_2 respectively. Recall that $\mathrm{OC}_j^{\mathrm{LP}}(x,t) = V_t^{\mathrm{DLP}}(x) - V_t^{\mathrm{DLP}}(x - A_j)$ denotes an approximate opportunity cost of product j at time t when the remaining capacity is x.

Proposition 7.2 *For any t, x, j, we have*

$$\mathrm{BP}_j(x,t) \leq \mathrm{OC}_j^{\mathrm{LP}}(x,t) \leq \mathrm{BP}_j(x - A_j, t).$$

Proof. By duality theory,

$$V_t^{\mathrm{DLP}}(x) = \alpha^\top x + \bar{d}^\top \beta + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \gamma^\ell,$$

$$V_t^{\mathrm{DLP}}(x - A_j) = \tilde{\alpha}^\top (x - A_j) + \bar{d}^\top \tilde{\beta} + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \tilde{\gamma}^\ell.$$

Note that both α, β and $\gamma^1, \ldots, \gamma^n$ and $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}^1, \ldots, \tilde{\gamma}^n$ are feasible solutions to the dual problem (8) of both P_1 and P_2. It shows that

$$V_t^{\mathrm{DLP}}(x) \leq \tilde{\alpha}^\top x + \bar{d}^\top \tilde{\beta} + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \tilde{\gamma}^\ell.$$

$$V_t^{\mathrm{DLP}}(x - A_j) \leq \alpha^\top (x - A_j) + \bar{d}^\top \beta + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \gamma^\ell.$$

This further implies that

$$\begin{aligned}
\mathrm{OC}_j^{\mathrm{LP}}(x,t) &\leq \tilde{\alpha}^\top x + (\bar{d}^\top \tilde{\beta} + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \tilde{\gamma}^\ell - [\tilde{\alpha}^\top (x - A_j) + (\bar{d}^\top \tilde{\beta} + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \tilde{\gamma}^\ell] \\
&= \tilde{\alpha}^\top A_j \\
&= \mathrm{BP}_j(x - A_j, t)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{OC}_j^{\mathrm{LP}}(x,t) &\geq \alpha^\top x + (\bar{d}^\top \beta + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \gamma^\ell - [\alpha^\top (x - A_j) + (\bar{d}^\top \beta + \sum_{\ell=1}^n \bar{d}_\ell (Q^\ell)^\top \gamma^\ell] \\
&= \alpha^\top A_j \\
&= \mathrm{BP}_j(x,t).
\end{aligned}$$

This completes the proof. ■

It is known that the objective function of the PNLP for the traditional network capacity management problem is concave and separable. Those fine properties pave the way for using some specialized nonlinear programming algorithms to solve the PNLP more efficiently [13]. In general we cannot prove that the objective function of the PNLP is concave with respect to either y or $z^1, \ldots, z^n$ unless some strong assumptions are imposed as demonstrated in the proposition below.

Proposition 7.3 *Suppose that each product ℓ, $r_\ell \geq \sum_{j=1}^n Q_j^\ell r_j$. Then the objective function of the PNLP is concave with respect to y, $z^1, \ldots, z^n$.*

Proof. To prove concavity of the objective function of the PNLP, we need only prove that its integrand

$$\sum_{j=1}^n r_j \min(y_j, d_j) + \sum_{\ell=1}^n \sum_{j=1}^n r_j \min\{z_j^\ell, Q_j^\ell(d_\ell - \min(y_\ell, d_\ell))\}$$

is concave. The above integrand can be rewritten as

$$\sum_{j=1}^n [r_j - \sum_{\ell=1}^n r_\ell Q_\ell^j] \min(y_j, d_j) + \sum_{\ell=1}^n \sum_{j=1}^n \min\{r_j z_j^\ell + r_\ell Q_\ell^j \min(y_j, d_j), r_j Q_j^\ell d_\ell\}.$$

The concave property of the integrand follows from the following facts: $r_j \geq \sum_{\ell=1}^n r_\ell Q_\ell^j$; the min-function of finitely many concave functions is still concave; the linear combination of several concave functions is still concave if coefficients are nonnegative. Therefore, the objective function of the PNLP is concave with respect to y, $z^1, \ldots, z^n$. ■

The following example shows that in general the objective function of the PNLP is not concave. Suppose there are two classes of products in a single-leg network. Assume that $r_1 = 100$, $r_2 = 1$, $Q_1^2 = 1$, and that the demand for both products is fixed at 10 and 100 respectively. Consider (y_1, y_2, z_1^2) to be three points $u_1 = (1, 0, 2)$, $u_2 = (1, 100, 2)$ and $u_3 = (1, 200, 2)$. Clearly, u_2 is the middle point of the line segment $[u_1, u_3]$. If the objective function f of the PNLP were concave, then $f(u_2) \geq 1/2(f(u_1) + f(u_3))$ would hold. However, it is easy to check by simple calculations that this is not the case. Therefore, f is not concave.

8 Numerical results

In this section, we present some numerical experiments for several booking schemes proposed in Section 3. Our main purpose was to test, for each booking scheme, how much the buy-up model enhances revenue performance compared with the non-buy-up model. All numerical experiments were carried out in MATLAB [14] on a Pentium M-1300 MHz PC. All linear programs were solved using `linprog.m` in MATLAB.

The numerical experiments were conducted on four different test examples. The first two airline examples T1 and T2 are from Higle and Sen [11]. In T1, the airline network consists of 7 cities and 7 legs with 8 itineraries. For each itinerary, there are two fare classes which result in 16 products in total. Test example T2 is similar to T1 but with a different network. In particular, T2 is a hub-spoke network with one hub, 20 spoke cities and 20 legs, and it has two fares and 50 itineraries (and hence 100 products). Additional information on revenue, mean demand, the coefficient of variation (which is equal to mean divided by standard deviation), leg capacities, and detailed network structures can be found from [11].

The last two test examples T3 and T4 are drawn from [6]. Example T3 is a hub-spoke network with one hub, 5 spoke cities and 10 legs, and it has two fares and 30 itineraries (and hence 60 products). Example T4 is a hub-spoke network with two hubs, 4 spoke cities and 10 legs, and it has two fares and 30 itineraries (and hence 60 products). For additional information on revenue, mean demand, leg capacities, and detailed network structures, see [6].

The simulation procedure we follow is described in Talluri and van Ryzin [19]. For each test example, we simulated the booking process 1000 times. In each simulation run, booking requests are randomly generated in two steps. In step 1, the number of requests for each product is randomly generated while in step 2, booking arrival times for each product are randomly generated. We used two different ways to simulate the booking process. In T1 and T2, the number of requests for each product is randomly generated according to a truncated normal distribution (left truncated at zero) with the given expected demand and the given coefficient of variation. Bookings start from 180 days prior to the flight departure time. Demand forecasts for all products are updated on day 180, 120, 60, 30, 14 prior to the departure time. Those five days are used to divide the booking horizon into 5 booking periods. A fixed percentage of demand as specified in [11] is assumed in each booking period for each product. In step 2, booking arrival times in each of 5 booking periods for each product are randomly generated according to a uniform distribution. In T3 and T4, the booking process is modelled as a non-homogeneous Poisson process, where the arrival intensity at time t has a beta distribution and the total number of arrivals has a gamma distribution. The booking is divided into 1000 units. Demand forecasts are updated on time 1000, 600, 400, 200 and 100 prior to the departure time. Higher fare customers arrive more often close to the end of the booking horizon while lower fare customers arrive more often early in the booking horizon. A detailed description of the booking process, used in T3 and T4, can be found in [6].

For all test examples, there are exactly two fare classes: high fare and low fare, for all itineraries. We assume that customers requesting a low-fare product **only** buy up for the corresponding high-fare product in the same itinerary, and customers requesting a high-fare product have a zero buy-up probability. Also we assume that the buy-up probability remains constant throughout the whole booking period in our main numerical experiments.

In each of 1000 simulation runs, all booking requests for each test example as generated above are processed (acceptance or rejection) based on a booking scheme. We have implemented the following five booking schemes:

- BLDLP: Partitioned booking limit policy based on the DLP; see Section 3.
- BPDLP: Bid-price policy based on the DLP; see Section 3.
- ABLDLP: Aggregated partitioned booking limit policy based on the DLP. In detail, a partitioned booking limit u_j for product j is obtained by aggregating y_j and $\sum_{\ell=1}^{n} z_j^{\ell}$. When a booking request for product j arrives, it is accepted if and only if the number of existing bookings of product j (including both initial and buy-up requests) is not more than $u_j - 1$. A buy-up request from product ℓ to product j is accepted if and only if the number of existing bookings of product j (including both initial and buy-up requests) is not more than $u_j - 1$.
- BPRLP: Bid-price policy based on the RLP. Similar to [19], the bid price is equal to the average of 30 samples of the dual prices of the RLP.
- OCDLP: Opportunity cost policy based on the DLP.

Neither the partitioned booking limit policy nor the bid-price policy based on the PNLP is implemented because computationally it is expensive to solve nonlinear programs and its revenue performance is not much better than BLDLP in the traditional network revenue management without buy-ups.

It is known [21] that revenue performance can often be enhanced through so-called re-optimization: resolve approximate problems with updated capacities and future demand, and then update either the partitioned booking limits or the bid prices. Following a similar line of thought, we reran approximation models five times, once at the beginning of each of five booking periods. Note that the RLP is re-optimized much more often than five times in [19] and one would expect that higher revenue values could be obtained with more frequent re-optimization. This is also generally true for the policy OCDLP; see [4].

In order to evaluate effects of buy-ups, we ran each booking scheme with different buy-up probabilities: 0, 0.1, 0.3, 0.5 and 0.8. When the buy-up probability is zero, the buy-up model reduces to the traditional network RM model without buy-ups. We should expect that as the buy-up probability increases, the revenue performance of the buy-up model should improve. The average total revenue over 1000 simulation runs for each booking scheme is used as the chief performance indicator in our experiments.

Tables 2, 3, 4 and 5 summarize average revenue and its 95% confidence interval with different buy-up probabilities for test examples T1, T2, T3 and T4 under five booking schemes. By comparing those tables, we make the following observations. No scheme completely outperforms other schemes across all buy-up probabilities and test examples, though BPRLP appears to be the best on average. This partially confirms the conclusion in [19]. ABLDLP consistently outperforms BLDLP (combinations of different buy-up probabilities and test examples). BPRLP consistently outperforms BPDLP except for p=0 in T2. BPRLP consistently outperforms ABLDLP except for p = 0, 0.1 in T1 and p = 0 in T2. OCDLP is sometimes as competitive as BPRLP (for example in T3), but it frequently produces much lower average revenues than other schemes. We conjecture that performance of OCDLP can improve drastically when re-optimization is done much more frequently.

The aim of our experiments is not to seek the best booking scheme, but to show that the buy-up model should improve the revenue performance of the traditional model without buy-up when buy-up does happen. This intuition is confirmed by our numerical results. Tables 2, 3, 4 and 5 demonstrate that for all five booking schemes and for test examples T1, T2, T3 and T4, the average revenue increases as the buy-up probability increases. This trend can be seen more easily from Figures 1, 2, 3, 4 and 5 where increase of average revenue in percentage of each buy-up booking scheme over the same scheme with zero buy-up probability is presented. The increment speed may vary depending on booking schemes and test examples. However, the overall trend of revenue increment is very similar and the revenue increment is significant which should not be ignored in practice. We note in passing feature that the revenue increment for T3 is quite often larger than T1, T2 and T4.

p	BLDLP	ABLDLP	BPDLP	BPRLP	OCDLP
0	292502	292503	270686	275315	255878
	[291215, 293789]	[291216, 293790]	[268864, 272508]	[273664, 276966]	[254021, 257734]
0.1	302029	308895	279895	291303	279237
	[300698, 303360]	[307539, 310251]	[278073, 281717]	[289574, 293032]	[277358, 281116]
0.3	335299	348506	334580	355100	327146
	[333734, 336864]	[346837, 350175]	[332634, 336526]	[352893, 357307]	[325113, 329179]
0.5	401487	422321	439028	442893	417019
	[399488, 403486]	[420121, 424521]	[436667, 441389]	[440550, 445236]	[414757, 419281]
0.8	501079	512162	505621	513267	474962
	[499155, 503003]	[510260, 514054]	[503694, 507548]	[511440, 515094]	[472269, 477655]

Table 2: Revenue and 95% confidence interval for T1.

p	BLDLP	ABLDLP	BPDLP	BPRLP	OCDLP
0	1379612	1379617	1387494	1361241	1316397
	[1376747, 1382474]	[1376752, 1382481]	[1383797, 1391191]	[1357156, 1365326]	[1311475, 1321319]
0.1	1428403	1467428	1421038	1476620	1404506
	[1425436, 1431370]	[1464510, 1470346]	[1417050, 1425025]	[1472728, 1480512]	[1399279, 1409732]
0.3	1604918	1685913	1705952	1788660	1714125
	[1601685, 1608151]	[1682496, 1689330]	[1702252, 1709652]	[1784541, 1792780]	[1709924, 1718325]
0.5	1917496	2029622	2112544	2121054	2093707
	[1913758, 1921234]	[2025338, 2033905]	[2108228, 2116861]	[2116706, 2125401]	[2089275, 2098140]
0.8	2259302	2343747	2363097	2377306	2285507
	[2255596, 2263008]	[2339958, 2347536]	[2359358, 2366837]	[237378, 2380827]	[2280876, 2290139]

Table 3: Average revenue and 95% confidence interval for T2.

p	BLDLP	ABLDLP	BPDLP	BPRLP	OCDLP
0	414551	414551	416487	418288	334513
	[414110, 414992]	[414110, 414992]	[416028, 416946]	[417798, 418779]	[333791, 335235]
0.1	418980	424411	420427	425069	404880
	[418494, 419465]	[423943, 424879]	[419973, 420881]	[424538, 425601]	[403984, 405775]
0.3	498912	522782	540191	540191	540191
	[498093, 499732]	[521819, 523746]	[539174, 541207]	[539174, 541207]	[539174, 541207]
0.5	663263	691873	709359	709359	709359
	[662314, 664212]	[690761, 692984]	[708205, 710513]	[708205, 710513]	[708205, 710513]
0.8	920471	943096	958068	958295	954829
	[919318, 921624]	[941906, 944287]	[956928, 959207]	[957147, 959443]	[933731, 955927]

Table 4: Average revenue and 95% confidence interval for T3.

p	**BLDLP**	**ABLDLP**	**BPDLP**	**BPRLP**	**OCDLP**
0	516689	516689	518207	518723	515085
	[516027, 517352]	[516027 517352]	[517448 518966]	[517959 519496]	[514279 519496]
0.1	557208	568906	577815	585095	565541
	[556455, 557961]	[568178, 569634]	[576744, 578886]	[584134, 586057]	[564335, 566746]
0.3	665556	676613	684717	684769	658555
	[664985, 666126]	[675977, 677248]	[684038, 68536]	[684067, 685471]	[657466, 659644]
0.5	755373	766760	775056	775465	761422
	[754721, 756024]	[766055, 767465]	[774264, 775847]	[774671, 776259]	[760324, 762520]
0.8	891358	902664	843900	912504	902963
	[890598, 892117]	[901848, 903480]	[843071, 844729]	[911625, 913383]	[902934, 903893]

Table 5: Average revenue and 95% confidence interval for T4.

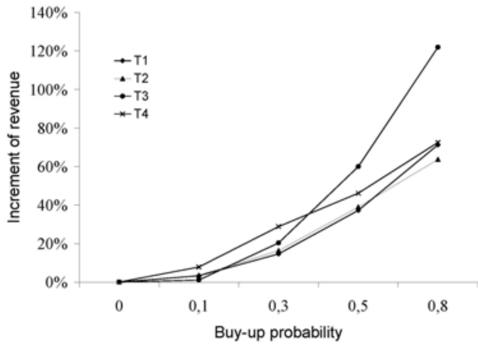

Figure 1: Revenue increment for BLDLP.

To generate more insights into the buy-up model, we analyze the results for T1 using the following metrics:

- How do load factors change when the buy-up probability increases?
- How do partitioned booking limits in the first period of the booking horizon change when the buy-up probability increases?
- How do buy-up acceptance ratios change when the buy-up probability increases? Here the buy-up acceptance ratio for a product is equal to the percentage of the booking requests which are initially rejected, then elect to buy up and are consequently accepted.
- How does the non-uniform buy-up probability affect performance of the buy-up model?

Regarding load factors, we did not observe any significant shifts when buy-up probabilities change. Therefore we do not present any table of results to illustrate this.

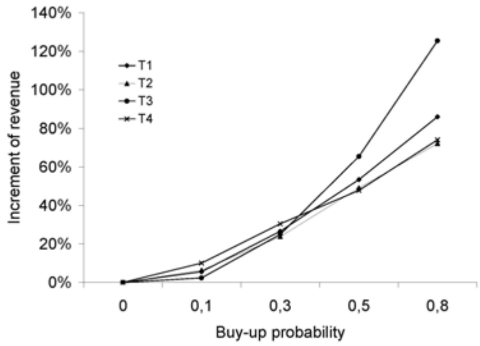

Figure 2: Revenue increment for ABLDLP.

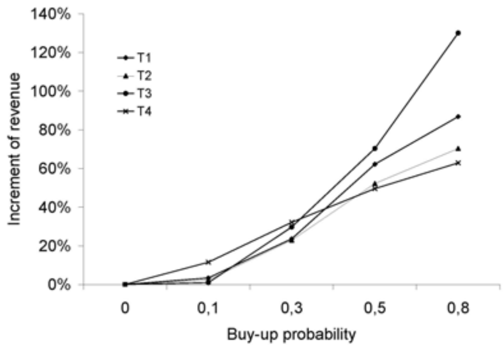

Figure 3: Revenue increment for BPDLP.

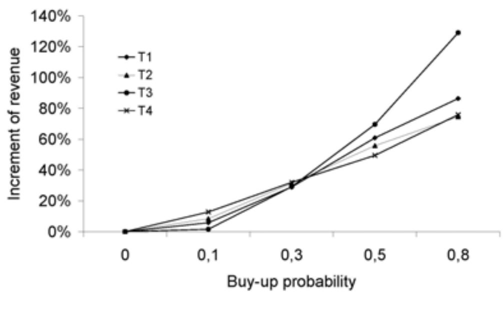

Figure 4: Revenue increment for BPRLP.

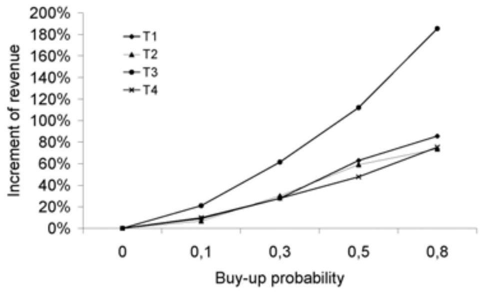

Figure 5: Revenue increment for OCDLP.

Product	Buy-up Probability				
	0.0	**0.1**	**0.3**	**0.5**	**0.8**
1	50	57.22	77.86	119.84	205
2	100	114.44	160	200	155
3	10	10	10	28.05	50
4	20	20	20	70	100
5	5	5	5	23.36	45
6	20	22.22	28.57	59.21	100
7	20	23.33	37.14	70	55
8	5	10	20	30	45
9	135	127.78	107.14	60.32	0
10	75	55.56	0	0	0
11	50	50	50	13.89	0
12	100	100	100	0	0
13	50	50	50	13.29	0
14	80	77.78	71.43	21.58	0
15	75	66.67	42.86	0	0
16	0	0	0	0	0

Table 6: Aggregated partitioned booking limits for the first period of T1.

Aggregated partitioned booking limits for the first period of test example T1 are shown in Table 6. It is not surprising to observe that the aggregated partitioned booking limit increases for all higher-fare products as the buy-up probability increases except for products 2 and 7 when the buy-up probability increases from 0.5 to 0.8. Intuitively, as the buy-up probability increases, more seats should be reserved for higher-value customers and lower-value customers who elect to buy up. In contrast and also as a complementary consequence, the aggregated partitioned booking limit decreases for all lower-fare products as the buy-up probability increases.

The buy-up acceptance ratio of ABDLP for test example T1 is displayed in Table 7. It shows that this ratio increases for most products as the buy-up probability increases. Note that booking requests for products $1, 2, \cdots, 8$ are lost forever when they are initially rejected since they are the most expensive products in their corresponding itineraries. Similar observations are obtained for other booking schemes for T1.

Product	Buy-up Probability				
	0.0	0.1	0.3	0.5	0.8
9		0.4	0.5	0.61	0.94
10		0.72	0.96	1	0.83
11		0.14	0.14	0.58	0.95
12		0.15	0.19	0.98	0.96
13		0.13	0.1	0.55	0.91
14		0.32	0.36	0.66	0.93
15		0.37	0.57	0.93	0.66
16		0.98	0.95	0.93	0.88

Table 7: The buy-up acceptance ratio of ABDLP for T1.

Finally we consider variable buy-up probabilities over time. But we still assume that the buy-up probability remains a constant in the same booking period for all lower-fare customers. In various approximation models (e.g. the DLP), we have to use a uniform buy-up probability Q_j^ℓ. Suppose there are N booking periods and the buy-up probability for lower-fare customers is q_t ($t = 1, \cdots, N$ where N is the index for the last booking period prior to the departure time). Then we approximate the average buy-up probability $\bar{q}_t$ from booking period t to booking period N using the following formula:

$$\bar{q}_t = \frac{\sum_{\tau=t}^{N} q_\tau \bar{d}_\tau}{\sum_{\tau=t}^{N} \bar{d}_\tau}, \qquad t = 1, \cdots, N \tag{22}$$

where $\bar{d}_t \in \mathbb{R}^n$ is the mean demand from booking period t onwards for the products in question. We make two remarks. Firstly when $\sum_{\tau=t}^{N} \bar{d}_\tau = 0$, we may assume that $\bar{q}_t = 1$ or any other probability as it does not matter when no booking request of such a product arrives. Secondly it turns out that buy-up probabilities for all lower-fare customers are the same for test example T1 as all lower-fare customers in T1 have the same mean demand distribution: $0.05, 0.25, 0.75, 0.0, 0.0$.

In our experiments, we assume that the variable buy-up probabilities are $(q_1, q_2, q_3, q_4, q_5)$ = (0, 0.1, 0.3, 0.5, 0.8). By formula (22), the new average approximate buy-up probabilities $(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4, \bar{q}_5)$ = (0.245, 0.25789, 0.3, 0.65, 0.8). Therefore, the overall average approximate buy-up probability for the whole booking horizon is 0.245 for all lower-fare customers. Table 8 summarizes the difference of revenue performance with a constant buy-up probability 0.245 and variable buy-up probabilities (0, 0.1, 0.3, 0.5, 0.8) for T1. A moderate revenue gain can be observed by replacing the constant buy-up probability by variable buy-up probabilities. Therefore, we would recommend using variable buy-up probabilities if variations are relatively large.

	BLDLP	**ABLDLP**	**BPDLP**	**BPRLP**	**OCDLP**
(I)	3245134	336229	320849	333779	310133
	[323041, 325987]	[334676,337783]	[318987,322711]	[331858,335699]	[308124,312142]
(II)	327879	341397	321216	343959	310579
	[326370, 329388]	[339792, 343003]	[319178, 323254]	[341805, 346113]	[308268, 312889]

Table 8: Revenue performance and confidence interval under a constant buy-up probability (I) and variable buy-up probabilities (II) for T1.

9 Concluding remarks

In this paper, we have proposed a buy-up model which allows us to explore the impact of customers' buy-up behavior. The buy-up model is defined using the dynamic programming formulation and approximated by various mathematical programming formulations which are much easier to solve. We have shown asymptotical optimality of both the partitioned booking limit policy and the bid-price policy that are based on various approximations. We have also shown why the bid price based on the RLP makes a sensible booking policy. We have numerically demonstrated through simulation that the buy-up model and its various approximations can generate significantly higher revenue than the traditional network revenue management models.

In the future, we would like to enhance our study by considering other issues. Firstly, in some airlines, overbooking is always treated as part of the inventory control process. It should be straightforward to include overbooking in our buy-up model though analysis may be relatively tedious. Secondly, it is known that virtual nesting control [24, 23] is another popular booking policy in addition to the partitioned booking limit policy and the bid-price policy. Therefore extending virtual nesting control for the buy-up model would merit further examination.

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