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# Long Term Spread Option Valuation and Hedging

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## Abstract

This paper investigates the valuation and hedging of spread options on two commodity prices which in the long run are cointegrated. For long term option pricing the spread between the two prices should therefore be modelled directly. This approach offers significant advantages relative to the traditional multi-factor spread option pricing model since the correlation between two asset returns is notoriously hard to model. In this paper, we propose one and two factor models for spot spread processes under both the risk-neutral and market measures. We develop pricing and hedging formulae for options on spot and futures spreads. Two examples of spread options in energy markets—the *crack spread* between heating oil and WTI crude oil and the *location spread* between Brent blend and WTI crude oil – are analyzed to illustrate the results.

**JEL classification:** G12

**Key words:** spread option, cointegration, mean-reversion, option pricing, energy markets

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## 1. Introduction

A *spread option* is an option written on the difference (*spread*) of two underlying asset prices  $S_1$  and  $S_2$  respectively. We consider *European* options with *payoff* the greater or lesser of  $S_2(T)-S_1(T)-K$  and 0 at maturity  $T$  and *strike price*  $K$  and focus on spreads in the commodity (especially energy) markets (both for spot and futures). In these markets spread options are usually based on differences between prices of the same commodity at two different locations (*location* spreads) or times (*calendar* spreads), between the prices of inputs and outputs (*production* spreads) or between the prices of different grades of the same commodity (*quality* spreads). The New York Mercantile Exchange (NYMEX) also offers tradable options on the heating oil/ crude oil or gasoline/crude oil spreads (*crack* spreads).

It is natural to model the spread by modelling each asset separately. Margrabe (1978) was the first to treat spread options and gave an analytical solution for strike price zero (the *exchange option*). Wilcox (1990) and Carmona and Durrleman (2003) use Bachelier's (1900) formula to analytically price spread options assuming the underlying prices follow arithmetic Brownian motions. It is more difficult to value a spread option if the two underlying prices follow geometric Brownian motions. Various numerical techniques have been proposed to price such an option. Rubinstein (1991) values the spread option in terms of a double integral. Dempster and Hong (2000) use the fast Fourier transform to evaluate this integral numerically. Carmona and Durrleman (2003) offer a good review of spread option pricing.

Many researchers have modelled a spread option by modelling the two underlying asset prices in the *risk neutral measure* as<sup>1</sup>

$$\begin{aligned}d\mathbf{S}_1 &= rS_1dt + \sigma_1S_1d\mathbf{W}_1 \\d\mathbf{S}_2 &= rS_2dt + \sigma_2S_2d\mathbf{W}_2 \\E d\mathbf{W}_1d\mathbf{W}_2 &= \rho dt.\end{aligned}\tag{1}$$

The *correlation*  $\rho$  plays a substantial rôle in valuing a spread option; trading a spread

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<sup>1</sup> Boldface is used throughout to denote random entities – here conditional on  $S_1$  and  $S_2$  having realized values  $S_1$  and  $S_2$  at time  $t$  which is suppressed for simplicity of notation.

option is equivalent to trading the correlation between the two asset returns. However, Kirk (1995), Mbanefo (1997) and Alexander (1999) have suggested that return correlation is very volatile in energy markets.

Thus assuming a *constant* correlation between two assets as in (1) is inappropriate for modelling. But there is another longer term relationship between two asset prices termed *cointegration* which has been little studied by asset pricing researchers. If a cointegration relationship exists between two asset prices the spread should be modelled *directly* for long term option pricing.

This is the topic of this paper which is organized as follows. Section 2 gives a brief review of price cointegration and the principal statistical tests for cointegration and for the mean reversion of spreads. Section 3 proposes one and two factor models of the underlying spread process in the risk-neutral and market measures and shows how to calibrate these models. Section 4 presents option pricing formulae for options on spot and futures spreads. Section 5 provides two examples in energy markets which illustrate the theoretical work and Section 6 concludes.

## **2. Cointegrated prices and mean reversion of the spread**

The value of a spread option is determined by the dynamic relationship between two underlying asset prices and the *correlation* of the corresponding *returns* time series is commonly understood and widely used. *Cointegration* is a method for treating the long-run equilibrium relationships between two asset *prices* generated by market forces and behavioural rules. Engle and Granger (1987) formalized the idea of integrated variables sharing an equilibrium relation which turns out to be either stationary or to have a lower degree of integration than the original series. They used the term cointegration to signify co-movements among trending variables which could be exploited to test for the existence of equilibrium relationships within the framework of fully dynamic markets.

The parameters in equation (1) can be calibrated using returns data but the cointegration relationship must be investigated with price data (Hamilton, 1994). In general, the return correlation is important for short term price relationships and the price cointegration for their long run counterparts. If two asset prices are cointegrated (1) is only useful for *short term* valuation even when the correlation between their returns is known exactly. Since we wish to treat long term spread option pricing we shall investigate the cointegration (long term equilibrium) relationship between asset prices. First we briefly explain the economic reasons why such a long-run equilibrium exists between prices of the same commodity at two different locations, prices of inputs and outputs and prices of different grades of the same commodity<sup>2</sup>.

The *law of one price* (or *purchasing power parity*) implies that cointegration exists for prices of the same commodity at different locations. Due to market frictions (trading costs, shipping costs, etc.) the same good may have different prices but the mispricing cannot go beyond a threshold without allowing market arbitrages (Samuelson, 1964). Input (raw material) and output (product) prices should also be cointegrated because they directly determine supply and demand for manufacturing firms. There also exists an equilibrium involving a threshold between the prices of a commodity of different grades since they are substitutes for each other.

If such long-term equilibria hold for these three pairs of prices cointegration relationships should be detected in the empirical data. Duan and Pliska (2003) model the log-price—rather than the price—cointegration between two US market indices. In *equity* markets investors are concerned with index *returns* rather than levels so this may be a good choice. However from the economic arguments it follows that the spread between two spot *commodity* prices reflects deviations from a general (possibly growth) equilibrium, e.g. the profits of producing (production spread), shipping (location spread) or switching (quality spread). Consistent with Jurek and Yang (2006) we model here *price* cointegration.

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<sup>2</sup> Although the models of this paper are applicable to calendar spreads we will not treat them here.

In empirical analysis economists usually use equations (2) and (3) to describe the cointegration relationship:

$$\mathbf{S}_{1t} = c_t + d\mathbf{S}_{2t} + \boldsymbol{\varepsilon}_t \quad (2)$$

$$\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_{t-1} = \omega\boldsymbol{\varepsilon}_{t-1} + \mathbf{u}_t, \quad (3)$$

where  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are the two asset prices and  $\mathbf{u}$  is a Gaussian disturbance. Engle and Granger (1987) demonstrate that a cointegration model is the same as an *error correction* model, i.e. the error term  $\boldsymbol{\varepsilon}_t$  in (2) must be *mean-reverting* (3). Thus a simple way to test the cointegration relationship is to test whether  $\omega$  is a significantly negative number in equation (3), i.e. whether the spread process is mean-reverting (Dickey-Fuller, 1979). Equation (2) can be seen as the dynamic equilibrium of an economic system. When  $\mathbf{S}_1$  and  $\mathbf{S}_2$  deviate from the long-run equilibrium relationship they revert back to it in the future.

For both location and quality spreads  $\mathbf{S}_1$  and  $\mathbf{S}_2$  should ideally follow the same trend, i.e.  $d$  should be equal to 1. However for production spreads such as the *spark spread* (the spread between the electricity price and the gas price)  $d$  may not be exactly 1. Usually 3/4 of a gas contract is equivalent to 1 electricity contract so that investors trade a 1 electricity / 3/4 gas spread which represents the profit of electricity plants (Carmona and Durrleman, 2003). Since gasoline and heating oil are cointegrated substitutes, the  $d$  value could be 1 for both the heating oil/crude oil spread and the heating oil/gasoline spread (Girma and Paulson, 1999). For our three spreads of interest — location, production and quality —  $d$  should be 1.

Letting  $\mathbf{x}_t$  denote the spread between two cointegrated spot prices  $\mathbf{S}_1$  and  $\mathbf{S}_2$  it follows from (2) and (3) in this case that

$$\mathbf{x}_t - \mathbf{x}_{t-1} = c_t - c_{t-1} - \omega(c_{t-1} - \mathbf{x}_{t-1}) + \mathbf{u}_t, \quad (4)$$

i.e., the spread of the two underlying assets is *mean-reverting*. No matter what the nature of the underlying  $\mathbf{S}_1$  and  $\mathbf{S}_2$  processes the spread between them can behave quite differently from their individual behaviour. This suggests modelling the spread

*directly* using an Ornstein-Uhlenbeck process for long-term option evaluation because the cointegration relationship has substantial influence in the long run. Such an approach gives at least three advantages over alternatives as it: 1) avoids modelling the correlation between the two asset returns, 2) catches the long-run equilibrium relationship between the two asset prices and 3) yields an analytical solution for spread options. For example, Jurek and Yang (2006) employ an Ornstein-Uhlenbeck process to model the spread between *Siamese twin equities*<sup>3</sup> and present an optimal asset-allocation strategy for spread holders.

### **2.1 Cointegration tests**

To test the cointegration of two asset prices, we first need to test whether each generates a *unit root* time series. In an efficient market asset prices singly will usually generate unit root (independent increment) time series because the current price should not provide forecasting power for future prices. If two asset price processes are unit root but the spread process is not, there exists a cointegration relationship between the prices and the spread will not deviate outside economically determined bounds.

The *augmented Dickey-Fuller* (ADF) test may be used to check for unit roots in asset price time series. The ADF statistic uses an ordinary least squares (OLS) auto regression

$$S_t - S_{t-1} = \delta_0 + \delta_1 \cdot S_{t-1} + \sum_{i=1}^p \delta_{i+1} (S_{t-i} - S_{t-i-1}) + \boldsymbol{\eta}_t \quad (5)$$

to test for unit roots, where  $S_t$  is the asset price at time  $t$ ,  $\delta_i$ ,  $i=0, \dots, p$ , are constants and  $\boldsymbol{\eta}_t$  is a Gaussian disturbance. If the coefficient  $\delta_1$  is negative and exceeds the critical value in Fuller (1976) then the null-hypothesis that the series has no unit root is rejected.

We can use an extension of (4) corresponding to (5) to test the cointegration

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<sup>3</sup> For details on Siamese twins see Froot and Dabora (1999).



relationship:

$$\begin{aligned} \mathbf{S}_{1t} &= \mathbf{c}_t + \mathbf{d} \times \mathbf{S}_{2t} + \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_{t-1} &= \chi_0 + \chi_1 \times \boldsymbol{\varepsilon}_{t-1} + \sum_{i=1}^p \chi_{i+1} (\boldsymbol{\varepsilon}_{t-i} - \boldsymbol{\varepsilon}_{t-i-1}) + \mathbf{u}_t. \end{aligned} \quad (6)$$

When  $\chi_1$  is significantly negative the hypothesis that cointegration exists between the two underlying asset price processes  $\mathbf{S}_1$  and  $\mathbf{S}_2$  is accepted (Hamilton, 1994).

## 2.2 Market measure mean reversion test

In Section 3 we will see that the mean-reverting property of the spot spread can be detected by examining the mean-reversion of futures spreads with a constant time to maturity. Empirically we estimate

$$\mathbf{F}(t + \Delta t, t + \Delta t + \tau) - \mathbf{F}(t, t + \tau) = \alpha + \beta \cdot \mathbf{F}(t, t + \tau) + \boldsymbol{\varepsilon}_t, \quad (7)$$

where  $\mathbf{F}(t, t + \tau)$  is the futures spread of maturity  $t + \tau$  observed at  $t$ ,  $\Delta t$  is the sampling time interval and  $\boldsymbol{\varepsilon}_t$  is a random disturbance. If  $\beta$  is significantly negative then the spot spread is deemed to be mean-reverting and a cointegration relationship is taken to exist between the two underlying asset prices. This method examines the evidence for mean-reversion in the *market* measure using historical futures prices data.

## 2.3 Risk-neutral measure mean-reversion test

We can use (*ex ante*) market data analysis to test whether investors *expect* the future spread to revert in the risk-neutral measure. This methodology focusses on relations between spread levels and the *spread term structure slope* defined as the change across the maturities of futures spreads. A negative relationship between the spot spread level (or short-term futures spread level) and the futures spread term structure slope shows that risk neutral investors expect mean-reversion in the spot spread. Indeed, since each futures price equals the trading date expectation of the delivery date spot price in the risk-neutral measure the current term structure of the futures spread reveals where investors expect the spot spread to be in future. Detecting an inverse relationship between current spread level and future slope supports a negative relationship in the risk neutral measure between the current spread level and its future

movement. Bessembinder *et al* (1995) attempt to discover *ex ante* mean reversion in commodity spot prices. To discover the negative relationship in our case we estimate

$$\mathbf{x}_L - x_S = \zeta + \gamma x_S + \boldsymbol{\varepsilon}, \quad (8)$$

where  $x_L$  and  $x_S$  are respectively long-end and short-end spread levels in the futures spread term structure and  $\boldsymbol{\varepsilon}$  is a noise term. If  $\gamma$  is significantly negative there is evidence that the spot spread is *ex ante* mean-reverting in the risk-neutral measure.

#### 2.4 Continuous time consequences

Regression models (7) and (8) allow the empirical examination of the mean-reverting properties of the spot spread in respectively the market and risk-neutral measures. Moreover by testing whether the spread process with  $d:=1$  is mean-reverting we can examine whether this ideal cointegration relationship holds in (2). If testing indicates mean-reversion then a cointegration relationship may be supposed and the spread process can be modelled directly. The underlying continuous time *spot spread* process  $\mathbf{x}_t$  should then follow the continuous time version of equation (4) in the *market* measure:

$$d\mathbf{x}_t = k(\psi(t) - x_t)dt + \sigma d\mathbf{W}, \quad (9)$$

where  $k$  is the mean reversion speed,  $\psi(t)$  is a function of physical time  $t$ ,  $\sigma$  is a constant volatility and  $\mathbf{W}$  is a Wiener process. The solution of (9) has the property that the variance of the spread  $\mathbf{x}_t$  will not blow up asymptotically in time and its unconditional variance is stationary.

The traditional two price spread process model does not possess these properties. By differencing the two equations in (1) we obtain the spread of the two contract prices as

$$d(\mathbf{S}_1 - \mathbf{S}_2) = r(\mathbf{S}_1 - \mathbf{S}_2)dt + \sigma(S_1, S_2, \rho)d\mathbf{W}, \quad (10)$$

where

$$\sigma(S_1, S_2, \rho) = \sqrt{\sigma_1^2 S_1^2 + \sigma_2^2 S_2^2 + 2\sigma_1\sigma_2 S_1 S_2 \rho}$$

is the instantaneous volatility of the spread at time  $t$ . Thus the price spread process

does not mean-revert in the risk neutral measure and the standard deviation of the spread increases over time and blows up asymptotically. We do not see this for spreads between cointegrated commodity prices in historical market data (Villar and Joutz, 2006).

Commodity and equity spread processes are different however. In the risk neutral measure any tradable equity portfolio without dividend payments - including spreads - should grow at the risk-free rate. Thus non-dividend paying stock spreads will not be mean-reverting in this measure. However in the case of physical commodities, especially those commodities which cannot be stored, the risk-neutral drift must encompass some form of stochastic *convenience yield*. We see this empirically in the mean-reverting properties of some commodity prices (Schwartz, 1997). Because of convenience yields spread processes for commodities can be quite different from those for equities.

### 3. Modelling the Spread Process

We have seen that we can model the spread process directly using an Ornstein-Uhlenbeck process if a cointegration relationship exists between the two underlying asset prices. If we wish to price contingent claims on the spread we must re-specify (9) in the risk neutral measure. We now present direct models of the spot and future spread processes with one and two factors.

#### 3.1 One factor model

First consider a one-factor model of the spot spread in the risk neutral measure specified by

$$dx_t = k(\theta - x_t)dt + \sigma d\mathbf{W}, \quad (11)$$

where  $\theta$  is a constant which represents the long-run *mean* of the spread process. For simplicity we assume that  $\theta$  does not depend on time.

Solving (11) given the starting time  $v$  and spread position  $x_v$  we obtain

$$\mathbf{x}_s = x_v e^{-k(s-v)} + \theta[1 - e^{-k(s-v)}] + e^{-ks} \int_v^s \sigma e^{kt} d\mathbf{W} \quad (12)$$

which follows a normal distribution at time  $s$  with mean

$$a_s = x_v e^{-k(s-v)} + \theta[1 - e^{-k(s-v)}] \quad (13)$$

and standard deviation  $b_s = \sqrt{\frac{1 - e^{-2k(s-v)}}{2k}} \sigma$ . (14)

Thus  $b_s \rightarrow \frac{\sigma}{\sqrt{2k}}$  as  $s \rightarrow \infty$  so that the spread standard deviation tends to a constant asymptotically.

Define  $F(t, T, x_t)$  as the *futures spread* ( the spread of two futures prices) of maturity  $T$  observed in the market at time  $t$  when the *spot spread* is  $x_t$ . In the risk neutral measure the spot spread process  $\mathbf{x}$  must satisfy the *no arbitrage condition*

$$E[\mathbf{x}_T | x_t] = F(t, T, x_t), \quad (15)$$

i.e. in the absence of arbitrage the conditional expectation with respect to the spot spread  $x_t$  of the out-turn spot spread at  $T$  in the risk-neutral measure is the futures spread observed at time  $t < T$ . This must hold because it is costless to enter a futures spread (long one future and short the other).

From (12) and the no arbitrage constraint (15) we have

$$F(t, T; x_t) = x_t e^{-k(T-t)} + \theta[1 - e^{-k(T-t)}] \quad (16)$$

From Ito's lemma it follows that the futures spread  $\mathbf{F}(t, T, x_t)$  with fixed maturity *date*  $T$  satisfies

$$d\mathbf{F}(t, T; x_t) = \frac{\partial F}{\partial x} dx_t + \frac{\partial F}{\partial t} dt = e^{-k(T-t)} \sigma d\mathbf{W}, \quad (17)$$

i.e. the futures spread is a *martingale* and its volatility decays exponentially in time to maturity  $(T-t)$ .

### 3.2 Two factor model

The mean reverting spot spread in (11) mainly reflects the short to medium term properties of the futures spread (Gabillon, 1995) with the volatility of the futures spread decaying exponentially with time to maturity. This is a term structure which is not flexible enough to match the volatility term structure observed in the market. The spot spread usually has an estimated strong mean-reversion speed so that the one factor model's estimated volatilities of future spreads with maturities longer than 2 or 3 years are quite close to zero while the observed spreads normally have quite noticeable volatility. Thus another factor  $y$  is needed to reflect the long-end movement of the futures spread term structure which relates to movements of fundamentals such as storage or shipping cost changes between the two commodities. For simplicity and consistent with market equilibrium we will assume that the long-run mean of the  $y$  process is zero in the risk neutral measure.

In the risk neutral measure the underlying *spot spread* process  $x$  and the long-run factor  $y$  follow

$$\begin{aligned}
 dx_t &= k(\theta + y_t - x_t)dt + \sigma d\mathbf{W} \\
 dy_t &= -k_2 y_t dt + \sigma_2 d\mathbf{W}_2 \\
 Ed\mathbf{W}d\mathbf{W}_2 &= \rho dt,
 \end{aligned} \tag{18}$$

where the latent factor  $y$  is a 0 mean-reverting process (if  $k_2$  is positive) representing the *long end* of the spread term structure. We can interpret the dynamics of the spot spread specified by (18) as reverting to a *stochastic* long run mean  $\theta + y_t$ . Since the volatility of the long end spread is usually much smaller than that of the short end  $\sigma_2$  should be quite small. Since fundamentals (e.g. storage costs) have slower speeds of adjustment, we can expect  $k_2$  to be much smaller than  $k$ . Obviously the long-end movement of the spread should not be *a priori* constrained to be mean-reverting and it could be Wiener process like in some circumstances. Given  $\sigma_2$ , the smaller the  $k_2$  value the closer the  $y$  process is to a Wiener process. Gabillon (1995) considered a similar model to (17) for oil futures contract prices.

Appendix 1 shows that the solution of (18) in the risk neutral measure is

$$\begin{aligned} \mathbf{x}_s = & x_v e^{-k(s-v)} + \theta[1 - e^{-k(s-v)}] + \frac{y_v k}{k - k_2} [e^{-k_2(s-v)} - e^{-k(s-v)}] \\ & + \frac{k\sigma_2}{k - k_2} \int_v^s [e^{-k_2(s-u)} - e^{-k(s-u)}] d\mathbf{W}_2(u) + \int_v^s e^{-k(s-t)} \sigma d\mathbf{W}(t). \end{aligned} \quad (19)$$

Define:

$$\begin{aligned} A_1 &:= \frac{\sigma^2}{2k} [1 - e^{-2k(s-v)}] \\ A_2 &:= \left( \frac{1}{2k_2} [1 - e^{-2k_2(s-v)}] + \frac{1}{2k} [1 - e^{-2k(s-v)}] - \frac{2}{(k + k_2)} [1 - e^{-(k_2+k)(s-v)}] \right) \cdot \frac{k^2 \sigma_2^2}{(k - k_2)^2} \\ A_3 &:= \frac{k\sigma_2\sigma}{k - k_2} \left( \frac{1}{k + k_2} [1 - e^{-(k+k_2)(s-v)}] - \frac{1}{2k} [1 - e^{-2k(s-v)}] \right). \end{aligned} \quad (20)$$

Then the standard deviation of  $\mathbf{x}$  at time  $s$  is

$$b_s = \sqrt{A_1 + A_2 + 2\rho A_3} \quad (21)$$

and  $\text{var}(\mathbf{x})$  is

$$\frac{\sigma^2}{2k} + \frac{\sigma_2^2}{2k_2(1 + k_2/k)} + \frac{\rho\sigma\sigma_2}{2(k + k_2)} \quad (22)$$

asymptotically if  $k_2$  is not zero. This is again because both the  $\mathbf{x}$  and  $\mathbf{y}$  processes are mean-reverting. If  $k_2$  is zero,  $A_2$  becomes

$$A_2' = \left\{ (s - v) + \frac{1}{2k} [1 - e^{-2k(s-v)}] - \frac{2}{k} [1 - e^{-k(s-v)}] \right\} \sigma_2^2. \quad (23)$$

In this case the standard deviation of the spread will grow with time  $s$  and blows up asymptotically. However, since  $\sigma_2$  is usually quite small, the speed of growth of the standard deviation is also quite small whether or not  $k_2$  is zero. This is consistent with the notion mentioned in Mbanefo (1997) that the spread standard deviation grows much more slowly than its underlying two legs. In summary the mean-reverting two factor model covers two cases of long-end movements: 1) a stationary  $\mathbf{y}$  factor and 2) a Brownian motion  $\mathbf{y}$  factor.

From the no arbitrage constraint (15) and (19), we have

$$F(t, T; x_t) = x_t e^{-k(T-t)} + \theta[1 - e^{-k(T-t)}] + \frac{y_t k}{k - k_2} [e^{-k_2(T-t)} - e^{-k(T-t)}]. \quad (24)$$

From Ito's lemma it follows that the *risk neutral* futures spread  $F(t, T)$  process with

fixed maturity date  $T$  satisfies

$$dF(t, T) = e^{-k(T-t)} \sigma d\mathbf{W} + \frac{k}{k-k_2} [e^{-k_2(T-t)} - e^{-k(T-t)}] \sigma_2 d\mathbf{W}_2 \quad (25)$$

and is thus a martingale. Its volatility is composed of two parts: the first relating to the one factor model and the second to the long run factor  $\mathbf{y}$ . As a result, if  $k_2$  is much smaller than  $k$  the volatility of the long-term futures spread will tend to decay only slowly to zero over time.

### 3.3 Spread process in the market measure

We will need a risk-adjusted version in order to calibrate the model presented above to market data. If we specify a risk premium process for  $\mathbf{x}$  and  $\mathbf{y}$  then the drift parts of our models can incorporate these risk premia in the market measure (Duffie, 1988). Previous studies assume constant risk premia when modelling Ornstein-Uhlenbeck processes (see, e.g. Hull & White (1990) and Schwartz (1997)) and so will we. Thus, the single-factor model for the spot spread process in the *market* measure follows

$$dx_t = [k(\theta - x_t) + \lambda]dt + \sigma d\mathbf{W}, \quad (26)$$

where  $\lambda$  is the *risk premium*. The two-factor model in the market measure follows

$$\begin{aligned} dx_t &= [k(\theta + y_t - x_t) + \lambda]dt + \sigma d\mathbf{W} \\ dy_t &= (-k_2 y_t + \lambda_2)dt + \sigma_2 d\mathbf{W}_2 \\ E d\mathbf{W} d\mathbf{W}_2 &= \rho dt, \end{aligned} \quad (27)$$

where  $\lambda$  and  $\lambda_2$  are the risk premia of the  $\mathbf{x}$  and  $\mathbf{y}$  processes respectively.

Again using Ito's lemma on the risk adjusted versions of (16) and (24) with (26) and (27) we obtain the futures spread process in the *market* measure for both models. For the one factor model the futures spread with a fixed maturity date  $T$  follows

$$dF(t, T) = \lambda e^{-k(T-t)} dt + e^{-k(T-t)} \sigma d\mathbf{W}. \quad (28)$$

For the two factor model (28) becomes

$$\begin{aligned}
d\mathbf{F}(t, T) = & \lambda e^{-k(T-t)} dt + \frac{k\lambda_2}{k-k_2} [e^{-k_2(T-t)} - e^{-k(T-t)}] dt + e^{-k(T-t)} \sigma d\mathbf{W} \\
& + \frac{k}{k-k_2} [e^{-k_2(T-t)} - e^{-k(T-t)}] \sigma_2 d\mathbf{W}_2.
\end{aligned} \tag{29}$$

From (28) and (29) the futures spreads with fixed maturity *date* are *not* mean-reverting in the market measure.

Although the models presented above are for *spot* spreads, it is not easy to observe directly the spot prices of a commodity and investors typically use the nearest maturity futures price to represent the spot price (Clewlow and Strickland, 1999). But since the futures spread with fixed maturity date is not mean-reverting for our models this makes it difficult to estimate the mean-reversion parameter of the spot spread. However we will now show that the futures spread with *constant time to maturity*  $\tau := T - t$  is mean-reverting in our models which can be used to determine the mean-reversion speed of the spot spread.

Using Ito's Lemma we find (see Appendix 2) that the process for the futures spread with a constant time to maturity can be specified in the market measure as follows.

One factor model:

$$d\mathbf{F}(t, t + \tau) = k[\theta + \frac{\lambda e^{-k\tau}}{k} - F(t, t + \tau)] dt + e^{-k\tau} \sigma d\mathbf{W}. \tag{30}$$

Two factor model:

$$d\mathbf{F}(t, t + \tau) = k[\theta + y_t e^{-k\tau} + \frac{\lambda e^{-k\tau}}{k} + \phi \lambda_2 - F(t, t + \tau)] dt + \sigma_3 d\mathbf{W}, \tag{31}$$

where  $\phi := \frac{e^{-k_2\tau} - e^{-k\tau}}{k - k_2}$  and  $\sigma_3 := \sqrt{e^{-2k\tau} \sigma^2 + k^2 \theta^2 \sigma_2^2 + 2\rho k \theta e^{-k\tau} \sigma \sigma_2}$  are constants.

From (30) and (31) a future spread with constant time to maturity is mean-reverting in both the one and two factor models with the *same* mean-reversion speeds as the spot process. Note that when  $\tau \rightarrow 0$ , (30) and (31) converge to (28) and (29) respectively.

### 3.4 Calibration



In energy markets contract maturities are usually ordered by monthly date. For example, crude oil futures contracts traded on the NYMEX are ordered by 30 consecutive future months and then by quarter up to 7 years. In order to check the mean-reverting property of the spot spread we can use the 1month (time to maturity) future spread with a monthly observation interval for data.

We use *maximum-likelihood estimation* (MLE) on the panel data of futures spread curves in the spirit of Chen and Scott (1993) and Pearson and Sun (1994). This method is commonly used in fixed income yield curve modelling (e.g. Duffie and Singleton, 1997; Dai, Singleton and Yang, 2006). Recently this technique has also been used in the estimation of convenience yield curve models (Casassus and Collin-Dufresne, 2005). Since the state variables are not directly observed in our data set, the Chen and Scott approach specifies these latent variables by solving expressions for some securities which are arbitrarily assumed to be priced without error in the market. The remaining securities are assumed to be priced with measurement errors. To illustrate the method let  $F(t, T_1)$  to  $F(t, T_5)$  represent 5 futures spreads available to determine the model parameters. For the one factor model we suppose the first futures spread at time  $t$  is observed without pricing error but  $F(t, T_2)$  to  $F(t, T_5)$  are priced with errors. Thus the model estimation equations are

$$\begin{aligned}
F(t, T_1) &= C_1 + D_1 x_t \\
F(t, T_2) &= C_2 + D_2 x_t + \mathbf{u}_{2t} \\
F(t, T_3) &= C_3 + D_3 x_t + \mathbf{u}_{3t} \\
F(t, T_4) &= C_4 + D_4 x_t + \mathbf{u}_{4t} \\
F(t, T_5) &= C_5 + D_5 x_t + \mathbf{u}_{5t},
\end{aligned} \tag{32}$$

where  $C_i := \theta[1 - e^{-k(T_i - t)}]$ ,  $D_i := e^{-k(T_i - t)}$  and  $\mathbf{u}_{2t}$  to  $\mathbf{u}_{5t}$  are joint normally distributed pricing errors. The *log-likelihood* function for all futures spreads at time  $t$  is given by

$$L_t := -\ln D_1 + \ln L_t^s + \ln L_t^e, \tag{33}$$

where  $\ln L_t^s$  is the log likelihood of the state variable  $x$  (taken as the one month future spread  $F(t, T_1)$ ) at time  $t$  and  $\ln L_t^e$  is the log likelihood of the other securities  $F(t, T_2)$  to  $F(t, T_5)$  with

$$\begin{aligned}
\ln L_t^s &:= -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln(V_s) - \frac{1}{2} \frac{(x_t - x_m)^2}{V_s} \\
V_s &:= \frac{1 - e^{-2k\Delta t}}{2k} \sigma^2 \\
x_m &:= x_{t-1} e^{-k\Delta t} + \left(\theta + \frac{\lambda}{k}\right) (1 - e^{-k\Delta t}) \\
L_t^e &:= -\frac{4}{2} \ln(2\pi) - \frac{1}{2} \ln(|\Omega|) - \frac{1}{2} u_t' \Omega^{-1} u_t
\end{aligned} \tag{34}$$

and  $\Delta t$  denotes the (1 month) observation interval. In (33)  $D_1$  is the coefficient  $e^{-k(T_i-t)}$  in the affine transformation (16) from  $x_t$  to  $F(t, T_i)$  and thus the Jacobian of this transformation is  $1/D_1$ . Since the first one month futures spread is priced without error its log-likelihood is determined by the log-likelihood of the state variable  $\ln L_t^s$  adjusted by the Jacobian multiplier  $1/D_1$ . In (34)  $V_s$  is the variance of the state variable conditional on  $x_{t-1}$ ,  $x_m$  is the mean of  $x_t$  conditional on  $x_{t-1}$  and  $\Omega$  is the covariance matrix for  $\mathbf{u}_t$ . The *total* log likelihood is  $\sum_t L_t$ , which is maximized to determine the parameters of the one factor model.

For the two factor model the corresponding expressions are

$$\begin{aligned}
F(t, T_1) &= C_1 + D_1 x_t + E_1 y_t \\
F(t, T_2) &= C_2 + D_2 x_t + E_2 y_t \\
F(t, T_3) &= C_3 + D_3 x_t + E_3 y_t + \mathbf{u}_{3t} \\
F(t, T_4) &= C_4 + D_4 x_t + E_4 y_t + \mathbf{u}_{4t} \\
F(t, T_5) &= C_5 + D_5 x_t + E_5 y_t + \mathbf{u}_{5t},
\end{aligned} \tag{35}$$

where  $C_i := \theta[1 - e^{-k(T_i-t)}]$ ,  $D_i := e^{-k(T_i-t)}$  and  $E_i := \frac{k}{k-k_2} [e^{-k_2(T_i-t)} - e^{-k(T_i-t)}]$ .

Defining  $J := \begin{bmatrix} D_1 & E_1 \\ D_2 & E_2 \end{bmatrix}$  the log-likelihood for the two factor model is

$$L_t := -\ln |J| + \ln L_t^s + \ln L_t^e, \tag{36}$$

where

$$\begin{aligned}
\ln L_t^s &:= -\ln(2\pi) - \frac{1}{2} \ln(V_x) - \frac{1}{2} \ln(V_y) - \frac{1}{2} \frac{(x_t - x_m)^2}{V_x} - \frac{1}{2} \frac{(y_t - y_m)^2}{V_y} \\
V_x &:= \frac{1 - e^{-2k\Delta t}}{2k} \sigma^2 + \left\{ \frac{1}{2k_2} [1 - e^{-2k_2\Delta t}] + \frac{1}{2k} [1 - e^{-2k\Delta t}] - \frac{2}{k + k_2} [1 - e^{-(k+k_2)\Delta t}] \right\} \frac{k^2 \sigma_2^2}{(k - k_2)^2} \\
&\quad + \frac{2\rho k \sigma_2 \sigma}{k - k_2} \left[ \frac{1}{k + k_2} (1 - e^{-(k+k_2)\Delta t}) - \frac{1}{2k} (1 - e^{-2k\Delta t}) \right] \\
V_y &:= \sigma_2^2 \left( \frac{1 - e^{-2k_2\Delta t}}{2k_2} \right) \\
x_m &:= x_{t-1} e^{-k\Delta t} + y_{t-1} \frac{k}{k - k_2} (e^{-k_2\Delta t} - e^{-k\Delta t}) + \frac{\lambda_2}{k_2} (1 - e^{-k\Delta t}) - \frac{\lambda_2 k}{k_2 (k - k_2)} (e^{-k_2\Delta t} - e^{-k\Delta t}) \\
&\quad + \left( \theta + \frac{\lambda}{k} \right) (1 - e^{-k\Delta t}) \\
y_m &:= e^{-k_2\Delta t} y_{t-1} + \frac{\lambda_2}{k_2} (1 - e^{-k_2\Delta t}) \\
L_t^e &:= -\frac{3}{2} \ln(2\pi) - \frac{1}{2} \ln(|\Omega|) - \frac{1}{2} u_t' \Omega^{-1} u_t. \tag{37}
\end{aligned}$$

In preliminary study we found that the estimated correlation  $\rho$  between the long end and the short end was insignificant. This makes economic sense in that the long end movements are slow and driven by fundamentals while the short-term movements which are random, fast and driven by market trading activities. That innovations in the long and short runs should be uncorrelated has been used to analyze the long-run and short-run components of stock prices (Fama and French, 1988). Routledge, Seppi and Spatt (2000) assert that the long run movements of commodity futures prices should have zero correlation with the short-run movements because the physical inventory can regenerate or renew in ‘stock out’ periods (Corollary 1.2, p.1304). Thus if the times to maturity of the futures data are long enough these correlation estimates should be zero. But due to data availability these estimates may not be estimated as insignificantly different from zero and then the first risk factor does not absorb enough of the short-term price movements (Schwartz and Smith, 2000). In our two factor model we will assume the correlation  $\rho$  to be zero.

#### 4. Spread option pricing and hedging

If the underlying asset price follows a Gaussian process the European *call* and *put*

prices with maturity T on this asset can be calculated respectively as

$$c = B \frac{b_s}{\sqrt{2\pi}} \exp\left[-\frac{(a_s - K)^2}{2b_s^2}\right] + B(a_s - K)\Phi\left(\frac{a_s - K}{b_s}\right) \quad (38)$$

$$p = B \frac{b_s}{\sqrt{2\pi}} \exp\left[-\frac{(a_s - K)^2}{2b_s^2}\right] - B(a_s - K)\Phi\left(-\frac{a_s - K}{b_s}\right), \quad (39)$$

where B is the price of a discount bond,  $a_s$  and  $b_s$  are respectively the mean and standard deviation of the underlying at maturity, and K is the strike price of the option (see Appendix 3). We have seen that the spread distribution at time T follows a normal distribution in both one factor and two factor models so that equations (38) and (39) can be used to price the spread option.

#### **4.1 Pricing within futures price maturities**

##### *Options on the spot spread*

Since the future spread is the expectation of the future spot spread in the risk-neutral measure the mean of the underlying asset at option maturity T can be obtained at current time t as

$$a_s = F(t, T). \quad (40)$$

Equations (14) and (21) show  $b_s$  for the one and two factor models respectively as constants given an initial time t and a fixed maturity date T. As spread option values with maturity T depend on F(t,T) through (40) investors can utilize futures of the same maturity date to hedge.

The *deltas* of calls and puts on the spread are given respectively by

$$\Delta_c = \frac{\partial c}{\partial a_s} = B\Phi\left(\frac{a_s - K}{b_s}\right) \quad (41)$$

$$\Delta_p = \frac{\partial p}{\partial a_s} = -B\Phi\left(-\frac{a_s - K}{b_s}\right). \quad (42)$$

Since a spread can be seen as long one asset and short the other simultaneously the delta hedge yields an *equal volume hedge*, i.e. long and short the same value of

commodity futures contracts. No matter how many factors are deemed to drive the futures price, only the corresponding maturity futures contracts are utilized to hedge the spread option providing they are available in the market.

*Option on the futures spread*

Define the option maturity as R, the futures maturity as T>R and the current time as t. The futures spread is a martingale so that its mean in the risk neutral measure is

$$a_s = E_Q[\mathbf{F}(R, T)] = F(t, T). \quad (43)$$

Its standard deviation in the one factor model is

$$b_s = \sqrt{\frac{e^{-2k(T-R)} - e^{-2k(T-t)}}{2k}} \sigma \quad (44)$$

and in the two factor model is

$$b_s = \sqrt{A_1^F + A_2^F + 2\rho A_3^F}, \quad (45)$$

where

$$\begin{aligned} A_1^F &= \frac{\sigma^2}{2k} [e^{-2k(T-R)} - e^{-2k(T-t)}] \\ A_2^F &= \frac{k^2 \sigma_2^2}{(k-k_2)^2} \left\{ \frac{1}{2k_2} [e^{-2k_2(T-R)} - e^{-2k_2(T-t)}] + \frac{1}{2k} [e^{-2k(T-R)} - e^{-2k(T-t)}] \right. \\ &\quad \left. - \frac{2}{(k+k_2)} [e^{-(k_2+k)(T-R)} - e^{-(k_2+k)(T-t)}] \right\} \\ A_3^F &= \frac{k\sigma\sigma_2}{k-k_2} \left\{ \frac{1}{k+k_2} [e^{-(k+k_2)(T-R)} - e^{-(k+k_2)(T-t)}] - \frac{1}{2k} [e^{-2k(T-R)} - e^{-2k(T-t)}] \right\}. \end{aligned} \quad (46)$$

Since R and T are known using (38) and (39)  $b_s$  is a constant in both models and (38) and (39) can be used to price call and put options on the spread at t. Again investors can use futures contracts with maturity T to hedge these options positions with equal volume hedges given by (41) and (42) respectively. The hedge-ratio difference between hedging a spot and a futures spread option arises from the difference in the  $b_s$

values (compare (14) and (21) with (44) and (45)).

If an option (on the spot spread) has a maturity longer than the corresponding futures traded in the markets (which is common for many real options), investors cannot use these methods to price and hedge the option. We study this situation next.

#### 4.2 Pricing beyond futures price maturities

Suppose at time  $t$  we know the value of the state variable  $x_t$  in the one factor model or  $x_t$  and  $y_t$  in the two factor model, then we can forecast the *mean spread* at time  $T > t$  using these state variables.

For the one factor model

$$a_s = x_t e^{-k(T-t)} + \theta [1 - e^{-k(T-t)}] \quad (47)$$

and for the two factor model

$$a_s = x_t e^{-k(T-t)} + \theta [1 - e^{-k(T-t)}] + \frac{y_t k}{k - k_2} [e^{-k_2(T-t)} - e^{-k(T-t)}]. \quad (48)$$

The standard deviations  $b_s$  will be the same as those in previous section.

If no futures contracts of long enough maturity are available, investors must use several short term futures to hedge the long term option, i.e. they need to hedge the individual factors underlying the futures contracts. There is quite a large literature on how to use short-term futures to hedge long-term ones, e.g. Brennan and Crew (1997), Neuberger (1998) and Hilliard (1999).

For the one factor model the *call delta* on the latent *spot spread* is

$$\Delta_x = \frac{\partial c}{\partial a_s} \frac{\partial a_s}{\partial x} = \Delta_c e^{-k(T-t)}, \quad (49)$$

where  $c$  is the price of a call option,  $\Delta_c$  is given by (41) and  $T$  is the option maturity. If the futures spread  $F(t, T_1)$  is utilized to hedge this option position the *hedge ratio* is

$$\Delta_F = \frac{\partial c}{\partial x} \frac{\partial x}{\partial F} = \Delta_c e^{-k(T-t)} e^{k(T_1-t)} = \Delta_c e^{-k(T-T_1)}. \quad (50)$$

Similarly the delta for puts on the spot spread is

$$\Delta_F = \frac{\partial p}{\partial x} \frac{\partial x}{\partial F} = \Delta_p e^{-k(T-t)} e^{k(T_1-t)} = \Delta_p e^{-k(T-T_1)}, \quad (51)$$

where  $\Delta_p$  is given by (42).

Applying the two factor model, investors must use two shorter term futures –  $F(t, T_1)$  and  $F(t, T_2)$  – to hedge long term options. Ideally  $T_1$  should be short (e.g. 1 month) and  $T_2$  the longest futures maturity available in the market. The call deltas on the latent two factors  $x$  and  $y$  are respectively

$$\Delta_x = \frac{\partial c}{\partial a_s} \frac{\partial a_s}{\partial x} = \Delta_c e^{-k(T-t)} \quad (52)$$

and

$$\Delta_y = \frac{\partial c}{\partial a_s} \frac{\partial a_s}{\partial y} = \Delta_c \times \frac{k}{k - k_2} [e^{-k_2(T-t)} - e^{-k(T-t)}]. \quad (53)$$

Suppose the futures spreads  $F(t, T_1)$  and  $F(t, T_2)$  are utilized to hedge the option position. Then to obtain the *delta neutral* hedge ratios  $n_1$  and  $n_2$  one must solve

$$\begin{aligned} \Delta_x + n_1 e^{-k(T_1-t)} + n_2 e^{-k(T_2-t)} &= 0 \\ \Delta_y + n_1 \frac{k}{k - k_2} [e^{-k_2(T_1-t)} - e^{-k(T_1-t)}] + n_2 \frac{k}{k - k_2} [e^{-k_2(T_2-t)} - e^{-k(T_2-t)}] &= 0. \end{aligned} \quad (54)$$

Since this paper focusses on ‘long’ term option valuation and hedging we should discuss the option maturities appropriate for the use of our valuation models. The answer is related to the mean-reversion speed  $k$ . The *decay half life*  $\ln 2/k$  of the mean-reverting spread process can be used to represent its mean-reversion strength. We propose that if an option time to maturity is longer than this half decay time, our methodology (both one factor and two factor models) is appropriate to evaluate the spread option. Also, we expect that the traditional spread option model (1) will over-value these longer term options because of the variance blow-up phenomenon previously discussed. Mbanefo (1997) noted that long-term spread options (longer

than 90 days) will be overvalued if mean-reversion of the spreads is not considered.

We note that jumps and stochastic volatility are not important in determining theoretical or empirical *long term* spread option prices (Bates, 1996; Pan, 2003) which allows models of this paper to remain parsimonious.

## 5. Examples

### *5.1 Crack spread: Heating oil/ WTI crude oil (CSHC)<sup>4</sup>*

The crack spread between heating oil and WTI crude oil (heating oil crack spread) represents the profit from refining heating oil from crude oil, i.e. the price of heating oil minus the price of crude oil. We have seen in Section 2 that a relative deviation between the (equilibrium) input and output price relationship could exist for short periods of time, but a prolonged large deviation will lead to the production of more end products until the output and input prices are nearer the long-term equilibrium relationship. Thus we expect the heating oil crack spread to be mean-reverting.

#### *Data*

The data for modelling crack spreads consist of NYMEX daily futures prices of WTI crude oil (CL) and heating oil (HO) from January 1984 to January 2005. The time to maturity of these futures ranges from 1 month to more than 2 years. In order to test for unit roots, a single monthly data point is collected on the first day of each month by taking the price of the futures contract with one month time to maturity. For example, if the trading day is 1 February 2000 then the futures contract taken for the time series is the 1 March 2000 contract.

We also create a long-end crack spread with time to maturity 1 year. The methodology is exactly the same as with the 1 month time series, but due to data unavailability we only use data from January 1989 to January 2005 to construct the long-end crack spread.

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<sup>4</sup> Abbreviations used in this section are NYMEX trading codes.



To calibrate the one-factor and two-factor spread models using (32) and (35), we calculate the monthly futures spreads with 5 futures contracts from January 1989 to January 2005. The time step  $\Delta t$  is chosen to be 1 month and the contracts chosen are 1 month, 6 month, 9 month, 12 month and 15 month time to maturity futures spreads.

#### *Unit root and cointegration tests*

First, we conduct the ADF test on the heating oil and crude oil prices using the longer time series. As noted in Section 3.4 we can test the mean-reversion of the spot spread by examining the 1 month futures spreads. Figure 1 shows the 1 month futures prices of crude oil and heating oil.

**Insert Figure 1 about here**

Table 1 shows the results of the ADF test of Section 2.1 on 1 month futures of crude oil and heating oil arising from estimating (5). In order to reject the hypothesis that a time series has a unit root the coefficient  $\delta_i$  must be significantly negative.

**Insert Table 1 about here**

From Table 1, we cannot reject the hypothesis that both crude oil and heating oil are unit-root time series (*cf.* Girma and Paulson (1999) and Alexander (1999)).

Next we test the spread time series directly by estimating (5) to find a very strong mean-reverting speed significant at the 1% level which suggests that cointegration does exist in the data. In other words, the mean-reversion of the spread does not appear to be caused by the separate mean-reversion of the heating and crude oil prices but by the long-run equilibrium (cointegration) between them which must be considered in long-run derivatives pricing of the crack spread. This agrees with Girma and Paulson (1999).

Historical spreads only offer us their market-measure characteristics but a regression relating the short term and long term spreads can give us the risk neutral spreads. We estimate the regression equation (8) using the 1 year crack spread as the long-end

futures spread and the 1 month crack spread as the short-end futures spread.

The results are given in Table 2 and the 1 year and 1 month crack spreads depicted in Figure 2.

**Insert Table 2 about here**

In Table 2 the estimate of  $\gamma$  is significantly less than zero. Since this test investigates the mean-reverting property of spreads in the risk-neutral measure, we conclude that an Ornstein-Uhlenbeck process is appropriate to model the spot process. Thus both the *ex ante* and *ex post* tests give evidence that the spot crack spread is mean-reverting in both the market and risk neutral measures.

**Insert Figure 2 about here**

#### *Model calibration*

In order to calibrate the (one-factor and two-factor) models we use the full data set and employ equations (32) and (35). From preliminary study we noticed that the heating oil price shows a seasonal pattern which is inherited in the crack spread. To eliminate the influence of seasonality, we used an equally weighted portfolio of 1 month and 6 month futures (opposite seasons) to determine the x factor and a similar portfolio of 9 month and 15 month futures to determine the y factor.

We then performed a MLE optimization to calibrate the one factor model. Table 3 lists the results. One can see that the standard deviations of the  $\theta$ ,  $\sigma$  and  $k$  estimates are quite small, i.e.  $\theta$ ,  $k$  and  $\sigma$  can be determined quite precisely, unlike the  $\lambda$  estimate. However we do not need the market prices of risk  $\lambda$  as input to option pricing. The asymptotic estimate of the standard deviation of the spread (when  $s$  goes to infinity in (14)) is \$2.03. Figure 3 shows the one month spread and the latent spot spread, which are very close to each other.

**Insert Table 3 about here**

**Insert Figure 3 about here**

**Insert Table 4 about here**

Similar to the situation in the one factor model, the market prices of risk  $\lambda$  and  $\lambda_2$  cannot be precisely estimated in the two factor model, but estimates of all the other parameters ( $\sigma$ ,  $\sigma_2$ ,  $k$ ,  $k_2$  and  $\theta$ ) can. The asymptotic estimate of the standard deviation of the spread is \$2.65, which is higher than that for the one factor model. This is easy to understand, since the one factor model only counts the short-term variance of the spread, while the two factor model takes account of both the long end (fundamentals) and the short end (trading activities). Also, assuming zero correlation between the two factors, we can examine the ratio of the long-end and the short-end variance –  $A_1:A_2$  in (20) – as time goes to infinity, which is about 1:1 in this example. Thus the asymptotic variance of the crack spread is nearly equally contributed by short-end (first factor) and long-end (second factor) movements. Since this ratio is quite high the second factor is obviously important in derivatives pricing and should not be omitted. Since the one factor model is nested in the two factor model by taking  $\sigma_2$ ,  $\lambda_2$  and  $y_v$  (the starting value of the  $y$  factor) to be zero, we can compare the differences in log-likelihood scores for each data set to see whether the additional parameters of the two-factor model provide a statistically significant improvement in that model's ability to explain the observed data. The relevant test statistic for this comparison is the chi-squared *likelihood ratio* test (Hamilton, 1994) with 3 degrees of freedom and the 99<sup>th</sup> percentile of this distribution is 11.34. Given that the log-likelihood scores increase by about 21, the improvements provided by the two factor model are quite significant.

**Insert Figure 4 about here**

Figure 4 shows the time evolution of the long and short factors with correlation coefficient 0.0056, which is not significant and thus consistent with the assumption that the correlation between the two factors is zero.

*Option valuation on the spot spread*

The decay half life is about half a year for the one factor model so that when valuing an option longer than half a year the methodology (using one factor or two factor models) in this paper is appropriate.

On 3 January 2005 the HO06N (heating oil future with maturity July 2006) contract had a value 44.2 (\$/Barrel); on the same day the CL06N (crude oil future with maturity July 2006) traded at 39.78 (\$/Barrel). By using the parameters in Tables 3 and 4, Table 5 gives the European option values on the *spot* crack spread with maturity July 2006 on this date. For comparison we also calculate option values from a model which ignores the cointegration effect. We can use the Black (1976) driftless GBM model to simulate both crude and heating oil futures price paths, and thus calculate the spread option value<sup>5</sup>. The average correlation coefficient (over a 20 year period) is 0.89 between heating and crude oil. We use call option values to compare the different models; it is then easy to obtain the put values by put-call parity.

**Insert Table 5 about here**

**Insert Table 6 about here**

From Table 5 we can see that the option value from the one factor model is typically smaller than that from the two-factor model and the latter is much smaller than that from the Black model. Since the Black model does not consider mean-reversion (the cointegration) of the spread, its spread distribution at maturity is wider than that of a cointegrated model and thus yields a larger option value. Put simply, a non-cointegrated model ignores the long-run equilibrium between crude and heating oil prices and thus over-prices the option. Since the two factor model accounts for long-term spread movement, it should yield a wider spread distribution at maturity and thus has a larger option value than the one factor model. In Table 6, both the one factor and two factor models yield an equal volume hedge but the Black model does not. As is well known, the less disperse the underlying terminal distribution, the more

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<sup>5</sup> Since there is no analytical solution when the strike is not zero one convenient way to calculate the option value is by Monte Carlo path simulation.

sensitive the option deltas are to the strike prices<sup>6</sup>. Thus the one-factor model yields the most sensitive deltas and the Black model has the least sensitive deltas among the three models.

## ***5.2 Location spread: Brent / WTI crude oil (LSBW)***

We define the LSBW location spread as the price of WTI crude oil (CL) minus the price of the Brent blend crude oil (ITCO). WTI is delivered in the USA and Brent in the UK.

### *Data*

NYMEX daily futures prices of WTI crude oil were described in the previous example. The daily Brent futures prices are from January 1993 to January 2005. The time to maturity of the Brent futures contracts range from 1 month to about 3 years. As in the previous example, monthly data is used to test for the unit root in Brent oil prices. We also create a monthly long-end LSBW spread with time to maturity of 1 year ranging from January 1993 to January 2005. In order to calibrate the one-factor and two-factor models, we calculate the monthly futures spread with 5 maturities from January 1993 to January 2005 at monthly intervals, i.e. the time step  $\Delta t$  in (30) and (31) is 1 month. The 5 contracts involved are 1 month, 3 month, 6 month, 9 month and 12 month futures spreads.

### *Unit root and cointegration tests*

As from the previous example we know that the WTI crude oil price follows a unit root process, in this example we need only conduct the ADF test on Brent crude oil prices. Figure 5 shows the 1 month futures prices of WTI crude oil and Brent blend. We again take the 1 month futures prices as representative of the spot price.

**Insert Figure 5 about here**

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<sup>6</sup> The *sensitivity* is defined as the ratio of the change of the deltas in the change of the strike prices.

Table 7 shows the results of ADF test estimating (5) on the 1 month futures of Brent blend. Similar to WTI crude oil, the Brent blend price is also a unit root process (since  $\delta_1$  is positive), but the LSBW location spread appears to be a mean-reverting process. This again suggests the existence of a long-run equilibrium in the data. To estimate future expectations of the spread, we estimate the regression equation (8). As in the previous example, we use the 1 year and 1 month LSBW futures spreads. The results are listed in Table 8. The 1 year and 1 month LSBW spread evolution is depicted in Figure 6.

**Insert Table 7 about here**

**Insert Figure 6 about here**

**Insert Table 8 about here**

From Table 8 we see that the estimate of  $\gamma$  is strongly negative, so that the market appears to expect the spot LSBW spread to be mean-reverting in the risk-neutral measure. Hence both the *ex ante* and *ex post* analyses support that the spot LSBW spread is mean-reverting so that mean-reversion should be accounted for in option pricing.

#### *Model Calibration*

We did not find evidence of seasonality in the LSBW spread. Hence we use the 1 month futures spread to back out the latent spot spread factor for the one factor model. For the two factor model we use the 1 month futures spread to estimate the short term  $x$  factor and an equally weighted portfolio consisting of the 9 month and 12 month futures spreads to estimate the long term  $y$  factor.

Tables 9 and 10 list the calibration results for the one factor and two factor models. We see that the asymptotic standard deviation of the spread are estimated to be \$1.60 and \$3.90 respectively for the one and two factor models. The ratio of long-end to the short-end variance ( $A_1:A_2$  in (9)) is 5:1 in the two factor model, i.e. the long end (second factor) movement of the spread accounts for much more variance than (first

factor) short end variation. Similar to the previous example the estimate of the asymptotic standard deviation from the two factor model is higher than that from the one factor model. From the Chi-squared (likelihood ratio) test the two factor model is very significantly better than the one factor model in explaining the observed LSBW spread data. The latent spot spread factor and the two factors (x and y) are shown in Figures 7 and 8 respectively. The correlation between the two factors is 0.04 which is again consistent with our zero correlation assumption.

**Insert Table 9 about here**

**Insert Table 10 about here**

**Insert Figure 7 about here**

**Insert Figure 8 about here**

#### *Option valuation on the spot spread*

The decay half life is about six months from the one factor model. Thus to model an option longer than six months the methods (one factor or two factor models) in this paper should be used.

On the day of 1 December 2003 the ITCO6Z (Brent blend crude oil future with maturity December 2006) contract had a value 24.62 (\$/Barrel); on the same day the CL06Z (WTI crude oil future with maturity December 2006) traded at 25.69 (\$/Barrel). By using the estimated parameters in Tables 9 and 10, Table 11 shows the European option value on the spot spread with maturity December 2006. Since the option is on the spot spread the hedging futures maturities should be the same as the option maturity – December 2006. The Brent and WTI crude oil contract prices both follow unit root processes so we may simulate both prices to calculate the non-cointegrated Black model's spread option value. Note that the average correlation between the Brent blend and WTI crude oils is 0.96.

**Insert Table 11 about here**

Table 11 shows that, similar to the previous example, the option value of the one factor model is typically smaller than that of the two-factor model and the latter is much smaller than the Black model. As before by ignoring cointegration the Black model tends to over-value the long term option. We obtain a similar pattern of deltas as in the previous example (see Table 12) and the explanation for this remains the same.

**Insert Table 12 about here**

## **6. Conclusion**

In this paper we have developed spread option pricing models in which the two price legs of the spread are cointegrated. Since the cointegration relationship is important for the long-run relationship between the two prices spread option evaluation should take account of this relationship if the option maturity is long. Assuming a cointegration relationship between the two underlying assets, we model the spread process *directly* using the Ornstein-Uhlenbeck process, i.e. we model directly the dynamic deviation from the long-run equilibrium which cannot be specified correctly by modelling the two underlying assets separately. We first specify risk-neutral processes for the spread and then determine the market processes by assuming constant risk-premia. We also propose two methods (*ex ante* and *ex post*) to test for mean-reversion of the spread process. Finally we give analytical solutions for the spread option price and deltas. In order to illustrate the theory, we study two examples — options on crack and location spreads respectively. Both market spread processes are found to be mean-reverting, which implies that their two price legs are cointegrated. From likelihood ratio tests the two factor model is found to be significantly better than the one factor model in explaining the crack and location spread data. The option values and Greeks from our cointegration models are quite different from those of standard models but are consistent with the practical observations of Mbanefo (1997). We are currently working on Lévy process versions of our models which may be more appropriate to commodity markets such as gas or



electricity and for shorter term option maturities generally.

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## APPENDIX 1 Solution of the Two Factor Model

Let  $z_{1t} := e^{kt} x_t$ . Then

$$\begin{aligned} dz_1 &= e^{kt} dx_t + ke^{kt} x_t dt \\ &= e^{kt} k(\theta + y_t - x_t) dt + e^{kt} \sigma d\mathbf{W} + ke^{kt} x_t dt \\ &= (\theta + y_t) e^{kt} k dt + e^{kt} \sigma d\mathbf{W}. \end{aligned} \quad (1.1)$$

Thus, given the starting time  $v$  and starting position  $x_v$ , at time  $s$

$$e^{ks} \mathbf{x}_s - e^{kv} x_v = \int_v^s (\theta + y) e^{kt} k dt + \int_v^s e^{kt} \sigma d\mathbf{W}(t) \quad (1.2)$$

$$\mathbf{x}_s = x_v e^{-k(s-v)} + \theta [1 - e^{-k(s-v)}] + e^{-ks} \int_v^s y e^{kt} k dt + e^{-ks} \int_v^s e^{kt} \sigma d\mathbf{W}(t). \quad (1.3)$$

Let  $z_{2t} := \exp(k_2 t) y$ . Then

$$\begin{aligned} dz_2 &= e^{k_2 t} dy + k_2 e^{k_2 t} y dt \\ &= e^{k_2 t} (-k_2 y) dt + e^{k_2 t} \sigma_2 d\mathbf{W}_2 + k_2 e^{k_2 t} y dt \\ &= e^{k_2 t} \sigma_2 d\mathbf{W}_2. \end{aligned} \quad (1.4)$$

Thus, given the starting time  $v$  and starting position  $x_v$ , at time  $t$

$$e^{k_2 t} \mathbf{y}_t - e^{k_2 v} y_v = \int_v^t e^{k_2 u} \sigma_2 d\mathbf{W}_2(u) \quad (1.5)$$

$$\mathbf{y}_t = y_v e^{-k_2(t-v)} + e^{-k_2 t} \int_v^t e^{k_2 u} \sigma_2 d\mathbf{W}_2(u) \quad (1.6)$$

Hence

$$\begin{aligned} \mathbf{x}_s &= x_v e^{-k(s-v)} + \theta [1 - e^{-k(s-v)}] + e^{-ks} \int_v^s y_t e^{kt} k dt + e^{-ks} \int_v^s e^{kt} \sigma d\mathbf{W}(t) \\ &= x_v e^{-k(s-v)} + \theta [1 - e^{-k(s-v)}] + ke^{-ks} \int_v^s y_t e^{-k_2(t-v)} e^{kt} dt + \\ &\quad e^{-ks} \int_v^s e^{kt} ke^{-k_2 t} \int_v^t e^{k_2 u} \sigma_2 d\mathbf{W}_2(u) dt + e^{-ks} \int_v^s e^{kt} \sigma d\mathbf{W}(t). \end{aligned} \quad (1.7)$$

By changing the integration order of the double integral in (1.7) we obtain

$$\begin{aligned} \mathbf{x}_s = & x_v e^{-k(s-v)} + \theta[1 - e^{-k(s-v)}] + \frac{y_v k}{k - k_2} [e^{-k_2(s-v)} - e^{-k(s-v)}] \\ & + \frac{k\sigma_2}{k - k_2} \int_v^s [e^{-k_2(s-u)} - e^{-k(s-u)}] d\mathbf{W}_2(u) + \int_v^s e^{-k(s-t)} \sigma d\mathbf{W}(t). \end{aligned} \quad (1.8)$$

Define:

$$\begin{aligned} A_1 &:= \frac{\sigma^2}{2k} [1 - e^{-2k(s-v)}] \\ A_2 &:= \left( \frac{1}{2k_2} [1 - e^{-2k_2(s-v)}] + \frac{1}{2k} [1 - e^{-2k(s-v)}] - \frac{2}{(k + k_2)} [1 - e^{-(k_2+k)(s-v)}] \right) \cdot \frac{k^2 \sigma_2^2}{(k - k_2)^2} \\ A_3 &:= \frac{k\sigma_2\sigma}{k - k_2} \left( \frac{1}{k + k_2} [1 - e^{-(k+k_2)(s-v)}] - \frac{1}{2k} [1 - e^{-2k(s-v)}] \right). \end{aligned} \quad (1.9)$$

Then the variance of  $\mathbf{x}_s$  at time  $s$  is  $A_1 + A_2 + 2\rho A_3$  and asymptotically  $\text{var}(\mathbf{x}_s)$

$$\text{becomes } \frac{\sigma^2}{2k} + \frac{\sigma_2^2}{2k_2(1 + k_2/k)} + \frac{\rho\sigma\sigma_2}{2(k + k_2)}, \quad (1.10)$$

a constant for  $k_2 \neq 0$ . This is because  $\mathbf{x}$  and  $\mathbf{y}$  are both mean-reverting processes.

Since  $E[\mathbf{x}_s | x_v] = F(v, s)$  holds in the risk neutral measure we have

$$F(v, s) = x_v e^{-k(s-v)} + \theta[1 - e^{-k(s-v)}] + \frac{y_v k}{k - k_2} [e^{-k_2(s-v)} - e^{-k(s-v)}]. \quad (1.11)$$

The risk neutral process for the futures spread  $F(t, T)$  follows

$$\begin{aligned} d\mathbf{F}(t, T) &= \frac{\partial F(t, T)}{\partial x_t} dx_t + \frac{\partial F(t, T)}{\partial y_t} dy_t + \frac{\partial F(t, T)}{\partial t} dt \\ &= e^{-k(T-t)} \sigma d\mathbf{W} + \frac{k}{k - k_2} [e^{-k_2(T-t)} - e^{-k(T-t)}] \sigma_2 d\mathbf{W}_2. \end{aligned} \quad (1.12)$$

## APPENDIX 2 Futures Spread with Constant Time to Maturity

Define  $\tau$  as the constant time to maturity and consider the futures spread process in the market measure. First, we treat the one factor model and express the spread as a function of  $t$  and  $\tau$ .

From (16), it follows easily that

$$F(t, t + \tau; x_t) = x_t e^{-k\tau} + e^{-k(t+\tau)} \left( \int_t^{t+\tau} k\theta e^{ku} du \right). \quad (2.1)$$

Applying Ito's lemma yields

$$dF(t, t + \tau) = e^{-k\tau} dx_t - [k e^{-k(t+\tau)} \int_t^{t+\tau} k\theta e^{ku} du] dt + e^{-k(t+\tau)} [k\theta e^{k(t+\tau)} - k\theta e^{kt}] dt. \quad (2.2)$$

Substituting for the integral from (2.1) we obtain

$$\begin{aligned} dF(t, t + \tau) &= e^{-k\tau} [k(\theta - x_t) + \lambda] dt + e^{-k\tau} \sigma d\mathbf{W} - [kF(t, t + \tau) - kx_t e^{-k\tau}] dt \\ &\quad + e^{-k(t+\tau)} [k\theta e^{k(t+\tau)} - k\theta e^{kt}] dt \\ &= k \left[ \theta + \frac{\lambda e^{-k\tau}}{k} - F(t, t + \tau) \right] dt + e^{-k\tau} \sigma d\mathbf{W}. \end{aligned} \quad (2.3)$$

When  $\tau \rightarrow 0$ , (2.3) yields the corresponding expression for the *spot* spread process (26).

For the two-factor model, similarly we have

$$F(t, t + \tau) = x_t e^{-k\tau} + e^{-k(t+\tau)} \int_t^{t+\tau} \theta e^{ku} k du + \frac{y_t k}{k - k_2} [e^{-k_2\tau} - e^{-k\tau}]. \quad (2.4)$$

Defining  $\phi := \frac{e^{-k_2\tau} - e^{-k\tau}}{k - k_2}$

$$\begin{aligned} dF(t, t + \tau) &= e^{-k\tau} dx_t + k\phi dy_t - k^2 e^{-k(t+\tau)} \left[ \int_t^{t+\tau} \theta e^{ku} du \right] dt \\ &\quad + k e^{-k(t+\tau)} [\theta e^{k(t+\tau)} - \theta e^{kt}] dt. \end{aligned} \quad (2.5)$$

Substituting for the integral from (2.4) we obtain

$$\begin{aligned}
d\mathbf{F}(t, t + \tau) &= e^{-k\tau} [k(\theta + y_t - x_t) + \lambda] dt + e^{-k\tau} \sigma d\mathbf{W} + k\phi(-ky + \lambda_2) dt \\
&\quad + k\phi\sigma d\mathbf{W} - [kF(t, t + \tau) - kx_t e^{-k\tau} - \phi k^2 y] dt + k[\theta - \theta e^{-k\tau}] dt \\
&= k[\theta + y_t e^{-k\tau} + \frac{\lambda e^{-k\tau}}{k} + \phi\lambda_2 - F(t, t + \tau)] dt + e^{-k\tau} \sigma d\mathbf{W} + k\phi\sigma_2 d\mathbf{W}_2. \quad (2.6)
\end{aligned}$$

Defining  $\sigma_3 := \sqrt{e^{-2k\tau} \sigma^2 + k^2 \phi^2 \sigma_2^2 + 2\rho k \phi e^{-k\tau} \sigma \sigma_2}$  equation (2.6) can be rewritten as

$$d\mathbf{F}(t, t + \tau) = k[\theta + y e^{-k\tau} + \frac{\lambda e^{-k\tau}}{k} + \phi\lambda_2 - F(t, t + \tau)] dt + \sigma_3 d\mathbf{W}. \quad (2.7)$$

Since  $\phi$  is a constant so is  $\sigma_3$ .

Since  $\phi = 0$ , when  $\tau=0$  equation (2.7) becomes the spot spread process equation (28).

Thus, from equations (2.3) and (2.7) one can see that if the spot spread is mean-reverting, both one factor and two factor models for futures spreads with constant time to maturity  $\tau$  will be mean-reverting.



### APPENDIX 3 Option Pricing When Underlying Terminal Prices Follow a Normal Distribution

Suppose at option maturity T an asset (or portfolio) price follows a normal distribution with mean  $a_s$  and standard deviation  $b_s$ .

Thus, any *European call option* struck at  $K$  can be priced as

$$c = BE_Q[\max(x - K, 0)] = B \int_K^{\infty} (x - K) \cdot \frac{1}{\sqrt{2\pi}b_s} \exp\left[-\frac{(x - a_s)^2}{2b_s^2}\right] dx, \quad (3.1)$$

where  $B$  is the discount bond price over the time remaining to exercise and the expectation subscript  $Q$  denotes the risk-neutral measure. Letting  $y := x - a_s$  we obtain

$$\begin{aligned} c &= B \int_{K - a_s}^{\infty} (y - K + a_s) \cdot \frac{1}{\sqrt{2\pi}b_s} \exp\left[-\frac{y^2}{2b_s^2}\right] dy \\ &= B \int_{K - a_s}^{\infty} y \frac{1}{\sqrt{2\pi}b_s} \exp\left[-\frac{y^2}{2b_s^2}\right] dy + B(a_s - K) \int_{K - a_s}^{\infty} \frac{1}{\sqrt{2\pi}b_s} \exp\left[-\frac{y^2}{2b_s^2}\right] dy \\ &= B \frac{b_s}{\sqrt{2\pi}} \exp\left[-\frac{(a_s - K)^2}{2b_s^2}\right] + B(a_s - K) \Phi\left(\frac{a_s - K}{b_s}\right), \end{aligned} \quad (3.2)$$

where  $\Phi(v)$  denotes the value of the standard normal cumulative distribution function from  $-\infty$  to  $v$ . Brennan (1979) obtains a similar option pricing formula for a normally distributed underlying asset.

By using put call parity, we can obtain the corresponding *European put option* price as

$$\begin{aligned} p &= c - B(a_s - K) \\ &= B \frac{b_s}{\sqrt{2\pi}} \exp\left[-\frac{(a_s - K)^2}{2b_s^2}\right] - B(a_s - K) \Phi\left(-\frac{a_s - K}{b_s}\right). \end{aligned} \quad (3.3)$$

If the strike price  $K$  is 0, which corresponds to ‘better to buy’ options (Margrabe, 1978), the call and put prices are given respectively by

$$c = B \frac{b_s}{\sqrt{2\pi}} \exp\left[-\frac{a_s^2}{2b_s^2}\right] + Ba_s \Phi\left(\frac{a_s}{b_s}\right) \quad (3.4)$$

$$p = B \frac{b_s}{\sqrt{2\pi}} \exp\left[-\frac{a_s^2}{2b_s^2}\right] - Ba_s \Phi\left(-\frac{a_s}{b_s}\right). \quad (3.5)$$

Thus the *deltas* of the call and put options respectively are

$$\Delta_c = \frac{\partial c}{\partial a_s} = B\Phi\left(\frac{a_s - K}{b_s}\right) \quad (3.6)$$

$$\Delta_p = \frac{\partial p}{\partial a_s} = -B\Phi\left(-\frac{a_s - K}{b_s}\right) \quad (3.7)$$

and the *gamma* of both options is

$$\Gamma_c = \Gamma_p = \frac{B}{\sqrt{2\pi}b_s} \exp\left[-\frac{(a_s - K)^2}{b_s^2}\right]. \quad (3.8)$$

Other Greeks are easily obtained from (3.2) and (3.3).

## Tables and Figures

**Table 1. ADF test for crude oil, heating oil and the spread between them**

	No. of observations	Lag (p)	$\delta_I$ or $\chi_I$	t value
Crude Oil	257	6	0.0201	1.03
Heating Oil	257	6	0.0188	1.13
Crack spread	257	6	-0.31	-3.97*

\* significant at the 1% level

**Table 2. Regression (8) parameter estimates for the crack spread**

	$\zeta$	$\Gamma$
Value	2.1644	-0.5505
t-stat	15.6970*	-16.3852*
No. of observations	154	
R <sup>2</sup>	57.37%	

\* significant at the 1% level

**Table 3. Parameter estimates for the one factor model**

	$\sigma$	k	$\Lambda$	$\theta$
Value	3.4525	1.4397	-0.7016	3.7167
Standard deviation	0.074219	0.104345	1.5417	0.0448
Log likelihood	-489			

**Table 4. Parameter estimates for the two factor model**

	$\sigma$	k	$\lambda$	$\theta$	$\sigma_2$	k <sub>2</sub>	$\lambda_2$
Value	5.023	3.0167	-0.0414	3.3021	1.6131	0.4045	0.9362
Standard deviation	0.0817	0.103	1.4102	0.0776	0.2386	0.2018	0.5487
Log likelihood	-468						

**Note:** We assume the correlation  $\rho$  between the two factors is zero

**Table 5. Comparison of crack option values**

Strike (\$)	2	3	4	5	6
One factor spread model	2.4140	1.6251	0.9826	0.5214	0.2377
Two factor spread model	2.4933	1.7425	1.1242	0.6604	0.3488
Black model	4.3930	3.8137	3.3162	2.9674	2.5197

**Table 6. Comparison of crack option deltas**

Strike (\$)	2		3		4		5		6	
	HO	CL	HO	CL	HO	CL	HO	CL	HO	CL
One factor spread model	0.84	-0.84	0.72	-0.72	0.55	-0.55	0.37	-0.37	0.21	-0.21
Two factor spread model	0.81	-0.81	0.69	-0.69	0.54	-0.54	0.39	-0.39	0.24	-0.24
Black Model	0.64	-0.56	0.57	-0.53	0.55	-0.43	0.43	-0.42	0.41	-0.36

**Table 7. ADF test for Brent and WTI crude oil and the location spread**

	No. of observations	Lag (p)	$\delta_1$ or $\chi_1$	t stat
Brent blend	152	6	0.04	2.48
WTI	257	6	0.0201	1.03
Location spread	152	6	-0.28	-3.45*

\* significant at the 1% level

**Table 8. Regression (7) parameter estimates for the location spread**

	$\zeta$	$\Gamma$
Value	0.9188	-0.7022
t-stat	11.5721*	-15.5476*
No. of observations	152	
R <sup>2</sup>	63.65%	

\* significant at the 1% level

**Table 9. Parameter estimates for the one factor model**

	$\sigma$	k	$\Lambda$	$\theta$
Value	2.5724	1.2928	0.6497	1.1902
Standard deviation	0.1811	0.0987	0.8779	0.028
Log likelihood	64.9			

**Table 10. Parameter estimates for the two factor model**

	$\sigma$	k	$\Lambda$	$\theta$	$\sigma_2$	$k_2$	$\lambda_2$
Value	2.588	1.3088	0.6031	1.0282	1.3975	0.0728	0.0178
Standard deviation	0.1629	0.0745	0.7886	0.0318	0.1161	0.0321	0.1791
Log likelihood	124.6						

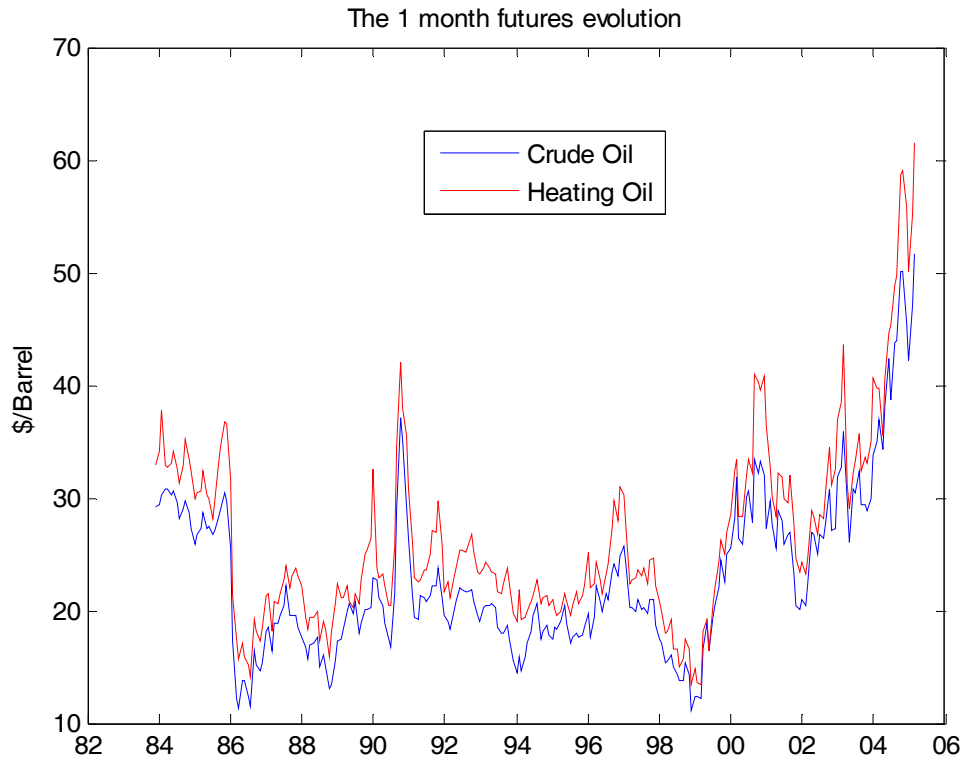
**Note:** We assume the correlation  $\rho$  between the two factors is zero

**Table 11. Comparison of spread option values**

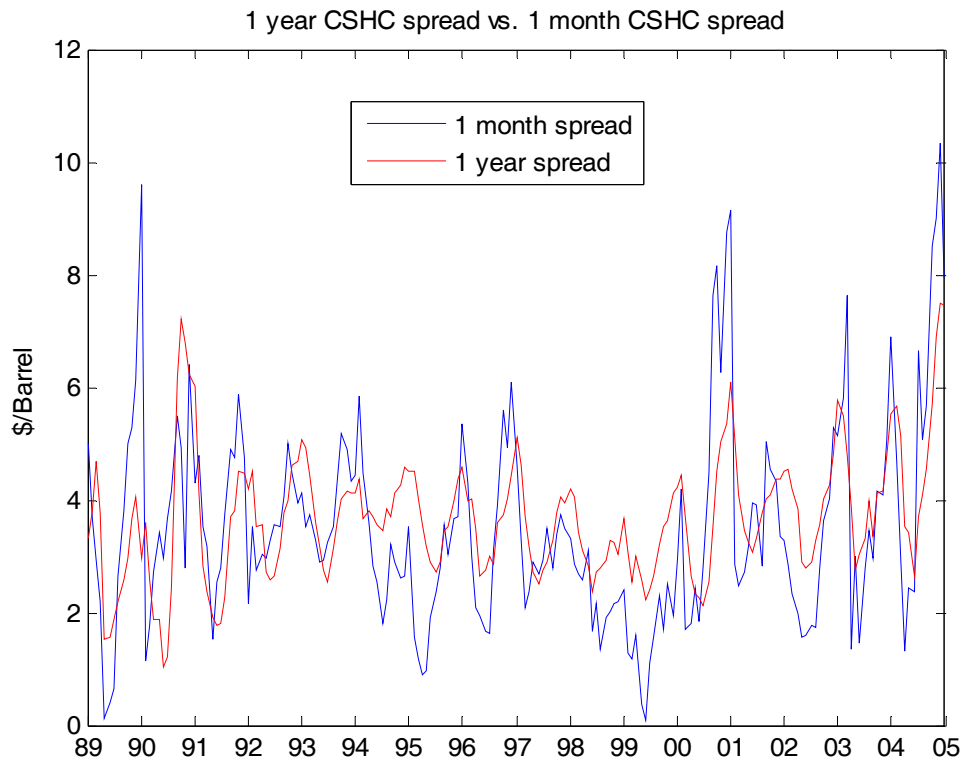
Strike (\$)	-1	0	1	2	3
One factor spread model	1.9592	1.1980	0.6157	0.2541	0.0809
Two factor spread model	2.1191	1.4383	0.8949	0.5034	0.2527
Black Model	2.5616	1.9821	1.5362	1.1041	0.8548

**Table 12. Comparison of spread option deltas**

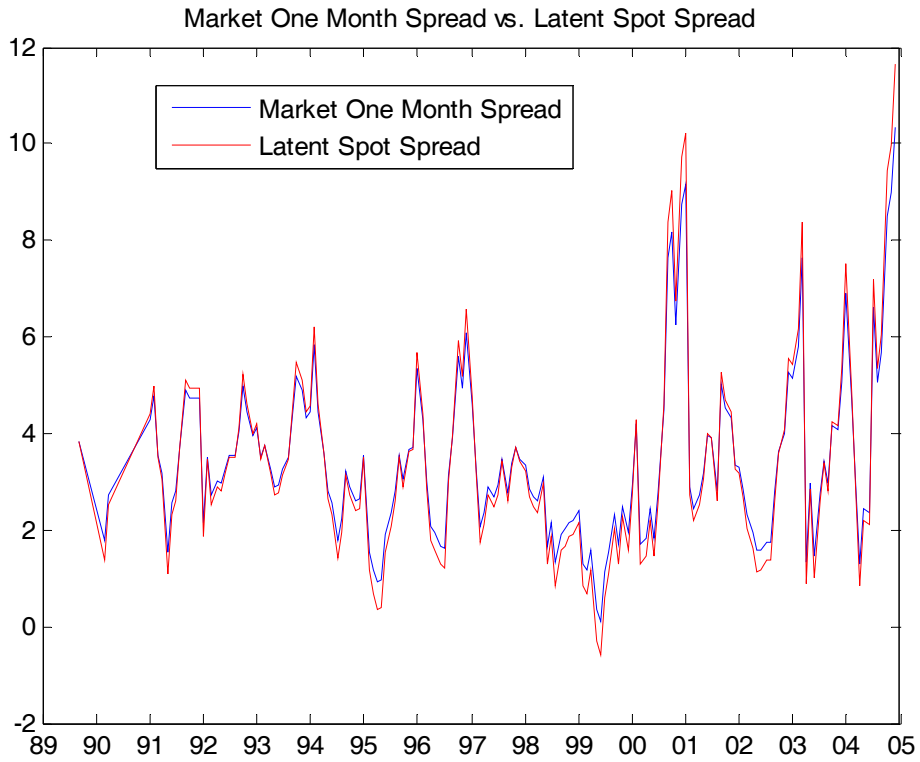
Strike (\$)	-1		0		1		2		3	
	CL	ITCO	CL	ITCO	CL	ITCO	CL	ITCO	CL	ITCO
Underlying										
One factor spread model	0.82	-0.82	0.68	-0.68	0.47	-0.47	0.26	-0.26	0.10	-0.10
Two factor spread model	0.74	-0.74	0.62	-0.62	0.47	-0.47	0.32	-0.32	0.19	-0.19
Black model	0.69	-0.63	0.55	-0.53	0.48	-0.43	0.42	-0.34	0.34	-0.30



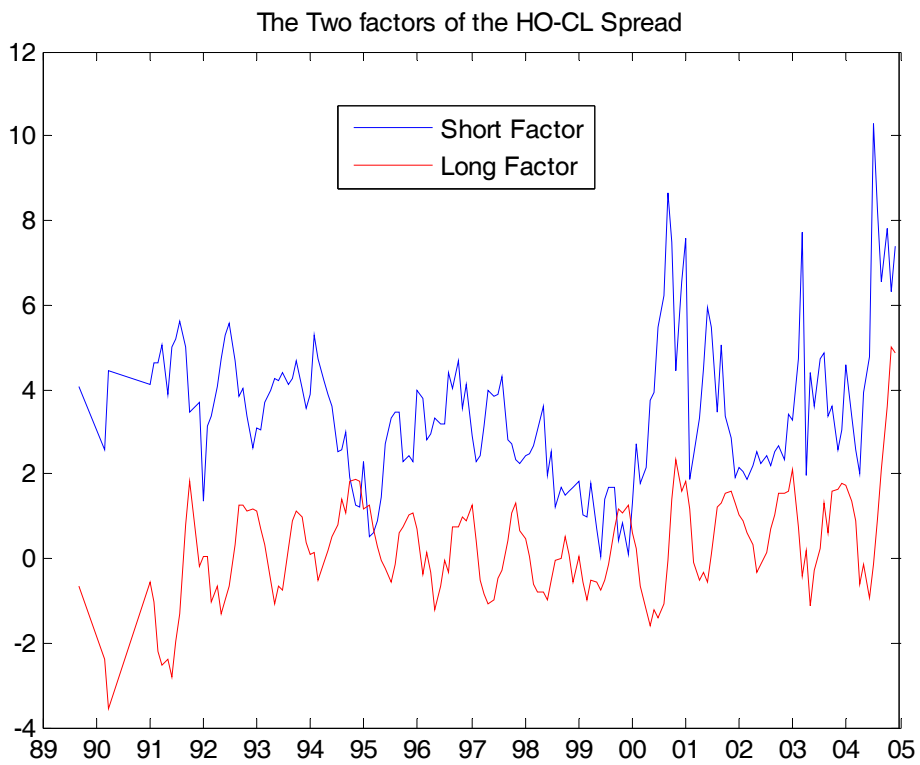
**Figure 1. The 1 month futures prices of crude oil and heating oil**



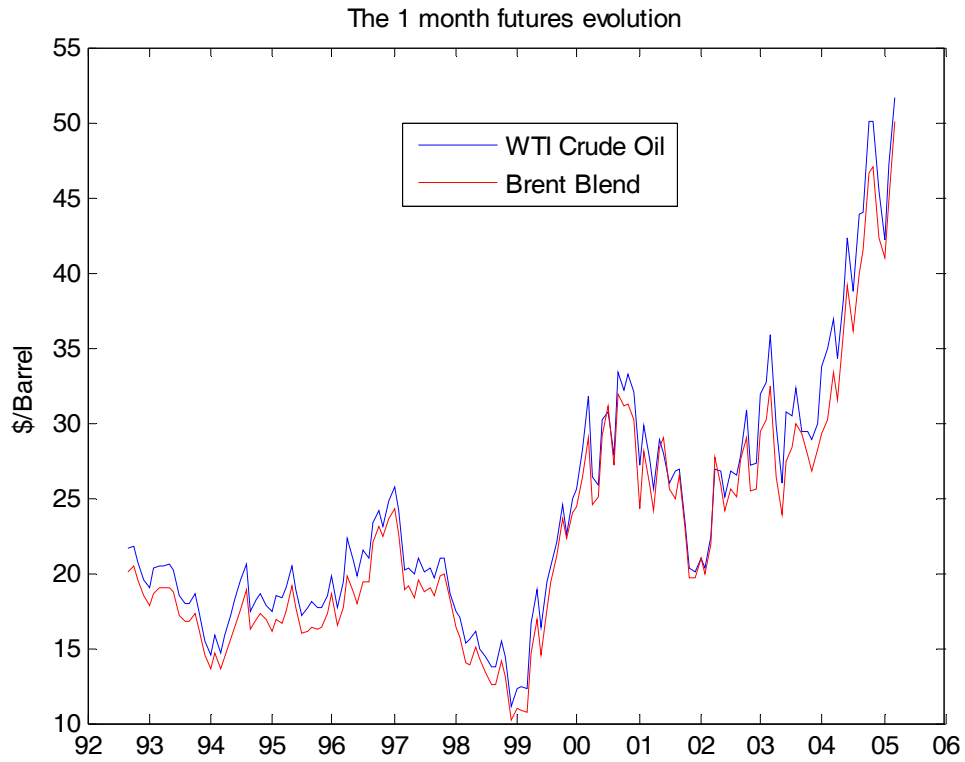
**Figure 2. The 1 month and 1 year crack spreads**



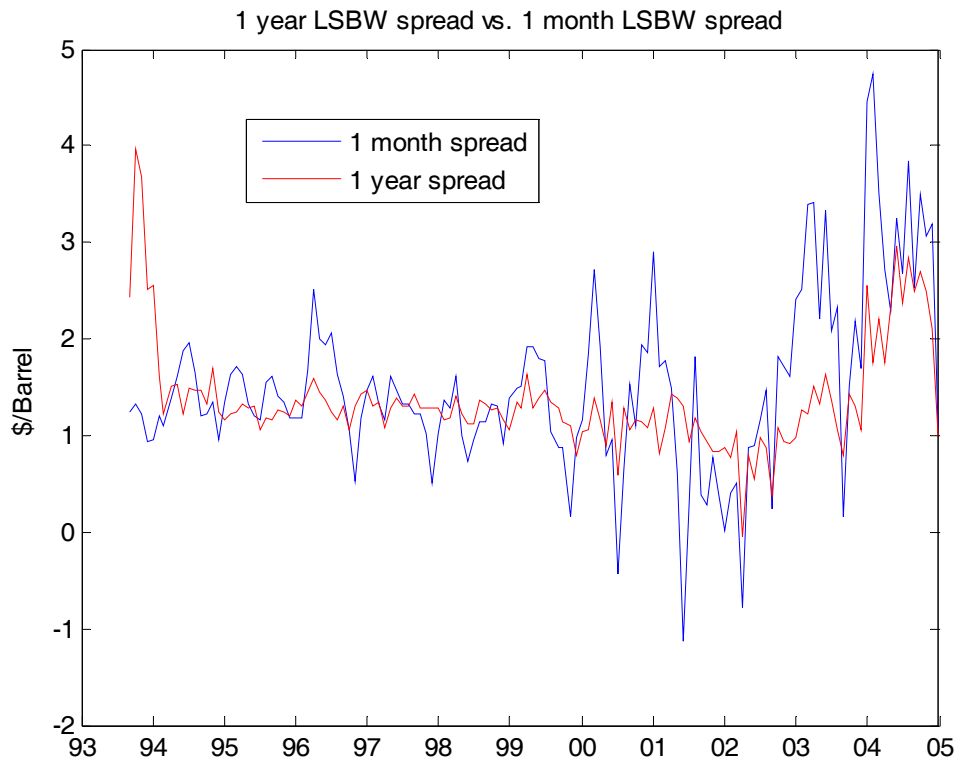
**Figure 3. The 1 month crack spread versus the latent spot spread**



**Figure 4. The two factors of the crack spread**

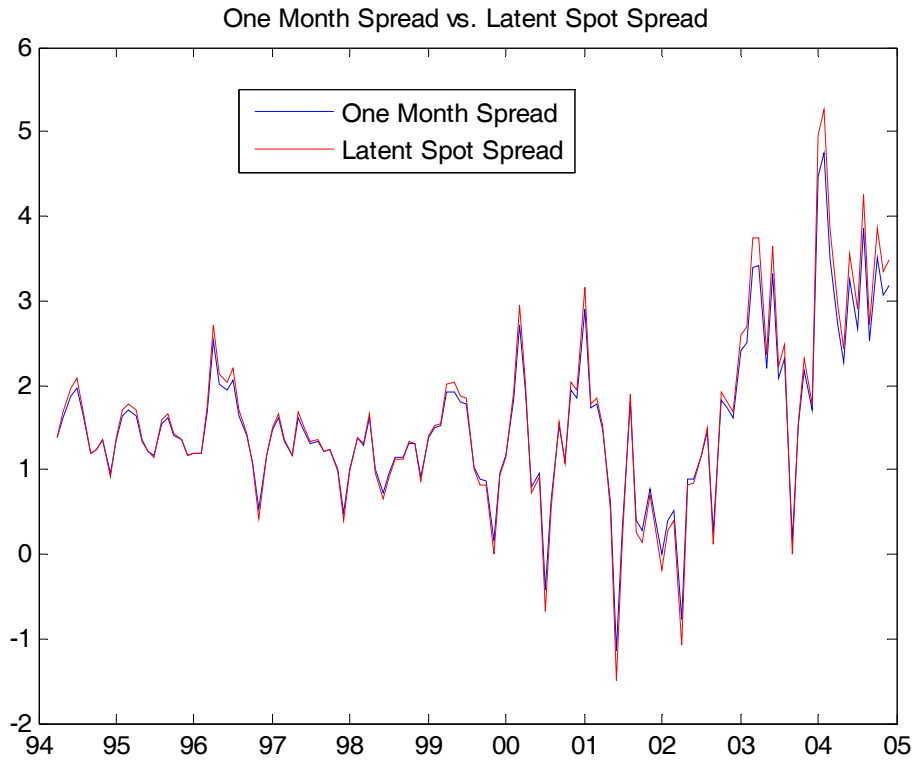


**Figure 5. The 1 month futures prices of WTI and Brent crude oil**

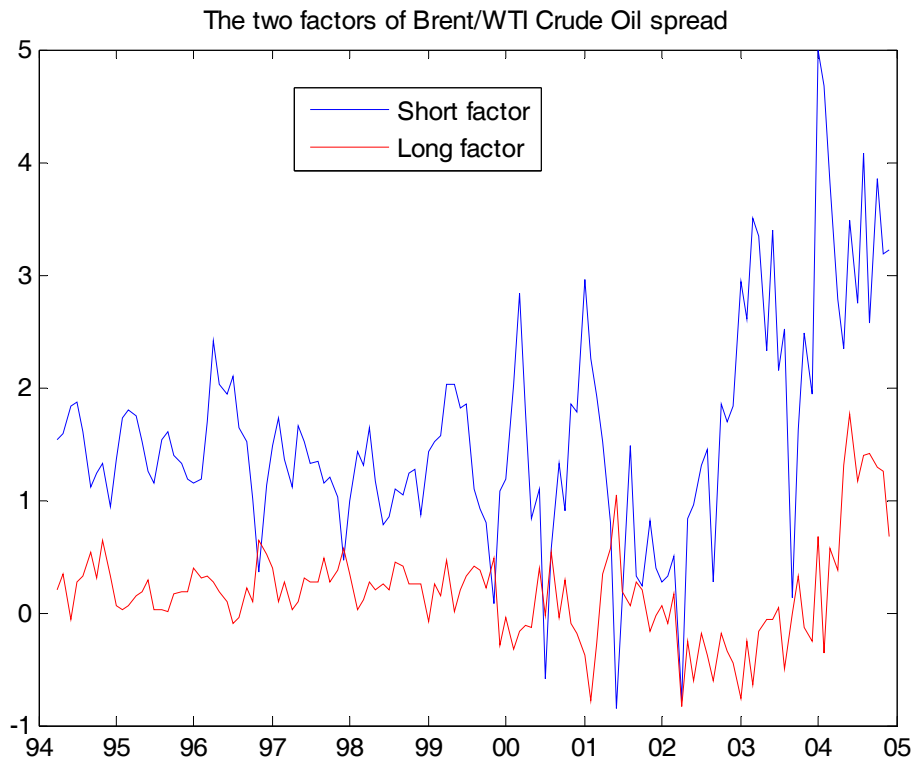


**Figure 6. The 1 month and 1 year location spreads**





**Figure 7. The 1 month location spread versus the latent spot spread**



**Figure 8. The two factors of crude oil location spread**