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Network Capacity Management Competition

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Abstract

We consider capacity management games between airlines who transport passengers over a joint airline network. Passengers are likely to purchase alternative tickets of the same class from competing airlines if they do not get tickets from their preferred airlines. We propose a Nash and a generalized Nash game model for such a capacity competition problem over a network. These two models are based on well-known deterministic linear programming and probabilistic nonlinear programming approximations for the non-competitive network capacity management problem. We prove the existence of a Nash equilibrium for both games and provide conditions to ensure the uniqueness of the Nash game. Our numerical results indicate that airlines can generate higher and more stable revenues from a booking scheme that is based on the combination of the partitioned booking-limit policy and the generalized Nash game model. The results also show that this booking scheme is robust irrespective of which booking scheme the competitor takes.

Keywords: Revenue management, capacity control, generalized Nash games, existence, uniqueness.

1 Introduction

Revenue management has become a prevailing concept in the airline and hospitality industries which provide perishable services or products. While many important components such as pricing, capacity management, overbooking, and choice modelling have been extensively studied in the past, demand forecasting and competition have not received enough attention although research on these topics is under way. For comprehensive treatments of revenue management theory and practice, see two recent excellent books [32, 37].

Most airlines in the world operate and compete fiercely over networks. They react to changes made by competing airlines in price, network structure and product design, among others. Because of competition, most airlines operate with very small profit margins. If competition is not carefully dealt with, small profit margins can diminish very quickly, as exemplified in the well-known failure of People Express [7].

Competition between airlines is complicated and multi dimensional. Possible competition elements are network settings, frequencies, timetabling, aircraft capacities, product offerings, pricing and capacity management, among others. In this paper, we focus on capacity availability.

It is well known [37] that the capacity management problem can be satisfactorily modeled by dynamic programming. The resulting dynamic programs are computationally intractable due to the curse of dimensionality and are often approximated by various simpler mathematical programs such as deterministic linear programs, probabilistic nonlinear programs, and randomized linear programs, see for example [3, 6, 9, 36, 39].

Capacity management becomes even more complicated when several airlines compete over a network. Conceptually, capacity management under competition can be formulated by dynamic programming. But we are content to model it using simplified models such as deterministic linear programs in order to avoid the curse of dimensionality. There is no doubt that approximate models can only give airlines sub-optimal solutions. Nevertheless, Dockner et al [10] argue "in regard to the assumption of payoff-maximizing behaviour, some researchers have suggested that players are only bounded rational: they satisfice rather than maximize. Satisficing behaviour means that a player is content with obtaining a certain level of payoff, not necessarily a maximal one".

Revenue management competition is only a recent addition to the vast game theory literature. Chapter 8 of the book [37] is devoted to revenue management competition. Cachon and Netessine [4] present an excellent review on game theory in the area of supply chain management. In particular, they review techniques for proving the existence and uniqueness of a Nash equilibrium for static, dynamic and cooperative games.

There are two notable features of revenue management competition that are different from traditional oligopolistic games. First, multiple products sold by the same airline share the same capacity. Second, unsatisfied demand for a particular product from one airline can be satisfied by other airlines; i.e., demand overflow is allowed because customers are willing to substitute another airline.

The first oligopolistic competition model with demand overflow between two firms is presented in [30] in a supply chain management setting where each firm produces a single product that can substitute similar products provided by competing firms. This work is extended in [20, 23, 24, 27]. Either deterministic or probabilistic rules to account for demand overflow are considered in [23, 27] while in [24], a choice model based on user's utility maximization is used for customers to decide from which firms they will purchase products.

Explicit duopoly capacity management competition in a single-leg setting is considered in [21, 28, 35] where multiple products sold by each firm share the same capacity. In all these papers, the existence and uniqueness of a Nash equilibrium are investigated.

Revenue management competition in both inventory and pricing is also studied by other researchers [1, 2, 8, 13, 14, 15, 16, 26, 31], where demand overflow is not considered. Rather they assume that demand for each product for each firm can be determined by prices whether demand is deterministic or stochastic.

We study capacity management games between several airlines in a network setting. To our knowledge, this has not been considered in the literature. Each airline sells multiple products as in traditional network revenue management problems [37]. Demand overflow is taken into account in a deterministic way as in [4, 27], i.e., a proportion of the customers, who do not get the tickets they want from their preferred airline, approach other airlines for similar products. We develop one Nash and one generalized Nash game model to represent capacity management games. It turns out that those games can be reformulated into either variational inequality or generalized variational inequality problems. We prove the existence of a Nash equilibrium for both games. We provide conditions to ensure the uniqueness of a Nash equilibrium for the

Nash game. We use an iterative algorithm for finding Nash equilibria of both game models. We conduct numerical experiments. Our numerical results indicate that airlines can generate higher and more stable revenues from a booking scheme that is based on the combination of the partitioned booking-limit policy and the generalized Nash game model. The results also show that this booking scheme is robust irrespective of which booking scheme the competitor takes.

The rest of the paper is organized as follows. In the next section, we propose a deterministic linear programming model and a probabilistic nonlinear programming model. In Section 3, we study both models in a game-theoretic framework. In particular we discuss the existence and uniqueness of Nash and generalized Nash equilibria. We conduct numerical experiments on performance of several booking schemes derived from both models in conjunction with either the partitioned booking-limit policy or the bid-price policy.

2 Game-Theoretic Models and Booking Policies

2.1 Basic notation and assumptions

I airlines, indexed by $i = 1, \cdots, I$, compete to provide services transporting customers over a joint airline network where each airline may provide services in part of the network. Suppose I airlines sell K classes of products, indexed by k, over the network for a particular future departure date. Each product is a combination of a fare class and an itinerary over the network. Let r^i be the unit price vector of dimension K for airline i. Since airlines do not necessarily provide identical services, some products may not be provided by all airlines. Without loss of generality, we may assume that $r^i_k > 0$ if product k is provided by airline i, and $r^i_k = 0$ otherwise. Let C^i be the remaining capacity vector of dimension M for airline i, where M is the number of legs over the network. Here each leg is a flight from one airport to another at a particular departure time. That is, two flights between two directly-connected airports are treated as two different legs. Let A^i be the leg-product incidence matrix for airline i, i.e., $A^i_{mk} = 1$ if and only if airline i sells product k and product k covers leg m. Clearly A^i is a matrix with M rows and K columns. If airline i does not sell product k, then column k of A^i contains all zero elements and $r^i_k = 0$. If airline i does not fly on leg m, then $C^i_m = 0$.

We make the following standard assumptions:

- The demand for one product is independent of that for another product over the network. However, we allow the demand for one product from one airline to be stochastically correlated to the demand for the same product from other airlines.
- Each customer is only interested in one particular product. Each customer makes a booking request from their preferred airline and with a certain probability, makes another booking request of the same product from another airline if their first booking request is rejected. If their second booking request is also rejected, then they become a lost customer to all airlines for this particular departure date.
- The prices of all products are fixed for all airlines.
- For modelling purposes, airline capacity C as well as capacity allocation is treated as a vector of non-negative real numbers.

2.2 The deterministic linear programming formulation

Assume that the primary demand for airline i is D^i. A rejected customer from airline i makes another booking request for the same product from other airlines. Suppose $o_k^{ij} \geq 0$ denotes the overflow rate of product k from airline i to airline j. That is, if a customer, who prefers airline i, is rejected for a booking request for product k, by airline i, then they would make a booking request of product k from airline j with a probability o_k^{ij}. Clearly, $\sum_{j \neq i} o_k^{ij} \leq 1$ for any i and k. We further assume that each customer has a unique airline for their initial preference.

Let $x^i \in \mathbb{R}^K$, denoting the partitioned booking limits, be decision variables for airline i. The total potential demand for airline i is $D^i + \sum_{j \neq i} o^{ji}[D^j - x^j]^+$, where $z^+ = \max\{0, z\}$ and max is taken componentwise for a vector z. This demand overflow approach has been used in [20, 23, 27, 30], where only a single product is offered by each of all players/airlines, and in [28], where two products are offered by each of two airlines. The total potential demand for airline i is made up from its own primary demand D^i and the overflow demand from other airlines, which is equal to $\sum_{j \neq i} o^{ji}[D^j - x^j]^+$. Assuming that partitioned booking limits x^{-i} for all other airlines other than i are given, airline i aims to determine its optimal partitioned booking limits by solving the following deterministic linear program (DLP):

$$
\begin{array}{lll}
\mathrm{DLP}^i_{\diamondsuit} \qquad & \max\limits_{x^i} & (r^i)^T x^i \\
& \text{s.t.} & A^i x^i \leq C^i, \\
& & x^i \leq D^i + \sum\limits_{j \neq i} o^{ji}[D^j - x^j]^+, \\
& & x^i \geq 0.
\end{array}
$$

Here subscript $\diamondsuit$ indicates that each airline first satisfies its primary demand and then accepts the overflow demand that cannot be satisfied by rival airlines. In $\mathrm{DLP}^i_{\diamondsuit}$, the objective is that airline i maximizes its total revenue. The first constraint states that the capacity on each leg must not be violated. The second constraint specifies that the total allocation to all airlines for each product must not exceed the demand for this product. The last constraint simply shows that the booking limits are nonnegative. One simple observation is that the sizes of DLPs for all airlines can be reduced by suitably removing the products/columns/variables that are not sold by a particular airline and the legs/rows that have zero remaining capacities or that are not covered by a particular airline. This is particularly useful in reducing computational time when solving DLPs. However, to keep notation simple, we shall not remove those columns or rows for the purpose of analysis because they do not affect our future analysis.

We can reformulate $\mathrm{DLP}^i_{\diamondsuit}$ into an equivalent nonlinear and nonsmooth program, whose feasible set depends only on the partitioned booking limit x^i of airline i:

$$
\begin{array}{ll}
\max\limits_{x^i} & (r^i)^T \min(x^i, D^i + \sum\limits_{j \neq i} o^{ji}[D^j - x^j]^+) \\
\text{s.t.} & A^i x^i \leq C^i, \\
& x^i \geq 0.
\end{array}
\tag{1}
$$

We can define another variation of (1). Let $\rho^i > r^i$ be a constant vector for each i.

$$\begin{aligned} \max_{x^i} \quad & (r^i)^T x^i + (\rho^i)^T \min(0, D^i + \sum_{j \neq i} o^{ji}[D^j - x^j]^+ - x^i) \\ \text{s.t.} \quad & A^i x^i \leq C^i, \\ & x^i \geq 0. \end{aligned} \tag{2}$$

The statement below shows that $\mathrm{DLP}^i_\diamondsuit$, (1) and (2) are equivalent.

Proposition 2.1 *Let $x^{-i} \geq 0$ be partitioned booking limits for all other airlines except for airline i.*

(a) If x^i is an optimal solution to $DLP^i_\diamondsuit$, then x^i is also an optimal solution to (1). Conversely, if x^i is an optimal solution to (1), then $\min(x^i, D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+)$ is an optimal solution to both $DLP^i_\diamondsuit$ and (1).

(b) x^i is an optimal solution to $DLP^i_\diamondsuit$ if and only if x^i is an optimal solution to (2).

Proof. (a) Suppose x^i is an optimal solution to $\mathrm{DLP}^i_\diamondsuit$. Then x^i is a feasible solution to (1). Therefore the optimal objective function value of $\mathrm{DLP}^i_\diamondsuit$ is not greater than that of (1). Conversely, suppose x^i is an optimal solution to (1). Then $\min(x^i, D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+)$ is a feasible solution to both $\mathrm{DLP}^i_\diamondsuit$ and (1), and $\min(x^i, D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+)$ is still an optimal solution to (1). Hence the optimal objective function value of (1) is not greater than that of $\mathrm{DLP}^i_\diamondsuit$.

Collectively, we have shown that $\mathrm{DLP}^i_\diamondsuit$ and (1) have the same optimal objective function value. Then it is obvious that any optimal solution to $\mathrm{DLP}^i_\diamondsuit$ is also an optimal solution to (1). Moreover, if x^i is an optimal solution to (1), then $\min(x^i, D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+)$ is an optimal solution to (1) because both x^i and $\min(x^i, D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+)$ are feasible to (1) and the objective function of (1) has the same value at both feasible solutions, and $\min(x^i, D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+)$ is an optimal solution to $\mathrm{DLP}^i_\diamondsuit$.

(b) We only need to prove that any optimal solution to (2) must satisfy $D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+ - x^i \geq 0$, i.e., any optimal solution to (2) must be a feasible solution to $\mathrm{DLP}^i_\diamondsuit$. By contradiction, assume that for an optimal solution $\bar{x}^i$ of (2), it holds that for some k, $\Delta_k \equiv D^i_k + \sum_{j\neq i} o^{ji}_k [D^j_k - x^j_k]^+ - \bar{x}^i_k < 0$. Define y^i such that $y^i_k = \bar{x}^i_k + \Delta_k$ and $y^i_\ell = \bar{x}^i_\ell$ for any $\ell \neq k$. Then $x^i \geq y^i \geq 0$ because $D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+ \geq 0$. This shows that y^i is still a feasible solution to (2). By simple calculations, we can prove that y^i is a better solution for (2) than $\bar{x}^i$, which is a contradiction. ■

The feasible sets of both (1) and (2) are simpler than that of $\mathrm{DLP}^i_\diamondsuit$ because (1) and (2) only involve the strategy variable x^i of airline i. Therefore, in the context of game theory, a generalized Nash game is converted into traditional Nash games. See the definitions of the Nash and generalized Nash games in the next section. However these equivalent formulations have a nonlinear (piecewise linear) and nonsmooth objective function (or payoff function in the context of game theory). The nonsmooth property of the payoff functions is undesirable from a computational point of view. However, these equivalent formulations are useful from an analysis point of view as we shall see in Section 3.2.

$\mathrm{DLP}^i_\diamondsuit$ is a linear program with respect to x^i for any given value for x^{-i}. However the constraint $x^i \leq D^i + \sum_{j\neq i} o^{ji}[D^j - x^j]^+$ is nonsmooth with respect to x^{-i} and hence nonsmooth with respect to the joint variable x. This nonsmooth property may pose difficulties for proposing computational methods for solving generalized Nash games. To address this potential drawback, we propose another reformulation for $\mathrm{DLP}^i_\diamondsuit$. Let $y^i, z^i \in \mathbb{R}^K$ be two auxiliary variables, and $\rho \in \mathbb{R}^K$ is a fixed parameter such that $\rho_k > \max_{i=1,\cdots,I}\{r^i_k\}$ for all k. We propose another linear programming formulation to $\mathrm{DLP}^i_\diamondsuit$.

$$
\begin{array}{lll}
\mathrm{DLP}^i_\oplus & \displaystyle\max_{x^i,z^i} & (r^i)^T x^i - \rho^T z^i \\
 & \text{s.t.} & A^i x^i \le C^i, \\
 & & x^i \le D^i + y^i, \\
 & & y^i = z^i + \displaystyle\sum_{j\neq i} o^{ji}(D^j - x^j), \\
 & & x^i, y^i, z^i \ge 0,
\end{array}
$$

where $\oplus$ indicates that nonsmooth functions used in $\mathrm{DLP}^i_\diamondsuit$ are replaced by smooth functions in $\mathrm{DLP}^i_\oplus$. The penalty term introduced in the objective function ensures that z^i are as small as possible in an optimal solution for $\mathrm{DLP}^i_\oplus$ and hence no additional demand will be induced artificially from the new formulation.

The proposition below shows that $\mathrm{DLP}^i_\diamondsuit$ and $\mathrm{DLP}^i_\oplus$ are indeed equivalent as far as optimal solutions are concerned when there are only two airlines in competition. For ease of exposition, we let I and II denote indices for two airlines.

Proposition 2.2

(a) For any optimal solution $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ *for* $\mathrm{DLP}^{\mathrm{I}}_\oplus$*, it must hold that* $y^{\mathrm{I}} = o^{\mathrm{II,I}}[D^{\mathrm{II}} - x^{\mathrm{II}}]^+$*, and* $z^{\mathrm{I}} = o^{\mathrm{II,I}}[x^{\mathrm{II}} - D^{\mathrm{I}}]^+$*, which are irrelevant to* x^{I}*.*

(b) If x^{I} *is feasible to* $\mathrm{DLP}^{\mathrm{I}}_\diamondsuit$*, then* $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ *is feasible to* $\mathrm{DLP}^{\mathrm{I}}_\oplus$*, where* $y^{\mathrm{I}} = o^{\mathrm{II,I}}[D^{\mathrm{II}} - x^{\mathrm{I}}]^+$*,* $z^{\mathrm{I}} = o^{\mathrm{II,I}}[x^{\mathrm{II}} - D^{\mathrm{I}}]^+$*. Conversely if* $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ *is feasible to* $\mathrm{DLP}^{\mathrm{I}}_\oplus$*, then* x^{I} *is feasible to* $\mathrm{DLP}^{\mathrm{I}}_\diamondsuit$*.*

(c) x^{I} *is an optimal solution for* $\mathrm{DLP}^{\mathrm{I}}_\diamondsuit$ *if and only if* $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ *is an optimal solution for* $\mathrm{DLP}^{\mathrm{I}}_\oplus$*, where* $y^{\mathrm{I}} = o^{\mathrm{II,I}}[D^{\mathrm{II}} - x^{\mathrm{II}}]^+$*, and* $z^{\mathrm{I}} = o^{\mathrm{II,I}}[x^{\mathrm{II}} - D^{\mathrm{II}}]^+$*.*

Proof. (a) Suppose $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ is an optimal solution for $\mathrm{DLP}^{\mathrm{I}}_\oplus$. Then $y^{\mathrm{I}} \ge o^{\mathrm{II,I}}[D^{\mathrm{II}} - x^{\mathrm{II}}]^+$ and $z^{\mathrm{I}} \ge o^{\mathrm{II,I}}[x^{\mathrm{II}} - D^{\mathrm{II}}]^+$ because $y^{\mathrm{I}} = z^{\mathrm{I}} + o^{\mathrm{II,I}}(D^{\mathrm{II}} - x^{\mathrm{II}})$, $y^{\mathrm{I}} \ge 0$, and $z^{\mathrm{I}} \ge 0$. We only need to prove that it does not happen that $y^{\mathrm{I}} > o^{\mathrm{II,I}}[D^{\mathrm{II}} - x^{\mathrm{II}}]^+$ or equivalently $z^{\mathrm{I}} > o^{\mathrm{II,I}}[x^{\mathrm{II}} - D^{\mathrm{II}}]^+$. Let us prove the result by contraction in two cases for a particular product k: either $D^{\mathrm{II}}_k - x^{\mathrm{II}}_k \ge 0$ or $D^{\mathrm{II}}_k - x^{\mathrm{II}}_k < 0$. We now assume $z^{\mathrm{I}}_k > o^{\mathrm{II,I}}[x^{\mathrm{II}}_k - D^{\mathrm{II}}_k]^+$.

When $D^{\mathrm{II}}_j - x^{\mathrm{II}}_k \ge 0$, construct a new vector $(\bar{x}^{\mathrm{I}}, \bar{y}^{\mathrm{I}}, \bar{z}^{\mathrm{I}})$ from $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ by changing the k-th element of x^{I}, y^{I}, and z^{I} such that $\bar{z}^{\mathrm{I}}_k = 0$, $\bar{y}^{\mathrm{I}}_k = y^{\mathrm{I}}_k - z^{\mathrm{I}}_k$, and $\bar{x}^{\mathrm{I}}_k = [x^{\mathrm{I}}_k - z^{\mathrm{I}}_k]^+$. $(\bar{x}^{\mathrm{I}}, \bar{y}^{\mathrm{I}}, \bar{z}^{\mathrm{I}})$ is also feasible to $\mathrm{DLP}^{\mathrm{I}}_\oplus$. Furthermore,

$$
\begin{array}{rcl}
r^{\mathrm{I}}_k \bar{x}^{\mathrm{I}}_k - \rho_k \bar{z}^{\mathrm{I}}_k & = & r^{\mathrm{I}}_k [x^{\mathrm{I}}_k - z^{\mathrm{I}}_k]^+ - \rho_k 0 \\
 & \ge & r^{\mathrm{I}}_k (x^{\mathrm{I}}_k - z^{\mathrm{I}}_k) \\
 & > & r^{\mathrm{I}}_k x^{\mathrm{I}}_k - \rho_k z^{\mathrm{I}}_k.
\end{array}
$$

This shows that $(\bar{x}^{\mathrm{I}}, \bar{y}^{\mathrm{I}}, \bar{z}^{\mathrm{I}})$ is a better solution than $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ for $\mathrm{DLP}^{\mathrm{I}}_\oplus$, which is a contradiction.

When $D^{\mathrm{II}}_k - x^{\mathrm{II}}_k < 0$, construct a new vector $(\bar{x}^{\mathrm{I}}, \bar{y}^{\mathrm{I}}, \bar{z}^{\mathrm{I}})$ from $(x^{\mathrm{I}}, y^{\mathrm{I}}, z^{\mathrm{I}})$ by changing the k-th element of x^{I}, y^{I}, and z^{I} such that $\bar{z}^{\mathrm{I}}_k = o^{\mathrm{II,I}}(x^{\mathrm{II}}_k - D^{\mathrm{II}}_k)$, $\bar{y}^{\mathrm{I}}_k = y^{\mathrm{I}}_k - z^{\mathrm{I}}_k + \bar{z}^{\mathrm{I}}_k \equiv 0$, and

$\bar{x}^{\rm I}_k = [x^{\rm I}_k - z^{\rm I}_k + \bar{z}^{\rm I}_k]^+$. By the assumption that $z^{\rm I}_k > o^{\rm II,I}[x^{\rm II}_k - D^{\rm II}_k]^+$, $z^{\rm I}_k > \bar{z}^{\rm I}_k$. Hence $y^{\rm I}_k > \bar{y}^{\rm I}_k \geq 0$, $x^{\rm I}_k \geq \bar{x}^{\rm I}_k \geq 0$. Moreover, $(\bar{x}^{\rm I}, \bar{y}^{\rm I}, \bar{z}^{\rm I})$ is also feasible to $\mathrm{DLP}^{\rm I}_{\oplus}$. We now compare the objective function values of $\mathrm{DLP}^{\rm I}_{\oplus}$ at two feasible solutions.

$$\begin{aligned} r^{\rm I}_k \bar{x}^{\rm I}_k - \rho_k \bar{z}^{\rm I}_k &= r^{\rm I}_k [x^{\rm I}_k - z^{\rm I}_k + \bar{z}^{\rm I}_k]^+ - \rho_k \bar{z}^{\rm I}_k \\ &\geq r^{\rm I}_k (x^{\rm I}_k - z^{\rm I}_k + \bar{z}^{\rm I}_k) - \rho_k \bar{z}^{\rm I}_k \\ &= r^{\rm I}_k x^{\rm I}_k - r^{\rm I}_k (z^{\rm I}_k - \bar{z}^{\rm I}_k) - \rho_k z^{\rm I}_k + \rho_k (z^{\rm I}_k - \bar{z}^{\rm I}_k) \\ &= r^{\rm I}_k x^{\rm I}_k - \rho_k z^{\rm I}_k + (\rho_k - r^{\rm I}_k)(z^{\rm I}_k - \bar{z}^{\rm I}_k) \\ &> r^{\rm I}_k x^{\rm I}_k - \rho_k z^{\rm I}_k. \end{aligned}$$

This shows that $(\bar{x}^{\rm I}, \bar{y}^{\rm I}, \bar{z}^{\rm I})$ is a better solution than $(x^{\rm I}, y^{\rm I}, z^{\rm I})$ for $\mathrm{DLP}^{\rm I}_{\oplus}$, which is a contradiction.

(b) This can be proved easily.

(c) Let $x^{\rm I}$ be an optimal solution for $\mathrm{DLP}^{\rm I}_{\Diamond}$. Suppose $(x^{\rm I}, y^{\rm I}, z^{\rm I})$ is not an optimal solution for $\mathrm{DLP}^{\rm I}_{\oplus}$, where $y^{\rm I} = o^{\rm II,I}[D^{\rm II} - x^{\rm II}]^+$ and $z^{\rm I} = o^{\rm II,I}[x^{\rm II} - D^{\rm II}]^+$. Let $(\bar{x}^{\rm I}, \bar{y}^{\rm I}, \bar{z}^{\rm I})$ be optimal to $\mathrm{DLP}^{\rm I}_{\oplus}$. By (a) $\bar{y}^{\rm I} = o^{\rm II,I}[D^{\rm II} - x^{\rm II}]^+ = y^{\rm I}$ and $\bar{z}^{\rm I} = o^{\rm II,I}[x^{\rm II} - D^{\rm II}]^+ = z^{\rm I}$. It follows that $(r^{\rm I})^T x^{\rm I} - \rho^T z^{\rm I} < (r)^{\rm I})^T \bar{x}^{\rm I} - \rho^T \bar{z}^{\rm I} = (r)^{\rm I})^T \bar{x}^{\rm I} - \rho^T z^{\rm I}$, which implies that $(r^{\rm I})^T x^{\rm I} < (r)^{\rm I})^T \bar{x}^{\rm I}$. By (b), $\bar{x}^{\rm I}$ is a feasible for $\mathrm{DLP}^{\rm I}_{\Diamond}$, which is better than $x^{\rm I}$. We arrive at a contradiction.

Let $(x^{\rm I}, y^{\rm I}, z^{\rm I})$ be an optimal solution for $\mathrm{DLP}^{\rm I}_{\oplus}$. Suppose $x^{\rm I}$ is not an optimal solution for $\mathrm{DLP}^{\rm I}_{\Diamond}$. and $\bar{x}^{\rm I}$ is optimal to $\mathrm{DLP}^{\rm I}_{\Diamond}$. Then $(r^{\rm I})^T x^{\rm I} < (r)^{\rm I})^T \bar{x}^{\rm I}$. By (b), $(\bar{x}^{\rm I}, y^{\rm I}, z^{\rm I})$ is feasible to $\mathrm{DLP}^{\rm I}_{\oplus}$, and also it is a better solution than $(x^{\rm I}, y^{\rm I}, z^{\rm I})$ for $\mathrm{DLP}^{\rm I}_{\oplus}$, which is a contradiction. Therefore $x^{\rm I}$ must be an optimal solution for $\mathrm{DLP}^{\rm I}_{\Diamond}$. ■

The following example shows that the statement in Proposition 2.2 cannot be extended to the case where there are more than two airlines in the game.

Example 2.1 *Suppose there are three airlines in the game and each airline provides exactly one product that covers a single leg shared by all three airlines. Assume* $r^1 = r^2 = r^3 = 1$, $C^1 = C^2 = C^3 = 30$, $D^1 = 20, D^2 = 10, D^3 = 30$, *and* $\rho = 2$, *which is greater than the maximum of* r^1, r^2 *and* r^3. *Consider* $(x^1, x^2, x^3) = (20, 20, 20)$. *It can be verified that given* $(x^2, x^3) = (20, 20)$, *the optimal solution for* $DLP^1_{\Diamond}$ *is* $x^1 = 30$ *while the optimal solution for* $DLP^1_{\oplus}$ *is* $(x^1, y^1, z^1) = (20, 0, 0)$.

2.3 The probabilistic nonlinear programming formulation

In $\mathrm{DLP}_{\Diamond}$, the stochastic nature of demand is entirely ignored. In the literature, several methods have been proposed to take the stochastic demand into account during the modelling process. One of those methods is based on so-called probabilistic nonlinear programming (PNLP). Let d^i represent the demand random variable for airline i. Recall the definitions of x^i and o^{ji}. An optimization problem for airline i is the following PNLP.

$$\begin{aligned} \mathrm{PNLP}^i_{\Diamond} \qquad & \max_{x^i} \ \mathbb{E}[(r^i)^T \min(x^i, d^i + \sum_{j \neq i} o^{ji}[d^j - x^j]^+)] \\ & \text{s.t.} \ \ A^i x^i \leq C^i, \\ & \qquad x^i \geq 0. \end{aligned}$$

Here $\sum_{j \neq i} o^{ji}[d^j - x^j]^+$ is the overflow demand to airline i from other airlines. We remark that the expectation is taken over the joint demand probability space of all airlines for each product.

There is a certain level of flexibility to model demand d^i. For example, we can either assume that d^i and d^j are stochastically independent for any $i \neq j$ or that d^i is a fixed proportion of the overall random demand d. In the latter, demand for the same product between two airlines is correlated. This does not contradict the demand independence assumption between different products made in Section 2.1.

When the demand d_k is a singleton for every product k for all airlines, then $\text{PNLP}^i_\Diamond$ reduces to an equivalent reformulation (1) of $\text{DLP}^i_\Diamond$.

2.4 Booking policies

Both booking limits and the dual prices or Lagrangian multipliers can be obtained from solving either $\text{DLP}_\Diamond$ or $\text{PNLP}_\Diamond$. These solutions are used to form booking policies such as partitioned booking-limit and bid-price controls [37].

In a partitioned booking-limit policy, a fixed amount of capacity of each resource is allocated to every product offered. The demand for each product has access only to its allocated capacity and no other product may use this capacity. In both $\text{DLP}_\Diamond$ and $\text{PNLP}_\Diamond$, these booking limits are set to be the respective optimal solutions x.

A bid-price control policy sets a threshold price or bid price for each resource in the network. Roughly this bid price is an estimate of the marginal cost of consuming the next incremental unit of the resource's capacity. When a booking request for a product arrives, the revenue from the request is compared to the sum of the bid prices of all the resources required by the product. If the revenue exceeds the sum of the bid prices, the request is accepted, provided all the resources associated with the requested product are still available; if not, the request is rejected.

In $\text{DLP}_\Diamond$, the optimal solution of dual variables associated with the capacity constraints, which are of the format $Ax \leq C$, are used as bid prices. Similarly the bid prices for $\text{PNLP}_\Diamond$ are the Lagrangian multipliers associated with the capacity constraints at their optimal solution.

3 Analysis of Game-Theoretic Models

In this section, we first formally define Nash and generalized Nash games based on the optimization models defined in the last section. We then define Nash and generalized Nash equilibrium points. In the end, we study the existence and uniqueness of (generalized) Nash equilibrium points.

We follow [29] to define generalized Nash games and generalized Nash equilibrium points. Since generalized Nash games have found many other applications in addition to revenue management competition considered in this paper, we shall use players and airlines interchangeably in a general context.

Let x^i be the strategy variable for player i, x^{-i} the joint strategy variable for all other players, and $x = (x^i, x^{-i})$ the joint strategy variable for all players. Let $\pi^i(x^i, x^{-i})$ be the payoff function for player i when the joint strategy is x. Assume that for any given strategies x^{-i} for all other players, player i must choose their strategy from their feasible set

$$\mathcal{K}^i(x^{-i}) = \{x^i : h^i(x^i) \leq 0, g^i(x^i, x^{-i}) \leq 0\},$$

where $h^i : \mathbb{R}^K \to \mathbb{R}^{N^i}$ and $g^i : \mathbb{R}^{K \times I} \to \mathbb{R}^{M^i}$. Here $h^i(x^i) \leq 0$ is a set of constraints that do not involve x^{-i}, and $g^i(x) \leq 0$ is a set of constraints that involve all strategy variables, which are usually called the joint/coupled constraints in the literature [33]. For any given strategy x^{-i} for other players, player i finds their best strategy $\bar{x}^i$ by solving the following maximization problem:

$$\begin{array}{ll} \max\limits_{x^i} & \pi^i(x^i, x^{-i}) \\ \text{s.t.} & x^i \in \mathcal{K}^i(x^{-i}). \end{array} \tag{3}$$

Definition 3.1 *A generalized Nash game defined by (3) for player i is to find $\bar{x}$, called a generalized Nash equilibrium, such that $\bar{x}^i \in \mathcal{K}^i(\bar{x}^{-i})$ is an optimal solution for player i when the strategies for all other players are fixed to be $\bar{x}^{-i}$.*

If there is no joint constraint in the game, i.e., $g^i(x) \leq 0$ diminishes for all i, then the generalized Nash game and a generalized Nash equilibrium reduce to a traditional Nash game and a Nash equilibrium respectively. The key difference between generalized Nash games and traditional Nash games is that the strategy space for a player may depend on other players' strategies in the former, but not in the latter, although the payoff functions in both types of games are allowed to be functions of other players' strategies.

The best response function is an important concept in game theory. For any given x^{-i}, the set of optimal solutions for (3) is called the response function for player i and is denoted by $\mathcal{BR}^i(x^{-i})$. To have a meaningful game, we often assume that $\mathcal{BR}^i(x^{-i})$ is singleton for any given x^{-i} and for all i. Consequently, $\bar{x}$ is a generalized Nash equilibrium if and only if $\bar{x}^i = \mathcal{BR}^i(\bar{x}^{-i})$ for all i. Intuitively, this optimality condition says that a player is not better off if they unilaterally change their strategy.

One can observe that the game based on $\text{DLP}^i_\Diamond$ for all airlines results in a generalized Nash game, while the game based on $\text{PNLP}^i_\Diamond$ results in a Nash game. The resulting two games are called $\text{DLP}_\Diamond$ game and $\text{PNLP}_\Diamond$ game. By Propositions 2.1, the $\text{DLP}_\Diamond$ game is equivalent to a traditional Nash game with nonsmooth and nonlinear payoff functions.

Generalized Nash games have been used for modelling competitive pricing in [31] where there is no network and demand does not overflow between players once prices are fixed. In [14], traditional Nash games have been used for proposing a unified game model combining pricing, capacity management, overbooking in a stochastic and dynamic framework for network revenue management. However, demand is not shared among competitors but the demand for individual airlines is determined by the prices of the same products offered by all airlines. Therefore, the game model in [14] is not an example of generalized Nash games. For other applications of generalized Nash games, see [11, 18, 29, 34].

3.1 Generalized Nash games and quasi variational inequalities

It is well known [12] that traditional Nash games are equivalent to variational inequality problems when the concerned payoff function for each player is continuously differentiable and concave with respect to its own strategies. Similarly, generalized Nash games are equivalent to quasi-variational inequality problems.

Given a point-to-point map Φ from $\mathbb{R}^m$ to itself and a point-to-set map $\mathcal{K}$ from $\mathbb{R}^m$ into subsets of $\mathbb{R}^m$, the quasi-variational inequality (QVI) problem is to find a vector $\bar{x} \in \mathcal{K}(\bar{x})$ such that

$$(y - \bar{x})^T \Phi(\bar{x}) \geq 0, \qquad \forall y \in \mathcal{K}(\bar{x}).$$

When $\mathcal{K}(x) \equiv \mathcal{K}$ for all x with $\mathcal{K}$ independent of x, the QVI reduces to the standard variational inequality (VI) problem.

The following results can be found from page 24 of [12].

Lemma 3.1 *Suppose the payoff function for player i in a generalized Nash game is continuously differentiable and concave with respect to x^i for all i. Let $\mathcal{K}(x) = \Pi_{i=1}^{I} \mathcal{K}^i(x^{-i})$ and*

$$\Phi(x) = -(\nabla_{x^1}\pi^1(x), \cdots, \nabla_{x^i}\pi^i(x), \cdots, \nabla_{x^I}\pi^I(x))^T$$

be a column vector.

(a) *$\bar{x}$ is a generalized Nash equilibrium if and only if $\bar{x}$ is a solution of the QVI defined by $\mathcal{K}(x)$ and Φ.*

(b) *Assume that $\mathcal{K}(x) \equiv \mathcal{K}$. Then $\bar{x}$ is a Nash equilibrium if and only if $\bar{x}$ is a solution of the VI defined by $\mathcal{K}$ and Φ.*

3.2 Existence of equilibrium

The existence of a Nash equilibrium (or generalized Nash equilibrium) for Nash games (or generalized Nash games) is an important topic in game theory. Without an equilibrium in a game, players do not know what strategy they should take. In many occasions, an equilibrium does exist. In this subsection, we provide conditions to ensure the existence of a Nash or a generalized Nash equilibrium for the two games proposed in Section 2.

The following results are well known in the literature by noting that concavity is preserved under the min-operator, limits, addition and hence under expectation and integration signs; see for example [4], and that the integral is continuously differentiable if the integrand is globally Lipschize continuous and continuously differentiable almost everywhere; see for example [17, 19].

Lemma 3.2

(a) For each airline i, the objective functions of $DLP_\diamondsuit$, (1) and (2) are all concave with respect to x^i.

(b) For each airline i, the objective function for $PNLP^i_\diamondsuit$ is concave with respect to x^i.

(c) Suppose d^i_k is a continuous random variable for all k and i. Then for any i, the objective function for $PNLP^i_\diamondsuit$ is continuously differentiable with respect to x.

Proposition 2.1 proves that an equivalence between $\text{DLP}_\diamondsuit$, (1) and (2). The lemma below shows equivalences between different generalized Nash and Nash games defined by those optimization problems. All results can be easily proved by the definitions of the generalized Nash equilibrium and the Nash equilibrium, and Proposition 2.1.

Lemma 3.3

(a) If $\bar{x}$ is a generalized Nash equilibrium for the $DLP_\diamondsuit$ game, then $\bar{x}$ is a Nash equilibrium for the Nash game defined by (1).

(b) $\bar{x}$ is a generalized Nash equilibrium for the $DLP_\diamondsuit$ game if and only if $\bar{x}$ is a Nash equilibrium for the Nash game defined by (2).

We are now ready to present existence results of a (generalized) Nash equilibrium for all four games.

Theorem 3.1

(a) There exists a generalized Nash equilibrium for the DLP$_\Diamond$ game.

(b) There exists a Nash equilibrium for the PNLP$_\Diamond$ game.

Proof. The results follow from Theorem 1 of [33], which states that a Nash equilibrium exists for a Nash game if the objective function for each player is concave with respect to their own strategy and continuous with respect to the strategies of all players and the strategy space for each player is convex and compact.

In light of Lemma 3.2 (b) and (c), the result in (b) follows from Theorem 1 of [33] directly. It is easy to verify that the conditions in Theorem 1 of [33] are satisfied for the Nash games defined by (2). Therefore the existence of a generalized Nash equilibrium of the DLP$_\Diamond$ game follows from Lemma 3.3(b). ■

3.3 Uniqueness of equilibrium

The uniqueness of the Nash equilibrium is another important topic in game theory. If there is a unique equilibrium, players can choose their strategies without vagueness. The importance of a unique equilibrium is highlighted in a statement by Cachon and Netessine [4]: "The obvious problem with multiple equilibria is that the players may not know which equilibrium will prevail. Hence, it is entirely possible that a non-equilibrium outcome results because one player plays one equilibrium strategy while a second player chooses a strategy associated with another equilibrium."

Let us recall a unique equilibrium result for the traditional Nash game [12].

Lemma 3.4 *Consider a Nash game in which the strategy space for every player is closed and convex. Assume that the negative Jacobian of Φ at any feasible solution x is positive definite. Then the Nash game has at most one Nash equilibrium.*

However, the result in Lemma 3.4 is not true in general for generalized Nash games although in [33] a uniqueness result of a normalized Nash equilibrium is proved for a generalized game in which all coupled constraints must be shared by all players.

In order to obtain the uniqueness of a Nash equilibrium for the Nash game based on Lemma 3.4, the payoff function must be twice continuously differentiable with respect to all strategy variables. Since the payoff functions in (1) and (2) for the DLP$_\Diamond$ game are nonsmooth, we are not able to establish the uniqueness of the Nash equilibrium for the DLP$_\Diamond$ game. Therefore we shall focus on the uniqueness of a Nash equilibrium for the PNLP game.

We now evaluate the Jacobian of the payoff functions of the PNLP$_\Diamond$ game. Netessine and Rudi [27] state that since the function under the expectation is integrable and has a bounded derivative, it satisfies the Lipschitz condition of order one, and hence the expectation and the derivative can be interchanged. This argument follows from a result in [17]. The payoff function for airline i is

$$\pi^i(x^i, x^{-i}) = E[\sum_{k=1}^{K} r_k^i \min(x_k^i, d_k^i + \sum_{j \neq i} o_k^{ji}[d_k^j - x_k^j]^+].$$

The first order derivatives of the payoff functions are

$$\frac{\partial \pi^i(x^i, x^{-i})}{\partial x^i_k} = r^i_k(1 - F_{d^i_k + \sum_{j\neq i} o^{ji}_k [d^j_k - x^j_k]^+}(x^i_k)), i = 1, \cdots, I, k = 1, \cdots, K,$$

where $F_{d^i_k + \sum_{j\neq i} o^{ji}_k [d^j_k - x^j_k]^+}$ is the cumulative probability distribution of random variable $d^i_k + \sum_{j\neq i} o^{ji}_k [d^j_k - x^j_k]^+$. The second order derivatives of the payoff functions are

$$\frac{\partial^2 \pi^i(x^i, x^{-i})}{(\partial x^i_k)^2} = -r^i_k f_{d^i_k + \sum_{j\neq i} o^{ji}_k [d^j_k - x^j_k]^+}(x^i_k),$$
$$i = 1, \cdots, I, k = 1, \cdots, K,$$
$$\frac{\partial^2 \pi^i(x^i, x^{-i})}{\partial x^i_k \partial x^i_\ell} = 0,$$
$$i = 1, \cdots, I, k = 1, \cdots, K, \ell = 1, \cdots, K, \ell \neq k,$$
$$\frac{\partial^2 \pi^i(x^i, x^{-i})}{\partial x^i_k \partial x^j_k} = -r^i_k o^{ji}_k f_{d^i_k + \sum_{m\neq i} o^{mi}_k [d^m_k - x^m_k]^+ | d^j_k > x^j_k}(x^i_k) \Pr(d^j_k > x^j_k),$$
$$i = 1, \cdots, I, j = 1, \cdots, I, k = 1, \cdots, K, j \neq i,$$
$$\frac{\partial^2 \pi^i(x^i, x^{-i})}{\partial x^i_k \partial x^j_\ell} = 0,$$
$$i = 1, \cdots, I, j = 1, \cdots, I, k = 1, \cdots, K, \ell = 1, \cdots, K, j \neq i, \ell \neq k.$$

The calculation of $\frac{\partial^2 \pi^i(x^i, x^{-i})}{\partial x^i_k \partial x^j_k}$ is slightly involved, but it can be done through the derivative definition. The Jacobian $\nabla\Phi(x)$ is a sparse matrix of $I \times K$ columns and $I \times K$ rows. We can rewrite $\nabla\Phi(x)$ as

$$\begin{pmatrix} N^{11} & \cdots & N^{1I} \\ \vdots & \vdots & \vdots \\ N^{I1} & \cdots & N^{II} \end{pmatrix}$$

where N^{ij} is a diagonal matrix in $\mathbb{R}^{K\times K}$ whose diagonal elements are

$$\left(\frac{\partial^2 \pi^i(x^i, x^{-i})}{\partial x^i_1 \partial x^j_1}, \cdots, \frac{\partial^2 \pi^i(x^i, x^{-i})}{\partial x^i_K \partial x^j_K} \right).$$

The sparseness of the Jacobian is due to the assumption that the demands between different products are statistically independent.

Under certain conditions, we are able to prove the uniqueness of the Nash equilibrium for the PNLP$_\Diamond$ game.

Theorem 3.2 *Suppose d^i_k is a continuous random variable for any k and i. Then a sufficient condition for the PNLP$_\Diamond$ game to have a unique Nash equilibrium is the following: (a) for any k and i, d^i_k has a large enough support interval which contains the projection of the feasible strategy space $\{x^i_k : A^i x^i \leq C^i, x^i \geq 0\}$ of airline i; (b) for any i and k, there exist two positive constants α^i_k and β^i_k such that $f_{d^i_k}(x)$ satisfies $\alpha^i_k \leq f_{d^i_k}(x) \leq \beta^i_k$ for x in its support interval; (c) for any i, j and k, o^{ji}_k is sufficiently small; (d) for any j and k, $\sum_{i\neq j} o^{ji}_k < 1$.*

Proof. By Lemma 3.4, it suffices to prove that Jacobian $\nabla\Phi(x)$ is positive definite for any x in the feasible set $\mathcal{K}$. In turn it is sufficient to prove that $\nabla\Phi(x)$ is both row and column diagonally dominant for any $x \in \mathcal{K}$. More precisely we need to prove the followings:

$$r_k^i f_{d_k^i + \sum_{j \neq i} o_k^{ji} [d_k^j - x_k^j]^+}(x_k^i) >$$

$$\sum_{j \neq i} r_k^i o_k^{ji} f_{d_k^i + \sum_{m \neq i} o_k^{mi} [d_k^m - x_k^m]^+ | d_k^j > x_k^j}(x_k^i) \Pr(d_k^j > x_k^j), \tag{4}$$

$$r_k^i f_{d_k^i + \sum_{j \neq i} o_k^{ji} [d_k^j - x_k^j]^+}(x_k^i) >$$

$$\sum_{j \neq i} r_k^j o_k^{ij} f_{d_k^j + \sum_{m \neq j} o_k^{mj} [d_k^m - x_k^m]^+ | d_k^i > x_k^i}(x_k^j) \Pr(d_k^j > x_k^j). \tag{5}$$

Note that for any i and k,

$$f_{d_k^i + \sum_{m \neq i} o_k^{mi} [d_k^m - x_k^m]^+ | d_k^j > x_k^j}(x_k^i) \Pr(d_k^j > x_k^j) \leq f_{d_k^i + \sum_{m \neq i} o_k^{mi} [d_k^m - x_k^m]^+}(x_k^i).$$

Inequality (4) is implied by condition (d). The right hand side of (5) is bounded above by

$$\sum_{j \neq i} r_k^j o_k^{ij} f_{d_k^j + \sum_{m \neq j} o_k^{mj} [d_k^m - x_k^m]^+}(x_k^j),$$

which in turn is bounded above by

$$\sum_{j \neq i} r_k^j o_k^{ij} \widehat{\beta}_k^j,$$

for any $x \in \mathcal{K}$. Here $\widehat{\beta}_k^j$ is an upper bound of the probability distribution function of random variable $d_k^j + \sum_{m \neq j} o_k^{mj} [d_k^m - x_k^m]^+$. By condition (b) and the convolution operation for the joint probability distribution of independent random variables, such a finite bound $\widehat{\beta}_k^j$ exists. Following the convolution operation again, $f_{d_k^i + \sum_{j \neq i} o_k^{ji} [d_k^j - x_k^j]^+}$ is bounded below by a positive constant $\widehat{\alpha}_k^i$. We have for any i, j and k,

$$r_k^i \widehat{\alpha}_k^i > \sum_{j \neq i} r_k^j o_k^{ij} \widehat{\beta}_k^j,$$

if o_k^{ji} is sufficiently small. Hence (5) holds. ■

A particular case of Theorem 3.2 is when $o_k^{ij} = 0$ for all i, j and k. In this case, there is no demand overflow and the existence of a unique Nash equilibrium is equivalent to a unique optimal solution for airline i's maximization problem. It is obvious that if the payoff function of airline i is strictly concave (equivalently matrix $-N^{ii}$ is positive definite) over its feasible region, then its optimal solution is unique. Furthermore, matrix N^{ii} is positive definite over its feasible region if probability distribution $f_{d_k^i}(x) > 0$ for any $x \in [0, d_{\max}]$, where $d_{\max}$ is an appropriate large positive number.

Unlike PNLP games, the following counter-example shows that the DLP$_\Diamond$ game can have multiple Nash equilibrium points.

Example 3.1 *Consider the DLP$_\Diamond$ game with two players I and II:*

$$\begin{array}{llcll}
\max\limits_{x_1^{\rm I},x_2^{\rm I}} & r^{\rm I}x_1^{\rm I}+r^{\rm I}x_2^{\rm I} & \quad & \max\limits_{x_1^{\rm II},x_2^{\rm II}} & r^{\rm II}x_1^{\rm II}+r^{\rm II}x_2^{\rm II} \\
\text{s.t.} & x_1^{\rm I}+x_2^{\rm I}\le\varepsilon & & \text{s.t.} & x_1^{\rm II}+x_2^{\rm II}\le\varepsilon, \\
& 0\le x_1^{\rm I}\le D_1^{\rm I}+o^{\rm II,I}[D_1^{\rm II}-x_1^{\rm II}]^+, & & & 0\le x_1^{\rm II}\le D_1^{\rm II}+o^{\rm I,II}[D_1^{\rm I}-x_1^{\rm I}]^+, \\
& 0\le x_2^{\rm I}\le D_2^{\rm I}+o^{\rm II,I}[D_2^{\rm II}-x_2^{\rm II}]^+, & & & 0\le x_2^{\rm II}\le D_2^{\rm II}+o^{\rm I,II}[D_2^{\rm I}-x_2^{\rm I}]^+.
\end{array}$$

Assume that ε is a small positive real number and $D_1^{\rm I}, D_2^{\rm I}, D_1^{\rm II}$, and $D_2^{\rm II}$ are much larger than ε. Then the optimal solution set of player I *is $\{(x_1^{\rm I}, x_2^{\rm I}) : x_1^{\rm I}+x_2^{\rm I}=\varepsilon, x_1^{\rm I}\ge 0, x_2^{\rm I}\ge 0\}$ because the unit prices for two products offered by player* I *are the same and the last two inequality constraints for player* I *are inactive no matter which strategies that player* II *takes. Likewise, the optimal solution set of player* II *is $\{(x_1^{\rm II}, x_2^{\rm II}) : x_1^{\rm II}+x_2^{\rm II}=\varepsilon, x_1^{\rm II}\ge 0, x_2^{\rm II}\ge 0\}$. This shows that the generalized Nash game has multiple solutions $(x_1^{\rm I}, x_2^{\rm I}, x_1^{\rm II}, x_2^{\rm II})$, which satisfy the following conditions: $x_1^{\rm I}+x_2^{\rm I}=x_1^{\rm II}+x_2^{\rm II}=\varepsilon$, $x^{\rm I}\ge 0$, $x^{\rm II}\ge 0$.*

The next example demonstrates that the DLP$_\Diamond$ game may have a unique Nash game.

Example 3.2 *Suppose both $C^{\rm I}$ and $C^{\rm II}$ are large positive numbers and demand $D^{\rm I}$ and $D^{\rm II}$ are small positive numbers. For simplicity, let us assume that there is exactly one product from each airline.*

Note that the capacity constraints $A^{\rm I}x^{\rm I}\le C^{\rm I}$ and $A^{\rm I}x^{\rm II}\le C^{\rm II}$ are inactive at any (generalized) Nash equilibrium for DLP$_\Diamond^{\rm I}$ and DLP$_\Diamond^{\rm II}$. On the other hand, the demand constraints are active at any (generalized) Nash equilibrium for DLP$_\Diamond^{\rm I}$ and DLP$_\Diamond^{\rm II}$. Then the response functions for player I and II are:

$$\mathcal{BR}^{\rm I}(x^{\rm II})=\begin{cases} D^{\rm I}+D^{\rm II}-x^{\rm II} & \text{if } x^{\rm II}\le D^{\rm II} \\ D^{\rm I} & \text{otherwise,}\end{cases}$$

$$\mathcal{BR}^{\rm II}(x^{\rm I})=\begin{cases} D^{\rm I}+D^{\rm II}-x^{\rm I} & \text{if } x^{\rm I}\le D^{\rm I} \\ D^{\rm II} & \text{otherwise.}\end{cases}$$

It is easy to see that $(D^{\rm I}, D^{\rm II})$ is the unique generalized Nash equilibrium for the DLP$_\Diamond$ game.

4 Computational experiments

In this section, we report our numerical experience for both game theoretical models proposed in this paper. We carried out our experiments based on two particular test examples drawn from [5] and their variations. All the computational experiments were conducted using MATLAB [25].

4.1 Test examples and booking schemes

In the first two examples, we assume that two airlines compete on capacity availability and both airlines are identical in terms of the product offerings, product prices, product demand parameters such as mean and standard deviation, and overflow rates. In Example 1, each airline operates over a hub-spoke network with one hub, 5 spoke cities and 10 legs. It has two fare classes for each of 30 itineraries (and hence 60 products). In Example 2, each airline operates over a hub-spoke network with two hubs, 4 spoke cities and 10 legs. It has two fare classes for each of 30 itineraries (and hence 60 products). For additional information on revenue, mean demand, leg capacities, and detailed network structures, see [5].

We also generated several more test examples which are variations of Example 1. In Example 3, we only changed product prices for the first airline. In particular, we assumed that the price for each product offered by the first airline is 3% more expensive than its original price while the second airline still uses its original prices in Example 1. In Example 4, we only changed the capacities of both airlines in Example 1. The capacity on each leg for the first airline was reduced by 10 seats and the capacity on each leg for the second airline was increased by 10 seats. Example 5 was generated from Example 1 by changing the demand parameter values. More precisely, the mean demand for each product offered by the first airline was reduced by two while the mean demand for each product offered by the second airline was increased by two. In Example 6, we made changes with regards to demand overflow rates. We assumed that the demand overflow rate from the first airline to the second is 0.75 and the demand overflow rate in the opposite direction is 0.25.

The simulation procedure that we followed is described in Talluri and van Ryzin [36]. For each test example, we simulated the booking process 500 times. In each simulation run, booking requests are randomly generated in two steps. In step 1, the number of requests for each product is randomly generated while in step 2, booking arrival times for each product are randomly generated. The booking process is modelled as a non-homogeneous Poisson process, where the arrival intensity at time t has a beta distribution and the total number of arrivals has a gamma distribution. The booking horizon is divided into 1000 units. Higher fare customers arrive more often close to the end of the booking horizon while lower fare customers arrive more often early in the booking horizon. A detailed description of the booking process can be found in [5].

In each of 500 simulation runs, all booking requests for each test example are processed (acceptance or rejection) based on a booking scheme. We implemented the following twelve booking schemes: `BLDLP`, `BPDLP`, `BLDLPSim`, `BPDLPSim`, `BLDLPMix`, `BPDLPMix`, `BLPNLP`, `BPPNLP`, `BLPNLPSim`, `BPPNLPSim`, `BLPNLPMix`, and `BPPNLPMix`. The meaning of each booking scheme based on DLP is given below and all booking schemes based on PNLP can be defined similarly.

- `BLDLP`: Partitioned booking-limit policy based on the partitioned booking limits obtained from $\text{DLP}_\Diamond$.
- `BPDLP`: Bid price policy based on the bid prices obtained from $\text{DLP}_\Diamond$.
- `BLDLPSim`: Partitioned booking-limit policy in which both airlines use the partitioned booking limits obtained from modified DLPs which ignore overflow demand (i.e., assuming $o_k^{21} = o_k^{21} = 0$ for all k).
- `BPDLPSim`: Bid-price policy in which both airlines use the bid prices obtained from modified DLPs which ignore overflow demand.
- `BLDLPMix`: Partitioned booking-limit policy in which the first airline uses the partitioned booking limits obtained from $\text{DLP}_\Diamond$ and the second airline uses the partitioned booking limits obtained from a modified DLP which ignores overflow demand.
- `BPDLPMix`: Bid-price policy in which the first airline uses the bid prices obtained from $\text{DLP}_\Diamond$ and the second airline uses the bid prices obtained from a modified DLP which ignores overflow demand.

Booking schemes `BLDLPSim`, `BPDLPSim`, `BLPNLPSim` and `BPPNLPSim` were tested in order to obtain insights when both airlines ignore demand overflow in the modified DLP and PNLP. Booking schemes BLDLPMix, BPDLPMix, BLPNLPMix and BPPNLPMix were tested in order to obtain insights when the first airline ignores demand overflow in the modified DLP and PNLP while the second airline takes the demand overflow into account when this airline solves $\text{DLP}_\Diamond$ and $\text{PNLP}_\Diamond$.

4.2 A computational algorithm

In all the booking schemes proposed in the previous subsection, each airline needs to use either partitioned booking limits or bid prices for processing booking requests. Both booking limits and bid prices are obtained by solving either the Nash game based on PNLP or the generalized Nash game based on DLP. As stated in Lemma 3.3, solving Nash games is equivalent to solving variational inequality problems and solving generalized Nash games is equivalent to solving quasi-variational inequality problems.

Many numerical algorithms have been developed for solving deterministic variational inequality problems and are well documented in [12]. Typically those algorithms require evaluations of the payoff functions of players and their high-order derivatives. However, evaluating the objective function of each airline as well as its high-order derivatives for PNLP is computationally expensive because it requires multi-dimensional integration. This motivates us to use a method that combines sample average approximation (SAA) and a Gauss-Seidel approach for solving the PNLP game.

Algorithms for solving quasi-varitional inequality problems are sporadic in the literature. When all players in the generalized Nash game share all coupled constraints, the so-called relaxation algorithm can be used to find a generalized Nash equilibrium under suitable conditions [38]. For a general quasi-variational inequality problem, a sequential penalty method is proposed in [29] and a Newton method is studied in [11]. The Newton method in [11] requires high-order derivatives which are not available for the $\text{DLP}_\Diamond$ game as nonsmooth terms with respect to strategies of competitors exist in the constraint of $\text{DLP}^i_\Diamond$. The method proposed below is close to the method of [29]. However, we do not apply a penalty approach to coupled constraints, rather we solve a linear program at each iteration.

We now present an outline of our algorithm for solving $\text{DLP}_\Diamond$ and $\text{PNLP}_\Diamond$ games.

`Algorithm 1.`

Step 1 Choose a starting point $x(n) = (x^1(n), x^2(n), \ldots, x^I(n))$. Let $n = 1$.

Step 2 For each player i at iteration $n+1$, finding $x^i(n+1)$ by solving an optimization problem (either DLP or PNLP) assuming $x^{-i}(n+1) = (x^1(n+1), \ldots, x^{i-1}(n+1), x^{i+1}(n), \ldots, x^I(n))$ is given.

Step 3 Check the convergence criterion. If $\|x(n+1) - x(n)\|$ is sufficiently small, then terminate the algorithm and $x(n)$ is an approximate (generalized) Nash equilibrium. Otherwise, set $n := n + 1$ and go to Step 2.

The above algorithm is proved to converge for Nash games with two players under suitable conditions [22]. To our knowledge, no one has studied convergence of the above algorithm for generalized Nash games. We shall leave convergence for a future investigation.

When solving the DLP game, the optimization problem defined in Step 2 is the linear program $\text{DLP}^i_\Diamond$, for which an optimal solution can be found easily in Matlab. When solving the PNLP game, the optimization problem defined in Step 2 is $\text{PNLP}^i_\Diamond$, which is a nonlinear program with the objective function being defined by a multi-dimensional integral. It is very computationally expensive to solve such an nonlinear program using an off-shelf software package. Therefore we employed SAA for $\text{PNLP}^i_\Diamond$. The SAA method we used is briefly described below.

First, randomly generate S samples of demand for all airlines. For sample s, demand is $d^i(s)$. Second, approximate the objective function of the $\text{PNLP}^i_\Diamond$ by a new function

$$f(x^i, x^{-i}) = \frac{\sum_{s=1}^{S}(r^i)^T \min(x^i, d^i(s) + \sum_{j\neq i} o^{ji}[d^j(s) - x^j]^+)}{S}.$$

Hence we obtain another optimization problem that approximates $\text{PNLP}^i_\Diamond$. Third, solve the new optimization problem. An optimal solution to the new optimization problem is an approximate solution to $\text{PNLP}^i_\Diamond$.

Introduce another variable $y^i(s)$ for each sample s. It turns out that the new optimization problem is equivalent to a linear program defined below:

$$\begin{aligned} \max_{x^i, y^i(s)} \quad & \sum_{s=1}^{S}(r^i)^T y^i(s) \\ \text{s.t.} \quad & A^i x^i \leq C^i, \\ & y^i(s) \leq x^i, \forall s, \\ & y^i(s) \leq d^i(s) + \textstyle\sum_{j\neq i} o^{ji}[d^j - x^j]^+, \forall s, \\ & x^i \geq 0, y^i(s) \geq 0, \forall s. \end{aligned}$$

We experimented with sample size $S = 50$ or $S = 80$. With different sample sizes, it is likely that different approximate solutions for $\text{PNLP}^i_\Diamond$ will result. However, our experiments showed that the sample size does not significantly alter the average total revenue for each player in the game.

4.3 Main computational results

	Example 1			Example 2		
	0.25	0.50	1.0	0.25	0.50	1.0
BLDLP	408,735	409,764	413,192	502,528	505,094	508,348
	408,654	419,347	412,745	501,696	504,253	507,725
BPDLP	356,793	357,261	357,684	489,001	489,001	489,001
	356,237	356,678	357,169	487,627	487,627	487,627
BLPNLP	303,955	319,633	343,610	364,234	377,948	407,965
	303,650	319,403	343,836	363,342	377,155	407,370
BPPNLP	356,760	361,064	361,613	407,463	408,600	410,080
	356,217	360,534	361,015	407,449	408,628	410,271
BLDLPSim	408,653	410,704	412,962	502,607	505,121	508,119
	408,463	410,553	412,867	501,694	504,195	507,342
BPDLPSim	356,797	357,316	357,684	489,001	489,001	489,001
	356,282	356,775	357,169	487,627	487,627	487,627
BLPNLPSim	273,571	274,235	275,113	352,571	348,333	348,678
	273,398	274,100	274,985	351,745	347,583	347,963
BPPNLPSim	356,764	357,233	357,684	407,425	408,504	410,080
	356,139	356,667	357,169	407,463	408,609	410,271
BLDLPMix	408,662	409,858	412,965	502,621	505,049	508,073
	408,634	419,365	412,772	501,785	504,291	507,762
BPDLPMix	356,725	357,246	357,684	489,001	489,001	489,001
	356,260	356,751	357,169	487,627	487,627	487,627
BLPNLPMix	273,072	273,572	274,095	352,327	348,061	348,400
	305,214	324,144	354,679	363,712	377,698	411,144
BPPNLPMix	356,759	354,061	352,003	407,479	408,474	410,080
	356,256	362,528	363,523	407,445	408,601	410,271

Table 1: Revenue comparisons between two airlines under different booking schemes.

Computational results for Examples 1 and 2 are shown in Table 1, where we report the average total revenue for each of two airlines under 12 different booking schemes. The results for each booking scheme are shown in two consecutive rows, in which the top row is for the first airline and the bottom row for the second airline. The results for Examples 1 and 2 are shown in the middle column and the right column respectively. For each example, we tested three different values for the overflow rate: $o_k^{ij} = 0.25$, 0.5 and 1.0 for all $i, j (i \neq j)$, and k. The same results are also depicted in Figure 1.

From both Table 1 and Figure 1, we can make the following observations.

(i) the partitioned booking-limit policy based on DLP performs better than the bid-price policy based on the same DLP for both airlines in both examples.

(ii) the partitioned booking-limit policy based on PNLP performs worse than the bid-price policy based on the same PNLP for both airlines in both examples.

(iii) for Example 1, the performance of the bid-price policy based on DLP is comparable to that of the bid-price policy based on PNLP, but for Example 2, the the bid-price policy based on DLP over-performs the same policy based on PNLP.

(iv) both airlines share revenue fairly evenly for all booking schemes except for `BLPNLPMix` and `BLPNLPMix`. This symmetric property indicates a symmetric equilibrium, which is indeed the case. Exceptions of `BLPNLPMix` and `BLPNLPMix` are logical since distortions are expected when two airlines use different booking schemes for processing booking requests.

(v) as the overflow rate increases, the average total revenues increase correspondingly for most booking schemes, which is intuitively correct because demand increases when the overflow rate increases. However, there are several exceptions such as `BLDLPMix` and `BPPNLPMix` in Example 1, which is once again not surprising because distortions are expected when two airlines use different equilibrium points to process booking requests.

(vi) in this symmetric case, it seems that the average total revenues for `BLDLP`, `BLDLPSim` and `BLDLPMix` are similar. The same can be observed for `BPDLP`, `BPDLPSim` and `BPDLPMix`, and for `BPPNLP`, `BPPNLPSim` and `BPPNLPMix`.

(vii) the booking schemes based on the partitioned booking-limit policy and DLP are the best among all booking schemes. The booking schemes based on PNLP are not competitive. This seems to confirm the existing observations in the literature, which states that the booking policies based on PNLP are not as competitive as the booking policies based on DLP in the monopoly setting.

Examples 3, 4 and 5 are variations of Example 1. These examples were designed to test how a change in product prices, capacities and demand affects the performance of different booking schemes. Numerical results for Examples 3 and 4 are shown in Figure 2 and for Example 5 on the left hand side of Figure 3. We make some additional observations.

(viii) Booking schemes based on the partitioned booking-limit policy and DLP still over-perform any booking scheme based on PNLP in Examples 3, 4 and 5.

(ix) Distortions are revealed on `BPDLPSim` and `BPDLPMix` in Examples 3 and 4, and on `BPDLP` in Example 5.

(x) Booking schemes `BLDLP`, `BLDLPSim` and `BLDLPMix` are stable and perform the best. The performance difference between those booking schemes is not significant and it cannot be said whether or not the difference is due to sampling errors.

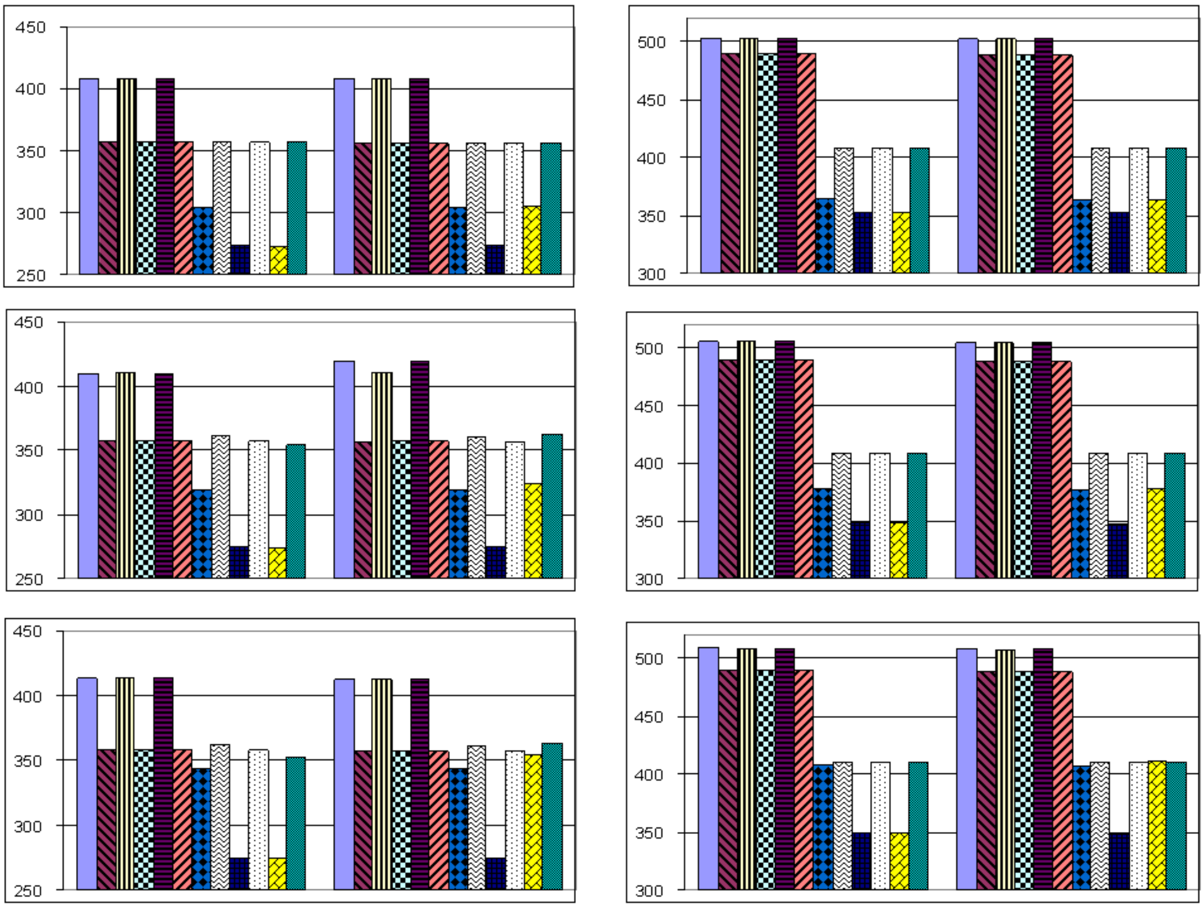

Figure 1: Comparison of average total revenues between two airlines under different booking schemes. The left hand side is for Example 1, where the three charts are for cases when the overflow rate is 0.25 (top), 0.5 (middle) and 1.0 (bottom) respectively. The right hand side is for Example 2, where the three charts are for cases when the overflow rate is 0.25 (top), 0.5 (middle) and 1.0 (bottom) respectively. In each chart, the average total revenues for the first airline are shown in the 12 bars on the left and for the second airlines in the 12 bars on the right. The order of booking schemes for each airline shown in all charts is: `BLDLP`, `BPDLP`, `BLDLPSim`, `BPDLPSim`, `BLDLPMix`, `BPDLPmix`, `BLPNLP`, `BPPNLP`, `BLPNLPSim`, `BPPNLPSim`, `BLPNLPMix`, and `BPPNLPMix`.

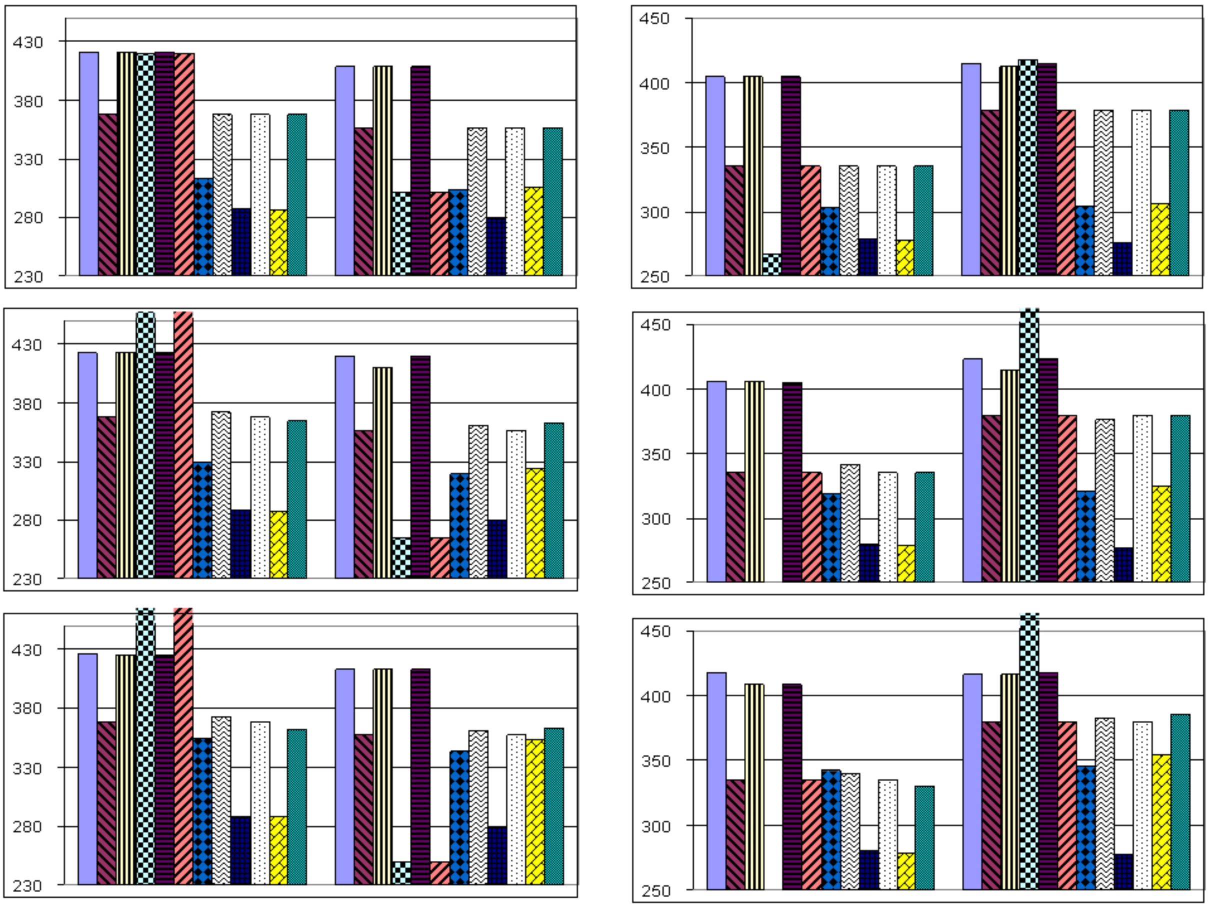

Figure 2: Comparison of average total revenues between two airlines under different booking schemes. The left hand side is for Example 3, and the right hand side for Example 4. See Figure 1 for the meanings of charts.

(xi) The revenue differences between two airlines correctly reflect the settings. In Example 3, the first airline charges higher than the second airline for each product. As expected, the first airline achieves a higher average total revenue than the second. In Example 4, the second airline has more capacity on each leg and hence has a higher average total revenue, which is confirmed by the numerical results. In Example 5, the second airline has more demand than the first and hence has a higher average total revenue, which is confirmed once again. Also, as a general trend in Examples 3, 4 and 5, the average total revenue increases slightly for both airlines when the overflow rate increases.

Example 6 is another variation of Example 1, in which we tested how asymmetric overflow rates affect the performance of different booking schemes. Numerical results for Example 6 are shown on the right top corner of Figure 3. We make a single observation for this example.

(xii) The booking schemes based on the partitioned booking-limit policy and DLP are the best. On the right hand side of Figure 3, we also plotted two diagrams on how overflow rates affect the average total revenue for the first airline in Examples 1 and 2 respectively. It can be seen that the overflow rate does not significantly improve the revenue for the first airline. We think this is due to the fact that the average demand matches the capacity fairly well, which indicates that many overflow booking requests are rejected at the second request. We expect that the revenue should improve significantly when the overflow rate is high and when the demand is small relative to the capacity.

Based on our limited numerical experiments and analysis, we conclude that the best approach among all 12 booking schemes proposed in this paper for network capacity management competition is the partitioned booking-limit policy based on DLP.

4.4 Additional computational results

In the previous subsection, we compared revenue performances of various booking schemes assuming both airlines employ the same booking scheme. In reality, different airlines may use different booking schemes. A natural question is how robust a booking scheme is with respect to different booking schemes employed by competing airlines. To answer this question, we conducted one more set of experiments in Examples 7 and 8, which are variations of Examples 1 and 2 respectively. The parameter values in Example 7 (or 8) are exactly as the same as those in Example 1 (or 2). In both Examples 7 and 8, we assumed that the first airline always used booking scheme `BLDLP`, which was numerically demonstrated to be the best booking scheme in the previous subsection, and the second airline used other different booking schemes. In particular, the second airline used the following 8 booking schemes: `BLDLP`, `BPDLP`, `BLDLPSim`, `BPDLPSim`, `BLPNLP`, `BPPNLP`, `BLPNLPSim`, and `BPPNLPSim`. Evidently, when the second airline uses booking scheme `BLDLP`, Example 7 is completely identical to Example 1.

The numerical results for Examples 7 and 8 are shown in Figure 4. The results show that the first airline receives a stable and similar average total revenue no matter what booking scheme the second airline takes. This observation is true for both Examples 7 and 8. On the other hand, the average total revenues for the second airline depend on booking schemes and are sometimes significantly lower than those obtained by the first airline even though both airlines compete in a symmetric game. We have also conducted experiments with variations of Examples 3, 4, 5 and 6, where the first airline always used booking scheme `BLDLP` and the second airline used other

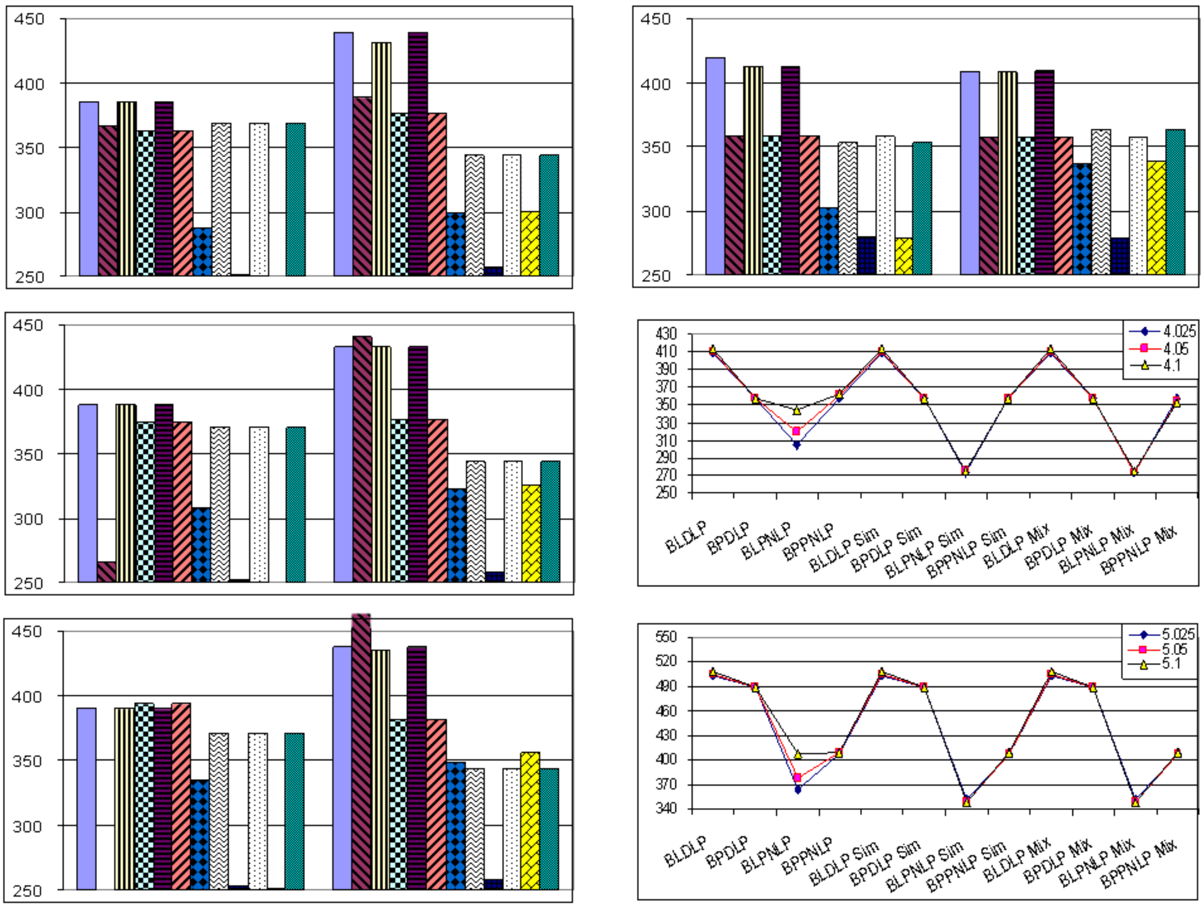

Figure 3: Comparison of the average total revenue between two airlines under different booking schemes in Examples 5 and 6 and impact of overflow rates on the average total revenue in Example 1. The left hand side is for Example 5. The bar chart at the right top corner is for Example 6. The two line charts on the right hand side are for Example 1. See Figure 1 for the meanings of charts.

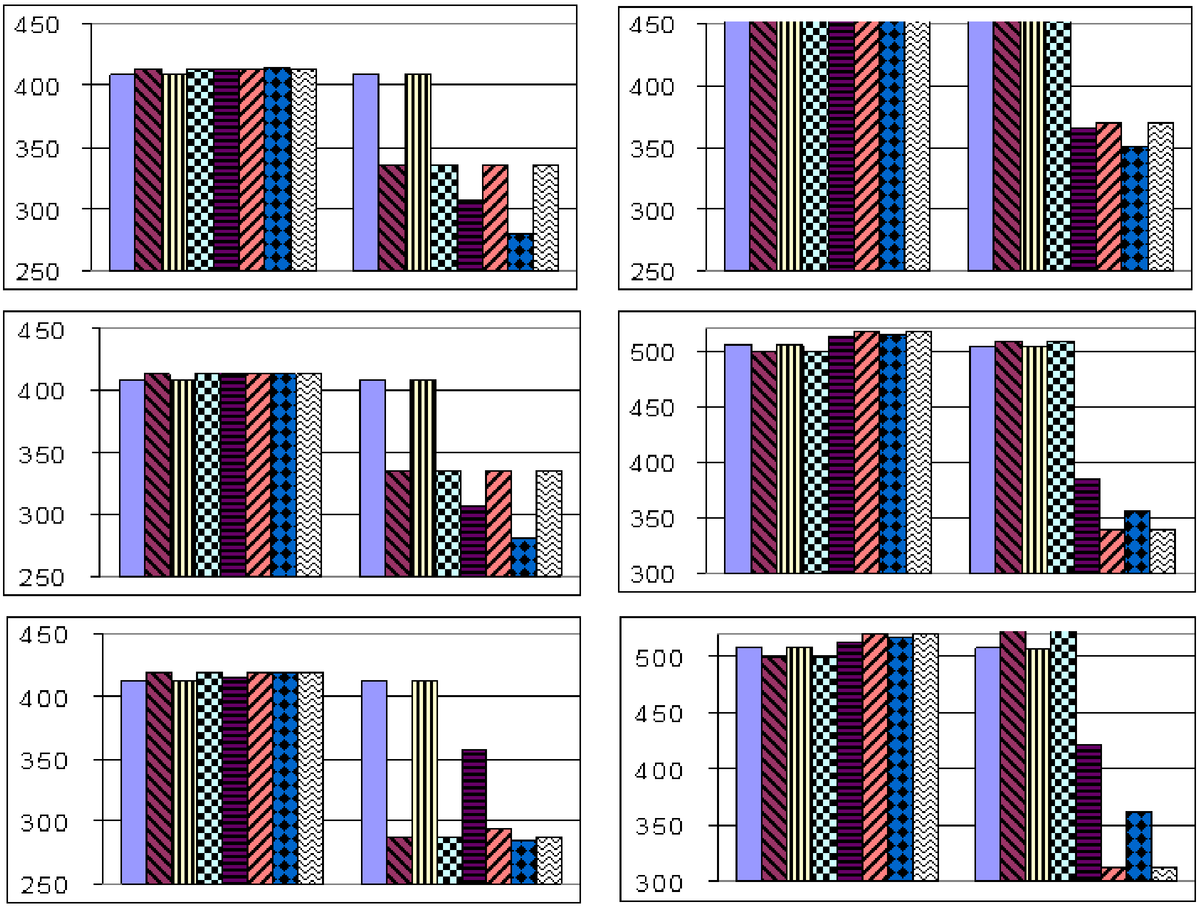

Figure 4: Comparison of the average total revenue between two airlines under different booking schemes in Examples 7 and 8. The three charts on the left hand side are for Example 7, and the three charts on the right hand side are for Example 8. The order of booking schemes for each airline shown in all charts is: `BLDLP`, `BPDLP`, `BLDLPSim`, `BPDLPSim`, `BLPNLP`, `BPPNLP`, `BLPNLPSim`, and `BPPNLPSim`. See Figure 1 for the meanings of charts.

booking schemes. The results are extremely similar to those of Examples 7 and 8, and hence are not reported here.

The conclusion we can make from the above additional numerical experiments is that booking scheme `BLDLP` is robust in the sense that the airline can still receive a stable and good average total revenue no matter which booking scheme is taken by competing airlines.

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