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Jiang, H., Netessine, S. and Savin, S.

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Author contact details are as follows:

Houyuan Jiang
Judge Business School
University of Cambridge
h.jiang@jbs.cam.ac.uk

Serguei Netessine
Wharton School of Business
University of Pennsylvania
netessine@wharton.upenn.edu

Sergei V Savin
Columbia Business School
New York
svs30@columbia.edu

Robust Newsvendor Competition

Houyuan Jiang[*], Serguei Netessine[†], and Sergei Savin[‡]

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Abstract

We analyze competition among newsvendors when the only information competitors possess about the nature of future demand realizations is the support of demand distributions. In such a setting, traditional expectation-based optimization criteria may not be adequate. In our analysis, we focus on several alternative criteria used in the robust optimization literature, such as relative and absolute regret, as well as worst-case performance. Using these robust criteria, we establish the unique Nash equilibrium solution for a (symmetric) game with an arbitrary number of players. In addition, we obtain closed-form, intuitive expressions for the optimal order quantities which allow us to gain insight into the nature of robust competition. We show that the ex-ante and ex-post versions of the competitive newsvendor problem are equivalent under the worst-case or the absolute regret or the relative regret criterion. Numerical analysis indicates that, among different robust approaches, absolute regret minimization offers the most sensible alternative when demand distribution is unknown.

Keywords: Robust optimization, newsvendor competition, absolute regret, relative regret.

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[*]Judge Business School, University of Cambridge.

[†]The Wharton School of Business, University of Pennsylvania.

[‡]Graduate School of Business, Columbia University.

1 Introduction

Classical operations management models often assume that a decision maker has complete knowledge of the distributions of uncertain parameters (typically, consumer demand for products or services). This is a sensible assumption for most mature products when historical demand information is available and there is sufficient reason to believe that future demand distribution can be forecasted using historical information. Even in cases when historical demand information is unavailable and the product is new, expert judgments can be solicited to predict the distribution of future demand. However, in many practical settings such an approach is either impractical or unreliable. For example, expert predictions of demand uncertainty are known to be biased (see, for example, Soll and Klayman [30]) and very little is known about experts' ability to predict the shape of the demand distribution. Realization of such limitations has led in recent years to the rapid development of a new kind of stochastic optimization paradigm which is based on the notion of "robustness." A robust solution typically ensures a certain level of performance irrespective of the underlying distributions of the involved random parameters.

The difficulty in forecasting demand for a single product is most extreme in multi-item problems, one example of which is stock-out-based demand substitution. In the classical incarnation of this problem consumers arrive with a product preference in mind (we call this the *primary* demand). If their primary product is out of stock, consumers may substitute it with one of the alternatives (the *secondary* demand). Demand substitution is pervasive, for example, in retail situations, and its importance is well-documented. A recent survey of retailers has found that, of the customers who do not find what they want on the shelf, 40% either defer the purchase or go to a competitor store to find the item (see Andraski and Haedicke [2]). Naturally, demand substitution needs to be accounted for when optimizing inventory management. For this purpose, extensive literature on demand substitution has been developed which relies on the knowledge of the joint probability of the primary demand distribution as well as the substitution behavior of consumers. Naturally, estimation of such joint multivariate demand distribution is a non-trivial task even in the presence of historical demand information. The reason is that the retailer is typically unable to observe whether the product has been purchased because it was the first-choice product or because the customer substituted some other out-of-stock product. These practical considerations drive our effort to apply a robust optimization approach to the multi-item inventory model with demand substitution.

As a basis of our analysis, we use the demand substitution model pioneered by McGillivray and Silver [18] and first studied in the competitive framework by Parlar [23]. In this model, consumers arrive with a product preference in mind, and a fixed proportion of these consumers wishing to purchase product i substitute product j for it if product i is out of stock. We focus on

the competitive, single-period version of this problem, which is often referred to as "competitive newsvendors." Instead of assuming that the primary demand distribution is known, we base our analysis only on the assumption that the support for the primary demand distribution is known. We then use several robust optimization criteria to find the Nash equilibrium of this game. Specifically, we use the maximin approach, the absolute (ex-post and ex-ante) regret minimization, and the relative (ex-post and ex-ante) regret minimization. We show that the maximin approach is unsatisfactory for analyzing this game because it results in the same solution with and without competition. On the other hand, both the absolute and the relative regret minimizations produce closed-form solutions which are amenable to interpretations. Moreover, we show that, although the absolute and the relative regret solutions differ, ex-ante and ex-post solutions coincide for each of these two regret alternatives. These solutions can be interpreted as intuitive modifications of the noncompetitive newsvendor solution. Numerical experiments indicate that the absolute regret minimization approach is quite sensible in that, while producing robust outcomes, it also results in solutions that are not far off from the solution obtained when demand distribution is known and the expected profit is maximized. Thus, we argue, robust optimization approaches are instrumental in gaining deeper insights into the newsvendor competition problem. The key contribution of this paper is in that it not only develops a methodology to study the newsvendor competition problem in settings where only the support of the demand distribution is known but also shows that this seemingly more difficult problem often possesses a solution more tractable than the solution of the problem with known demand distribution.

2   Literature Survey

Two research streams that are closely related to our work study, respectively, demand substitution under competition and robust stochastic optimization. Studies of demand substitution in the context of single-period models with demand uncertainty were pioneered by McGillivray and Silver [18]. A large number of papers followed this cornerstone work but we only survey papers in this stream that focus on strategic interactions under demand substitution. Parlar [23] is the first paper to study demand substitution under competition for two players. Wang and Parlar [32] extend this analysis to three players while Netessine and Rudi [20] generalize it to an arbitrary number of players. Kraiselburd et al. [14] study contracting in a supply chain when retailers compete through demand spillovers. Netessine and Zhang [22] generalize this analysis to complementary products, and Netessine and Shumsky [21] analyze competition with spillovers between two airlines that segment customers into two classes. In addition to the

widely studied basic model of McGillivray and Silver [18], other, more sophisticated models of demand substitution have been proposed (e.g., Lippman and McCardle [16], Mahajan and van Ryzin [17], and Bernstein and Federgruen [6]). However, in our analysis we focus on the most basic model of demand substitution. Two papers, Anupindi et al. [3] and Kok and Fisher [13], offer demand estimation procedures for substitution models but under centralized inventory management. These papers indicate that accurate demand estimation is far from trivial even for centralized inventory management of substitutable products.

The extant literature offers several robust optimization approaches. The earliest is probably the maximin approach of Scarf [29], who investigates a firm's decision to select an order quantity to maximize its worst-case profit under demand uncertainty. Scarf assumes that only mean and standard deviation of future demand is known. Scarf [29] and, later, Gallego and Moon [10] demonstrate that the worst-case distribution is discrete with two mass points and obtain the expressions for the optimal order quantity and for the resulting profit. To correct for the overly conservative nature of such an approach to profit maximization, Ben-Tal and Nemirovski [5] and, later, Bertsimas and Sim [8] employ the notion of "budget of uncertainty."

An example of a less conservative robust optimization criterion is the so-called minimax-regret introduced by Savage [28]. Under this criterion, a firm minimizes the maximum absolute regret of making a suboptimal decision. In the stochastic inventory model the absolute regret can be defined before or after the demand realization which results, respectively, in ex-ante or ex-post versions of regret minimization. Morris and Yi [19], Kasugai and Kasegai [12], and Vairaktarakis [31] study the notion of minimax absolute ex-post regret for the newsvendor problem. Perakis and Roels [24] and Yue et al. [33] study the minimax absolute ex-ante regret newsvendor problem and derive the optimal order quantities in the presence of limited demand information, such as the moments (mean and variance) or the shape (support, symmetry and unimodality) of the demand distribution. Perakis and Roels [25] point out that the minimax absolute ex-ante regret approach parallels the entropy maximization studied by Jaynes [11] and hence is intuitively appealing. Eren and Maglaras [9] use entropy maximization to update the booking limits for a revenue management problem while obtaining demand information.

A third variety of the robust profit optimization deals with the notion of relative regret. Zhu et al. [34] derive the optimal order quantities in the newsvendor model under the relative regret criterion when support of the distribution and either mean or the standard deviation of demand are known. Ball and Queyranne [4] use relative regret in the context of a single-leg revenue management problem. This work is extended by Lan et al. [15] who consider both relative and absolute regrets and propose new static and dynamic booking control policies for a single-leg, multiple-fare class problem in cases when only upper/lower bounds on demand are available.

As is evident from our survey, analysis of robust policies has been almost exclusively limited to the monopolistic setting with a couple of exceptions. Aghassi and Bertsimas [1] introduce the notion of robust games based on the worst-case analysis. They show the existence of mixed-strategies Nash equilibria with or without private information, compute mixed-strategies Nash equilibria, and provide comparisons between robust and Bayesian games. Another exception is Perakis and Roels [26], who study a two-echelon supply chain under unknown demand distribution and price-only contract. This paper is, perhaps, the closest to ours, because in one instance (section 6) Perakis and Roels study a supply chain with symmetric competing newsvendors. But Perakis and Roels [26] assume that demand distribution possesses the increasing generalized failure rate property whereas we assume that the support of the distribution is known. Furthermore, they do not model demand overflow/substitution and instead reallocate excess demand using an approach similar to the one in Lippman and McCardle [16]. In another related paper, Lan et al. [15] study the single-leg revenue management problem under both the absolute and the relative regret minimization criteria and, similar to our work, assume that only the support of the demand distribution is known.

3   The Model

Consider a market populated by N newsvendors, each selling a different product. For newsvendor $i = 1, ..., N$, we denote the product selling price and procurement costs by p_i and c_i, respectively. We assume that customers arrive with a product preference in mind so that each newsvendor faces random primary demand denoted by D_i with the support $[A_i, B_i]$. Moreover, if product i is out of stock, a proportion o_{ji} of customers unsatisfied by newsvendor i "spills over" to (substitutes) newsvendor j. We will assume throughout that $\sum_{j \neq i} o_{ji} \leq 1$ for any $i = 1, ..., N$. Given the set of product order quantities $\mathbf{Q} = (Q_1, ..., Q_N)$ selected by newsvendors at the beginning of the period and the set of potential demand distributions $\mathbf{F} = (F_1, ..., F_N)$, the total expected profit for newsvendor i is given by

$$\Pi_i(\mathbf{Q}, \mathbf{F}) = -c_i Q_i + p_i E_{\mathbf{F}} \left[\min(D_i^E, Q_i) \right], \tag{1}$$

where $D_i^E = D_i + \sum_{j \neq i} o_{ij}(D_j - Q_j)^+$ is the effective demand for newsvendor i, which accounts for the demand spillover from other newsvendors. For convenience, we will denote by $\left[A_i, B_i^E \right]$ support of demand distribution for D_i^E where $B_i^E = B_i + \sum_{j \neq i} o_{ij}(B_j - Q_j)^+$. We will use $\mathcal{D}$ to denote the domain for the demand distributions $\mathbf{F}$. Note that (1) reduces to the classical newsvendor profit functions when $o_{ij} = 0$ for all $j \neq i$. For the game-theoretic analysis below, it is convenient to introduce the following commonly used "$i-$centered" notation for

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$\mathbf{Q} = (Q_1, ..., Q_N)$ and $\mathbf{F} = (F_1, ..., F_N)$: $\mathbf{Q} = (Q_i, \mathbf{Q}_{-i})$ and $\mathbf{F} = (F_i, \mathbf{F}_{-i})$. In traditional analysis of the newsvendor game (see Netessine and Rudi [20]) newsvendor i selects Q_i to maximize her expected profit by solving the following maximization problem:

$$\max_{Q_i \geq 0} \Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}), \forall i \tag{2}$$

in which $\mathbf{Q}_{-i}$ and $\mathbf{F}$ are assumed to be given. Here we revisit the solution of this problem:

Proposition 1 *For the newsvendor game defined in (2):*

(a) *(Netessine and Rudi [20]) For any known continuous demand distribution $\mathbf{F}$, there exists a Nash equilibrium solution $\mathbf{Q}^*$ satisfying the following optimality equations:*

$$P(D_i \leq Q_i^*) - P\left(D_i < Q_i^* < D_i + \sum_{j \neq i} o_{ij}(D_j - Q_j^*)^+\right) = (p_i - c_i)/p_i, \forall i. \tag{3}$$

(b) *Consider a symmetric two-player game with uniformly distributed demand: i.e., let $A_i = A$, $B_i = B$, $c_i = c$, $p_i = p$, and $o_{ij} = \gamma$, $i, j = 1, 2$. Then the unique Nash equilibrium is*

$$Q^* = \begin{cases} B + (B - A)\left(1 - \sqrt{1 + 2\gamma c/p}\right)/\gamma, & \text{if } (2 + 3\gamma)p \geq 2c(1 + \gamma)^2, \\ A + (B - A)\sqrt{2\gamma(p - c)/((1 + 2\gamma)p)}, & \text{otherwise.} \end{cases} \tag{4}$$

In particular, when $\gamma = 0$, $Q^* = (c/p)A + (1 - c/p)B$.

We state Proposition 1 to illustrate two points. First, if demand distribution is known, the equilibrium solution is quite complex and can be stated only implicitly: one must solve a system of simultaneous equations (3) with each equation involving multidimensional integrals over the regions that themselves depend on the values of decision variables. Likewise, any parametric sensitivity analysis is quite complex in this case because it requires implicit differentiation of the system of equations. The second part of Proposition 1 illustrates that, when the simplest possible distributional form (the uniform distribution), as well as problem symmetry is assumed, the problem becomes more tractable, although even in this case one has to worry about different solutions in certain parameter ranges. This example with uniform demand distribution is useful because, without the effect of competition through demand spillovers, absolute regret minimization in the newsvendor problem coincides with the solution that uses uniform demand distribution (see also Vairaktarakis [31]). As will be evident shortly, this is not the case in competitive models.

The rest of the paper is organized as follows. Section 4 studies a newsvendor game under the worst-case (or maximin) criterion. We prove that, under this criterion, the ex-ante maximin

problem and the ex-post maximin problem are equivalent and then show that the solution with and without competition is the same. In section 5.1, we introduce a newsvendor game in which participants minimize the absolute ex-ante regret and find its solution. In section 5.2, we prove the equivalence between the newsvendor games under the absolute ex-post and ex-ante regret minimization criteria. In section 6.1 we analytically solve a minimax relative ex-post regret problem, and in section 6.2 we show that a minimax relative ex-ante regret problem is equivalent to a maximin relative ex-post regret problem. Computational comparisons between the robust approaches and the traditional approach which maximizes expected profits are presented in section 7. We conclude with a summary of our findings in section 8.

4 The Maximin (Worst-Case) Approach

We begin with the most conservative of all robust optimization approaches, the maximin criterion. Under the ex-ante maximin approach, newsvendor i determines the optimal order quantity by solving the following optimization problem:

$$\max_{Q_i \geq 0} \left(\min_{\mathbf{F} \in \mathcal{D}} \left(-c_i Q_i + p_i E_{\mathbf{F}}[\min(D_i^E, Q_i)] \right) \right), \tag{5}$$

hence the term "maximin." Clearly, this approach is very conservative in that the newsvendor seeks protection against the worst possible outcome. Likewise, under the ex-post maximin approach, newsvendor i determines the optimal quantity by solving the following optimization problem:

$$\max_{Q_i \geq 0} \left(\min_{\mathbf{D} \in [\mathbf{A}, \mathbf{B}]} \left(-c_i Q_i + p_i \min(D_i^E, Q_i) \right) \right). \tag{6}$$

The following proposition establishes equivalence of these two formulations and finds the Nash equilibrium order quantities.

Proposition 2 *(a) The optimal value for the ex-ante maximin problem (5) is attained at a point in the interval $[A_i, B_i^E]$.*

(b) *The optimal value for the inner minimization problem of (5) is achieved at $\mathbf{F}$ such that for newsvendor i, F_i has a discrete distribution with a unit impulse at A_i.*

(c) *Both ex-ante and ex-post maximin problems are equivalent in the sense that for newsvendor i, both problems have the same objective function with respect to Q_i and hence the same unique Nash equilibrium solution, such that $Q_i^* = A_i$, $\forall i$.*

We see that the Nash equilibrium solution under the maximin criterion is somewhat unappealing: essentially, each firm's stocking quantity is set at the point of lowest possible demand

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realization. This solution essentially ignores demand substitution and competition – indeed, it is easy to demonstrate that a stand-alone newsvendor would stock inventory in a similar way. Moreover, any N newsvendors under the centralized inventory management would stock inventory in the exact same way. We conclude that the overly conservative nature of the maximin criterion may limit its applicability when modeling competitive situations such as ours.

5 Absolute Regret Criterion

In this section we apply the absolute regret criterion to the newsvendor competition problem. This robust optimization approach was studied in the context of the classical newsvendor problem by Vairaktarakis [31] and further developed by Perakis and Roels [24].

5.1 Absolute Ex-Ante Regret

We define the absolute ex-ante regret for newsvendor i as

$$\Delta_i^{\mathrm{ea}}\left(\mathbf{Q}, \mathbf{F}\right) = \max_{\widehat{Q}_i \geq 0} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) - \Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}) \right). \tag{7}$$

Given this definition, the minimax absolute ex-ante regret minimization problem for newsvendor i can be stated as follows:

$$R^{\mathrm{ea}}\left(\mathbf{Q}_{-i}\right) = \min_{Q_i \geq 0} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\Delta_i^{\mathrm{ea}}\left(\mathbf{Q}, \mathbf{F}\right) \right) \right). \tag{8}$$

Reverting the order of the two maximizations in (8), we obtain an equivalent formulation:

$$\min_{Q_i \geq 0} \left(\max_{\widehat{Q}_i \geq 0} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) - \Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}) \right) \right) \right)$$

$$= \min_{Q_i \geq 0} \left(\max_{\widehat{Q}_i \geq 0} \left(c_i(Q_i - \widehat{Q}_i) + p_i \max_{\mathbf{F} \in \mathcal{D}} \left(E_{\mathbf{F}}\left[\min(D_i^E, \widehat{Q}_i) \right] - E_{\mathbf{F}}\left[\min(D_i^E, Q_i) \right] \right) \right) \right). \tag{9}$$

The feasible regions for both Q_i and $\widehat{Q}_i$ in the optimization problems (7) and (9) can be reduced as shown in the following lemma:

Lemma 1 (a) The optimal value for the optimization problem with respect to $\widehat{Q}_i$ is attained at a point in the interval $[A_i, B_i^E]$, $\forall i$.

(b) The optimal value for the optimization problem with respect to Q_i is attained at a point in the interval $[A_i, B_i^E]$, $\forall i$.

In view of this result, we can simplify the problem, which now becomes

$$\min_{A_i \le Q_i \le B_i^E} \left(\max_{A_i \le \widehat{Q}_i \le B_i^E} \left(c_i(Q_i - \widehat{Q}_i) + p_i \max_{\mathbf{F} \in \mathcal{D}} \left(E_{\mathbf{F}} \left[\min(D_i^E, \widehat{Q}_i) \right] - E_{\mathbf{F}} \left[\min(D_i^E, Q_i) \right] \right) \right) \right). \tag{10}$$

In the following proposition we solve the inner maximization problem in (10) analytically.

Proposition 3 *Consider two sets of order quantities $\{Q_i\}$ and $\left\{ \widehat{Q}_i \right\}$ such that $A_i \le Q_i \le B_i^E$ and $A_i \le \widehat{Q}_i \le B_i^E$ for all i, and define the set of values $\left\{ \widehat{D}_i \right\}$ such that*

$$\widehat{Q}_i = \widehat{D}_i + \sum_{j \ne i} o_{ij} (\widehat{D}_j - Q_j)^+. \tag{11}$$

Further, consider a particular joint probability distribution $\widehat{\mathbf{F}}$ for the demand of all newsvendors such that $\widehat{F}_i$ is a unit impulse probability distribution with mass at $\widehat{D}_i$ for all i. Then $\widehat{\mathbf{F}}$ is the optimal solution to the inner maximization problem of the minimax absolute ex-ante regret problem (10): $\max_{\mathbf{F} \in \mathcal{D}} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) - \Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}) \right).$

This finding is consistent with Perakis and Roels [24] and Bertsimas and Popescu [7], who show that the inner maximization problem in (10) is equivalent to a moment-bound problem so that the distribution achieving the maximum regret is discrete with a single mass point. If we use this observation, the following proposition identifies the Nash equilibrium in this problem and is the key result of this section.

Proposition 4 *(a) The best response function of newsvendor i is uniquely determined by $Q_i^* = \frac{c_i}{p_i} A_i + \frac{p_i - c_i}{p_i} B_i^E$.*

(b) There exists a Nash equilibrium solution satisfying the following system of nonlinear and non-smooth equations:

$$Q_i^* = \frac{c_i}{p_i} A_i + \frac{p_i - c_i}{p_i} \left(B_i + \sum_{j \ne i} o_{ij} (B_j - Q_j^*)^+ \right), \forall i. \tag{12}$$

Note that (12) generalizes the solution to the classical newsvendor problem under the regret minimization criterion, which can be obtained by letting $o_{ij} \equiv 0, \forall i, j$. In this case, $Q_i^{NV} = \frac{c_i}{p_i} A_i + \frac{p_i - c_i}{p_i} B_i, \forall i$, a solution that coincides with that of Vairaktarakis [31]. This solution can be thought of as the weighted average of the lower and the upper bounds on demand distribution. As Vairaktarakis notes, this solution can also be obtained by assuming that demand distribution is uniform on $[A_i, B_i]$. Note, however, that the competitive newsvendor solution differs from the solution obtained in Proposition 1 that uses the uniform demand distribution. Instead, the

upper bound B_i is increased to B_i^E to account for substitution. It is easily verified that each competing newsvendor stocks a higher quantity than Q_i^{NV} because of the higher upper bound on demand distribution. As is evident from Proposition 4, absolute regret minimization allows us to obtain a simple yet intuitive solution for the competitive newsvendor problem. However, even this simple solution is implicit, since equilibrium decisions appear both on the left- and the right-hand side of (12). In two special cases we are able to obtain closed-form solutions for this game with an arbitrary number of players.

Proposition 5 *(a) Suppose that*

$$\sum_{j \neq i} \frac{o_{ij} c_j}{p_j} (B_j - A_j) \leq \frac{c_i}{p_i - c_i} (B_i - A_i), \forall i. \tag{13}$$

Then $Q_i^* \leq B_i$ and the unique Nash equilibrium $\mathbf{Q}^*$ can be obtained as follows. Define

$$\mathbf{A} = \begin{vmatrix} 1 & o_{12}(1 - \frac{c_1}{p_1}) & \ldots & o_{1N}(1 - \frac{c_1}{p_1}) \\ o_{21}(1 - \frac{c_2}{p_2}) & 1 & \ldots & o_{2N}(1 - \frac{c_2}{p_2}) \\ \ldots & \ldots & \ldots & \ldots \\ o_{N1}(1 - \frac{c_N}{p_N}) & o_{N2}(1 - \frac{c_N}{p_N}) & \ldots & 1 \end{vmatrix},$$

$$\mathbf{B}_i = \begin{vmatrix} 1 & o_{12}(1 - \frac{c_1}{p_1}) & \ldots & o_{1,i-1}(1 - \frac{c_1}{p_1}) & \sigma_{i1} & \ldots & o_{1N}(1 - \frac{c_1}{p_1}) \\ o_{21}(1 - \frac{c_2}{p_2}) & 1 & \ldots & o_{2,i-1}(1 - \frac{c_2}{p_2}) & \sigma_{i2} & \ldots & o_{2N}(1 - \frac{c_2}{p_2}) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ o_{N1}(1 - \frac{c_N}{p_N}) & o_{N2}(1 - \frac{c_N}{p_N}) & \ldots & o_{N,i-1}(1 - \frac{c_N}{p_N}) & \sigma_{iN} & \ldots & 1 \end{vmatrix},$$

where $\sigma_{ij} = \frac{p_i B_i - c_i(B_i - A_i)}{p_i} + (1 - \frac{c_N}{p_N}) \sum_{j \neq i} o_{ij} B_j$. Then the equilibrium solution is:

$$Q_i^* = \frac{\det \mathbf{B}_i}{\det \mathbf{A}}, \forall i. \tag{14}$$

(b) There exists at least one newsvendor i_0 such that $Q_{i_0}^* \leq B_{i_0}$.

(c) Consider a symmetric game, such that $A_i = A$, $B_i = B$, $c_i = c$, $p_i = p$, and $o_{ij} = o$, $\forall i, j$, and let $\widehat{p} = p + o(p - c)(N - 1)$. Then, the unique Nash equilibrium is given by

$$Q_i^* = \frac{c}{\widehat{p}} A + \frac{\widehat{p} - c}{\widehat{p}} B, \forall i. \tag{15}$$

In (a), we have a closed-form solution to the game in a case when each optimal stocking quantity does not exceed the upper bound on the primary demand distribution. This assumption is quite reasonable in settings where substitution proportions o_{ij} are relatively small. In these

settings, the solution is obtained by solving a system of linear equations. In (b) we show that at least one stocking quantity will not exceed the upper bound on the primary demand distribution. This result is intuitive: all newsvendors together would not stock more than the upper bound on the aggregate demand. Finally, in (c) we have a simple closed-form solution for the symmetric game. Such a simple solution arises from considering cases (a) and (b) together: when the problem is symmetric, it is easy to demonstrate that all equilibrium order quantities will not exceed upper bounds on the primary demand and, therefore, the solution in (a) applies. Furthermore, this solution takes a particularly intuitive form: it replicates the simple newsvendor solution but with an adjusted price $\widehat{p}$ that accounts for demand substitution. This dependence results in simple comparative statics: the equilibrium stocking quantity is clearly increasing in $\widehat{p}$ and hence it is also increasing in substituting fraction o and the number of newsvendors $(N-1)$, approaching the upper bound B on demand distribution. To conclude this section, we obtain a closed-form solution for a problem with arbitrary cost/revenue parameters but with only two newsvendors.

Proposition 6 *Let $N = 2$, and for $i, j = 1, 2$, $i \neq j$, define*

$$\overline{o}_{ij} = \left(\frac{c_i/p_i}{1 - c_i/p_i} \right) \frac{(B_i - A_i)}{(B_j - A_j)\, c_j/p_j}. \tag{16}$$

Note that it is impossible to have both $o_{ij} > \overline{o}_{ij}$ and $o_{ji} > \overline{o}_{ji}$.

(a) If $o_{ij} \leq \overline{o}_{ij}$ and $o_{ji} \leq \overline{o}_{ji}$, the unique Nash equilibrium is

$$Q_i^* = \frac{\frac{c_i}{p_i}A_i + \left(1 - \frac{c_i}{p_i}\right)B_i + \left(1 - \frac{c_i}{p_i}\right)\frac{c_j}{p_j}o_{ij}(B_j - A_j) - \left(1 - \frac{c_i}{p_i}\right)\left(1 - \frac{c_j}{p_j}\right)o_{ij}o_{ji}B_i}{1 - \left(1 - \frac{c_i}{p_i}\right)\left(1 - \frac{c_j}{p_j}\right)o_{ij}o_{ji}}, \quad i = 1, 2. \tag{17}$$

Furthermore, $Q_i^* \leq B_i$ and $Q_j^* \leq B_j$.

(b) If $o_{ij} \leq \overline{o}_{ij}$ and $o_{ji} > \overline{o}_{ji}$, the unique Nash equilibrium is

$$
\begin{aligned}
Q_i^* &= \frac{c_i}{p_i}A_i + \left(1 - \frac{c_i}{p_i}\right)B_i, \\
Q_j^* &= \frac{c_j}{p_j}A_j + \left(1 - \frac{c_j}{p_j}\right)B_j + \left(1 - \frac{c_j}{p_j}\right)o_{ji}\frac{c_i}{p_i}(B_i - A_i).
\end{aligned} \tag{18}
$$

Furthermore, $Q_i^* \leq B_i$ and $Q_j^* > B_j$.

The last proposition provides additional insights into the equilibrium solution for asymmetric newsvendors. Threshold values for substitution fractions (16) play the same role as condition (13): when both substitution fractions are small enough, as in case (a), both competing

newsvendors select stocking quantities exceeding the newsvendor solution but lower than the upper bound on the primary demand. The solution in this case is obtained by evaluating (14) and the resulting expression is quite easy to interpret. The first two terms in the numerator $\frac{c_i}{p_i}A_i + \left(1 - \frac{c_i}{p_i}\right)B_i$ reflect the classical newsvendor solution that the newsvendor would follow when faced with the primary demand only. The third term $\left(1 - \frac{c_i}{p_i}\right)\frac{c_j}{p_j}o_{ij}(B_j - A_j)$ accounts for demand overflow from retailer j to retailer i. Clearly, the stocking quantity is higher if more customers are willing to substitute (higher o_{ij}), if support of demand distribution for retailer j is bigger (higher $B_j - A_j$), if retailer i earns a higher margin or if retailer j earns a lower margin (because in this case retailer j stocks less and there is a higher likelihood of overflow demand). Finally, the fourth term $\left(1 - \frac{c_i}{p_i}\right)\left(1 - \frac{c_j}{p_j}\right)o_{ij}o_{ji}B_i$ indicates that the stocking quantity is reduced because of substitution from retailer i to retailer j resulting in a higher stocking quantity for retailer j and, as a result of competitive interactions, lower demand and stocking quantity for retailer i. In case (b), when substitution fractions are asymmetric (one higher and one lower than the thresholds), one of the newsvendors elects to stock more than the upper bound on its primary demand. Thus, the second newsvendor does not expect any demand spillovers and elects to stock the classical newsvendor quantity Q_i^{NV}. Clearly, in this case the solution does not depend on one of the substitution fractions, o_{ij}. This simple and intuitive reflection of competitive interactions is not available in newsvendor competition models that assume knowledge of demand distribution (Netessine and Rudi [20], Lippman and McCardle [16]).

5.2 Absolute Ex-Post Regret

As opposed to the ex-ante regret, the ex-post regret is determined after the demand realization is known. In this subsection we demonstrate that the ex-post absolute regret minimization problem results in the same solution as the ex-ante absolute regret minimization problem and hence the same insights apply. In particular, it is easy to see that, given inventory levels $\mathbf{Q}_{-i}$ for all other newsvendors and demand realization $\mathbf{D} = (D_1, \cdots, D_N)$, the best policy for newsvendor i is to order a product quantity $D_i^E = D_i + \sum_{j \neq i} o_{ij}(D_j - Q_j)^+$ in order to match supply with demand exactly. This policy produces $(p_i - c_i)D_i^E$ in profits and any deviation from this amount will be the ex-post regret. Thus, we can formally define the absolute ex-post regret for newsvendor i as

$$\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D}) - \Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D}) = (p_i - c_i)D_i^E - p_i \min(D_i^E, Q_i) + c_i Q_i, \qquad (19)$$

where $\Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D}) = p_i \min(D_i^E, Q_i) - c_i Q_i$ and $\widehat{Q}_i = D_i^E$. We then define the minimax absolute ex-post regret minimization problem for newsvendor i as

$$\min_{Q_i \geq 0} \left(\max_{\mathbf{D} \in [\mathbf{A}, \mathbf{B}]} \left(\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D}) - \Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D}) \right) \right). \qquad (20)$$

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Proposition 7 concisely states the solution to (20):

Proposition 7 *There exists a Nash equilibrium solution to the minimax absolute ex-post regret problem satisfying $Q_i^* = \frac{c_i}{p_i} A_i + \frac{p_i - c_i}{p_i} B_i^E$, $\forall i$. Moreover, minimax absolute ex-post and ex-ante regret problems (20) and (8) are equivalent.*

As is evident from the last result, the solution to the absolute ex-post regret minimization problem is equivalent to the solution to the absolute ex-ante regret minimization problem and therefore all results (Propositions 4-6) remain valid in the absolute ex-post regret setting as well.

6 Relative Regret Criterion

In this section we apply the relative regret minimization criterion to the newsvendor competition problem. This robust optimization approach was first applied to the newsvendor model by Zhu et al. [34].

6.1 Relative Ex-Post Regret

The relative ex-post regret is defined after observing demand realization. In this case, given inventory levels $\mathbf{Q}_{-i}$ for all competitors and demand realization $\mathbf{D} = (D_1, \cdots, D_N)$, the best policy for newsvendor i is to order a quantity of D_i^E resulting in $(p_i - c_i)D_i^E$ in profits for newsvendor i. We therefore define the relative ex-post regret for newsvendor i as

$$\frac{\Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D})}{\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D})} = \frac{p_i \min(D_i^E, Q_i) - c_i Q_i}{(p_i - c_i)D_i^E}, \tag{21}$$

where $\widehat{Q}_i = D_i^E$ and $\Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D})$ is the same as in Section 5.2. The relative ex-post regret defined here is similar to the definition in Ball and Queyranne [4]. We further define the maximin relative ex-post regret minimization problem for newsvendor i as follows:

$$\max_{A_i \leq Q_i \leq B_i^E} \left(\min_{\mathbf{D} \in \mathcal{D}} \left(\frac{\Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D})}{\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D})} \right) \right). \tag{22}$$

The analysis of this problem is presented below.

Proposition 8 *For the newsvendor game defined by (22), the best response function of newsvendor i is uniquely determined by $Q_i^* = \frac{p_i A_i B_i^E}{(p_i - c_i)A_i + c_i B_i^E}$, and a Nash equilibrium solution satisfies the following system of nonlinear and non-smooth equations:*

$$Q_i^* = \frac{p_i A_i \left(B_i + \sum_{j \neq i} o_{ij}(B_j - Q_j^*)^+ \right)}{(p_i - c_i)A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij}(B_j - Q_j^*)^+ \right)}, \forall i. \tag{23}$$

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We note that the solution for the monopolist newsvendor problem under the relative regret minimization criterion can be obtained by letting $o_{ij} \equiv 0, \forall i, j$, resulting in $Q_i^{NV} = \frac{p_i A_i B_i}{(p_i - c_i) A_i + c_i B_i}$. Clearly, the Nash equilibrium solution preserves the basic form of this solution but expands the support of demand distribution: instead of $[A_i, B_i]$ we have $[A_i, B_i^E]$. This generalization is similar to the outcome of the absolute regret minimization, and it is intuitively appealing. One can easily verify that $Q_i^* \geq Q_i^{NV}, \forall i$, so that competitive newsvendors stock more inventory than in the monopoly case because they face demand distribution with the higher upper bound (which now accounts for spillovers). Although the solution in (23) is intuitive, it is still given in an implicit form because equilibrium stocking quantities appear both on the left- and right-hand side of the optimality conditions. As in the absolute regret minimization problem, we can obtain simpler solutions in two special cases.

Proposition 9 *(a) If*

$$p_i A_i \leq c_i B_i \left(1 + \frac{B_i - A_i}{\sum_{j \neq i} o_{ij} \frac{c_j B_j (B_j - A_j)}{(p_j - c_j) A_j + c_j B_j}} \right), \forall i, \tag{24}$$

then $Q_i^* \leq B_i$ and the Nash equilibrium is defined by the following system of nonlinear equations:

$$Q_i^* = \frac{p_i A_i \left(B_i + \sum_{j \neq i} o_{ij} (B_j - Q_j^*) \right)}{(p_i - c_i) A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij} (B_j - Q_j^*) \right)}, \forall i. \tag{25}$$

Furthermore, the Nash equilibrium is unique.

(b) There exists at least one newsvendor i_0 such that $Q_{i0}^* \leq B_{i0}$.

(c) Suppose the game is symmetric so that $A_i = A$, $B_i = B$, $c_i = c$, $p_i = p$, and $o_{ij} = \gamma/(N-1), \forall i, j$. Then, the unique Nash equilibrium is given by

$$Q_i^* = \frac{(1 + \gamma)(pA + cB) - cA - \sqrt{((1 + \gamma)(pA + cB) - cA)^2 - 4\gamma(1 + \gamma)pcAB}}{2\gamma c}. \tag{26}$$

In (a), we provide simple conditions ensuring that the equilibrium stocking quantity does not exceed the upper bound on the primary demand distribution. Similar to the absolute regret minimization case, these conditions are easily satisfied if substitution proportions o_{ij} are relatively small. The difference is that, when relative regret is minimized, this condition may not even depend on the substitution fraction. To explain why, notice that, when $p_i/c_i < B_i/A_i$, condition (24) trivially holds. Intuitively, when $p_i/c_i < B_i/A_i$, support of the demand distribution is relatively wide (B_i/A_i is high) and the profit margin is relatively small (p_i/c_i is small). Thus, no matter how big demand overflow due to competition may be, the newsvendor

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will never stock more than B_i. This is the unique feature of the relative regret minimization problem because the regret in this case is measured relative to the baseline. However, unlike in the absolute regret minimization case, this condition is not sufficient to obtain an equilibrium solution in a closed form: it transforms only nonlinear and non-smooth optimality conditions (23) into nonlinear but smooth optimality conditions (25) which can easily be solved numerically and in some cases analytically. Moreover, in this case the Nash equilibrium is unique. In (b) we see that condition (24) is always satisfied for at least one newsvendor because all newsvendors taken together would not stock more than the upper bound on the aggregate demand. Finally, when all newsvendors are symmetric, (a) and (b) together imply that (25) is the unique solution which, because of the symmetry assumption, becomes a simple equation that we solve in closed form. When there are two newsvendors in the game, we provide a closed-form solution for arbitrary cost and revenue parameters as shown below.

For $i, j = 1, 2, i \neq j$, define

$$
\bar{o}_{ij} = \begin{cases} \left(\dfrac{(p_j - c_j)A_j + c_j B_j}{c_j B_j (B_j - A_j)} \right) \left(\dfrac{c_i B_i (B_i - A_i)}{p_i A_i - c_i B_i} \right), & \text{if } \dfrac{p_i}{c_i} > \dfrac{B_i}{A_i}, \\ \infty, & \text{otherwise,} \end{cases}
$$

$$
\alpha_i^+ = \frac{E_i E_j - G_i o_{ji} + G_j o_{ij} + \sqrt{(E_i E_j - G_i o_{ji} + G_j o_{ij})^2 + 4E_i o_{ji} G_i E_j}}{2 E_i o_{ji}},
$$

$$
\alpha_i^- = \frac{E_i E_j - G_i o_{ji} + G_j o_{ij} - \sqrt{(E_i E_j - G_i o_{ji} + G_j o_{ij})^2 + 4E_i o_{ji} G_i E_j}}{2 E_i o_{ji}},
$$

where $E_i = (p_i/c_i - 1) A_i + B_i + o_{ij}(B_j - p_j/c_j A_j)$, and $G_i = p_i/c_i(p_i/c_i - 1)A_i^2$. As it is easy to see, since $o_{ij} \leq 1$ and $o_{ji} \leq 1$, $o_{ij} > \bar{o}_{ij}$ and $o_{ji} > \bar{o}_{ji}$ cannot both hold.

Further, for $i, j = 1, 2, i \neq j$, define

$$
Q_m^{\alpha+} = \frac{p_m A_m}{c_m} + \alpha_m^+, m = 1, 2,
$$

$$
Q_m^{\alpha-} = \frac{p_m A_m}{c_m} + \alpha_m^-, m = 1, 2,
$$

$$
Q_i^{NV} = \frac{p_i A_i B_i}{(p_i - c_i) A_i + B_i},
$$

$$
Q_i^{\beta} = \frac{p_i A_i (B_i ((p_j - c_j)A_j + c_j B_j) + o_{ij} c_j B_j (B_j - A_j))}{((p_i - c_i)A_i + c_i B_i)((p_j - c_j)A_j + c_j B_j) + c_i c_j o_{ij} B_j (B_j - A_j)}.
$$

Proposition 10 *Let $N = 2$, $i, j = 1, 2$ and $i \neq j$.*

(a) *Suppose that $o_{ij} > \bar{o}_{ji}$ and $o_{ji} \leq \bar{o}_{ji}$. Then (Q_i^{β}, Q_j^{NV}) is the unique Nash equilibrium.*

(b) *Suppose that $o_{ij} \leq \bar{o}_{ij}$ and $o_{ji} \leq \bar{o}_{ji}$. Then for $m = 1, 2$, $\alpha_m^+ \leq 0$ and $\frac{p_m A_m}{c_m} + \alpha_m^+ \leq B_m$, $(Q_i^{\alpha+}, Q_j^{\alpha+})$ is a Nash equilibrium. Also, for $m = 1, 2$, $\alpha_m^- \leq 0$ and $\frac{p_m A_m}{c_m} + \alpha_m^- \leq B_m$, $(Q_i^{\alpha-}, Q_j^{\alpha-})$ is a Nash equilibrium.*

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(c) *Suppose that $o_{ij} \leq \bar{o}_{ij}$ and $o_{ji} \leq \bar{o}_{ji}$. Then $(Q_i^{\alpha-}, Q_j^{\alpha-})$ is the unique Nash equilibrium if*

$$B_j \geq \frac{p_j A_j}{c_j} \qquad \text{and} \qquad o_{ij} \leq \frac{(\frac{p_i}{c_i} - 1)A_i + B_i}{\frac{p_j A_j}{c_j} - B_j} \tag{27}$$

hold simultaneously when $i = 1$ and $j = 2$ and when $i = 2$ and $j = 1$.

Similar to the result of Proposition 6 for the absolute regret minimization, we derive conditions for the existence of different equilibria using thresholds for substitution fractions. In the case of relative regret we must define thresholds more complexly because, as we saw in the previous proposition, conditions for $Q_i^* < B_i$ may not depend on the substitution fractions. To reflect this observation, we let $\bar{o}_{ij} = \infty$ (so that $o_{ij} \leq \bar{o}_{ij}$ is trivially satisfied) when $p_i/c_i \leq B_i/A_i$. With the exception of this caveat, the equilibrium solution for the duopoly is similar to the absolute regret minimization case. When both substitution fractions are small enough as in Proposition 10 (b), competing newsvendors select stocking quantities that are higher than the newsvendor solution Q_i^{NV} but lower than the upper bound on the primary demand. If, however, substitution fractions are asymmetric as in Proposition 10 (a) (one higher and one lower than the thresholds), then in a typical Nash equilibrium (Q_i^{β}, Q_j^{NV}), one competitor to stocks more than the upper bound on demand causing the other competitor stock at exactly the newsvendor solution amount.

For asymmetric newsvendor competition with $N = 2$, unlike for the absolute regret criterion, we are unable to prove the uniqueness of the Nash equilibrium for the relative regret criterion. However, under the relative regret criterion, uniqueness can be guaranteed for some special cases; see Proposition 10 (a) and (c). A particular observation from Proposition 10 (c) is that $(Q_i^{\alpha-}, Q_j^{\alpha-})$ is the unique Nash equilibrium when both o_{ij} and o_{ji} are sufficiently small. In our computational experiments, we have not identified any examples that had multiple Nash equilibria or any examples that had $(Q_i^{\alpha+}, Q_j^{\alpha+})$ as a Nash equilibrium.

6.2 Relative Ex-Ante Regret

We define the relative ex-ante regret for newsvendor i as

$$\delta_i^{ea}(\mathbf{Q}, \mathbf{F}) = \frac{\max_{\widehat{Q}_i \geq 0} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) \right)}{\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})}, \tag{28}$$

where $\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})$ is defined in Section 5.1. Using this definition, we further define the minimax relative ex-ante regret problem for newsvendor i as

$$\min_{Q_i \geq 0} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\delta_i^{ea}(\mathbf{Q}, \mathbf{F}) \right) \right). \tag{29}$$

Proposition 11 concisely states the solution to (29):

Proposition 11 *There exists a Nash equilibrium solution to the minimax relative ex-ante regret problem satisfying $Q_i^* = \frac{p_i A_i B_i^E}{(p_i - c_i) A_i + B_i^E c_i}$. Moreover, the maximin relative ex-post problem (22) and the minimax ex-ante regret problem (29) are equivalent.*

As is evident from the last result, the relative ex-ante regret minimization problem is equivalent to the relative ex-post regret minimization problem and therefore all results for the relative ex-post regret minimization game (Propositions 8-10) will be valid for the relative ex-ante regret minimization game.

7 Numerical Comparison of Different Approaches

In this section, we numerically compare all robust approaches analyzed in the previous sections with the expected profit optimization approach. The goal is to identify the robust approach that results in solutions that are close to the expected profit maximization approach while using limited demand information. We begin by focusing on a symmetric game. The values of problem parameters for both newsvendors for our basic example are: $A = 20$, $B = 70$, $p = 20$, $c = 10$, and $\gamma = 0.5$. We assume that the demands for both newsvendors are uniform.

We used four different optimization criteria: maximizing the expected profit (`Exp-Rev`), minimizing the maximum absolute regret (denoted `Abs-Reg`), minimizing the maximum relative regret (`Rel-Reg`), and maximizing the worst-case profit (`Maximin`). For each of four criteria, we calculated values for four performance measures: order quantities (`Solution`), the expected profit (`ExpRev`), the absolute regret (`AbsReg`), and the relative regret (`RelReg`). Using the parameters above as a base scenario, we then vary support of the demand distribution, retail price, and the substitution parameter. To be concise, we summarize results of these numerical experiments in the Appendix (Table 1 describes parameter changes and Table 2 describes results) and only highlight the main observations here.

Naturally, the newsvendor can achieve the highest expected profit among all criteria if he/she orders based on the `Exp-Rev` criterion. We observe that the `Abs-Reg` approach performs very well, i.e., it leads to the Nash equilibrium order quantities and profits which are very close to the `Exp-Rev` approach even though this method uses only demand support information. The performance of the `Rel-Reg` approach is considerably worse but can still be called satisfactory. In contrast, the `Maximin` approach performs poorly due to its overly conservative nature.

To obtain further insights into the performance of all four robust approaches, we conducted a detailed sensitivity analysis of our results with respect to the changes in the values of the price, the demand support and the overflow rate. In particular, in Figure 1 we display the

computational results for the four performance measures obtained by systematically varying the product price from 20 to 30 units while keeping all other parameters constant as in our basic example. These four figures reflect that, no matter how problem parameters change, the absolute regret minimization approach still performs remarkably close to the expected profit maximization, while the relative regret minimization is a distant second (indeed, when it comes to the optimal order quantity, the gap is quite large). The worst-case approach performs much worse largely because the optimal order quantity for it does not change with price. It is interesting to see that for methods other than `Maximin`, the (absolute or relative) regret is quite insensitive to price increases since the order quantity depends on price.
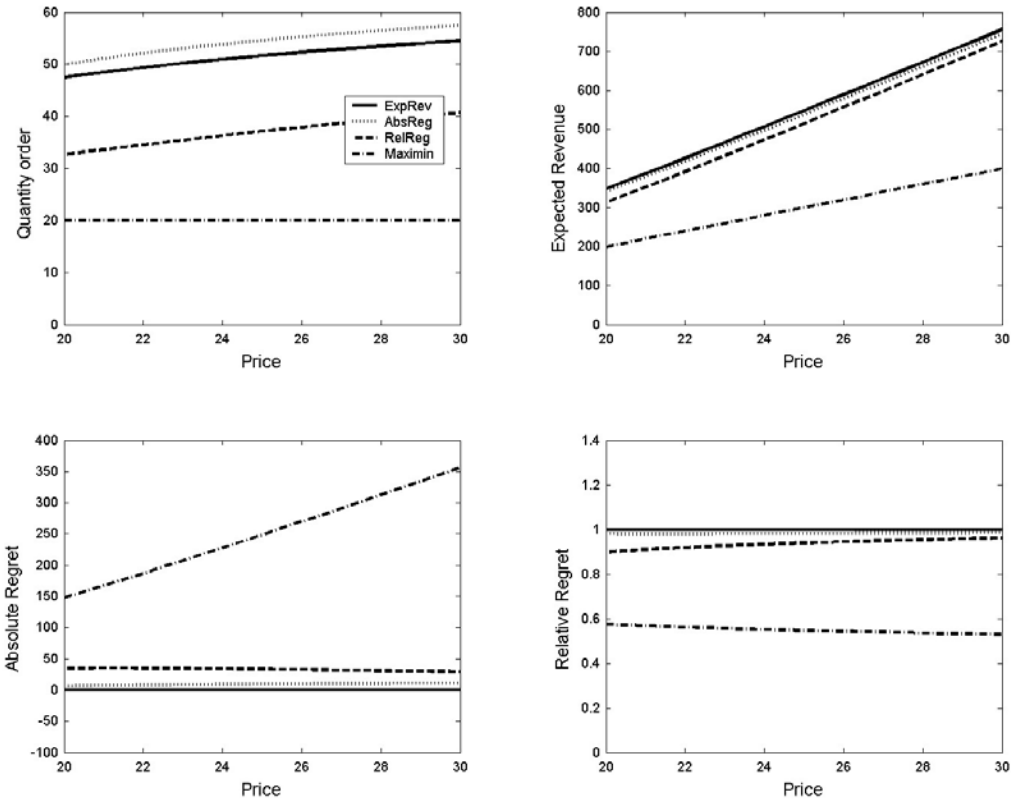

Figure 1: Price sensitivity analysis for a symmetric game.

Figure 2 illustrates the sensitivity of our results with respect to changes in the upper bound of demand support. Since the equilibrium for the `Maximin` approach is insensitive to the upper bound of demand distribution, its performance quickly worsens as B increases. Likewise, the equilibrium order quantity for the `Rel-Reg` case appears to be quite insensitive to changes in
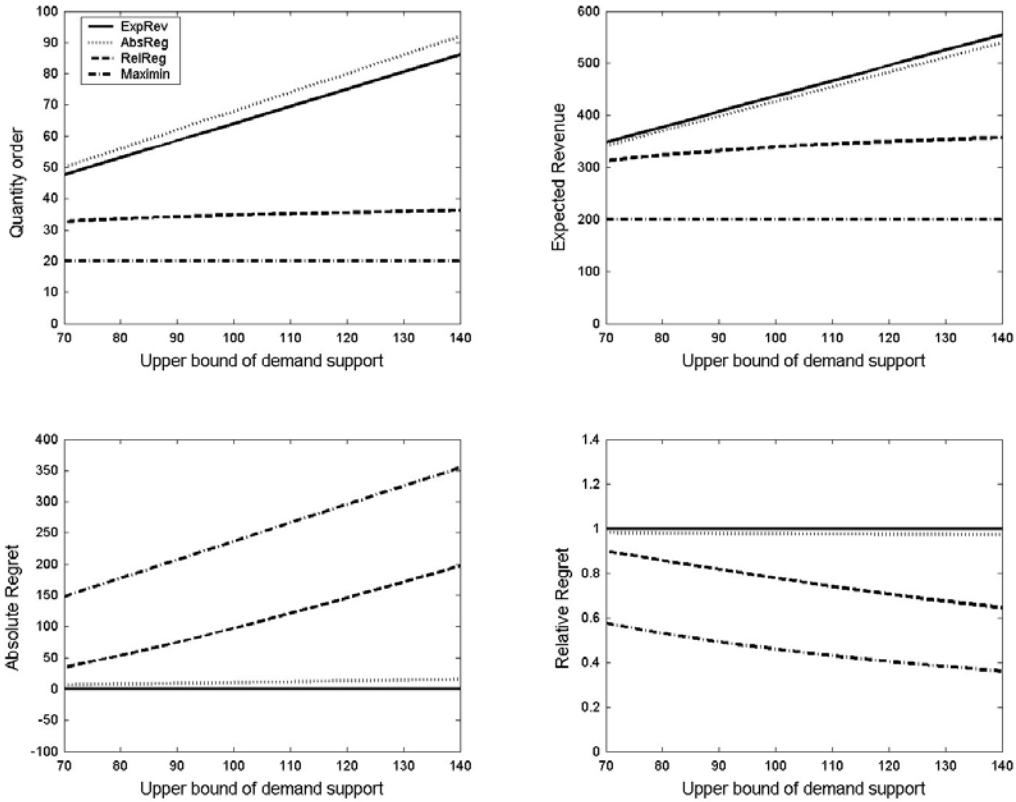
Figure 2: Demand support sensitivity analysis for a symmetric game.

the upper bound for parameters that we use, and therefore this approach performs considerably worse relative to expected profit maximization. Thus, the increase in the upper value of the demand support interval has a consistent positive impact on both the order quantities and the expected profit for `Exp-Rev` and `Abs-Reg`, but not on `Maximin` or `Rel-Reg`. The absolute regret minimization approach once again results in only a moderate decrease in profits and very small deviation in order quantities.

Finally, Figure 3 shows the effects of changes in the overflow rate on our performance measures. It appears that, for very large overflow rates, the performance of the absolute regret minimization approach worsens, but it is still quite good. On the other hand, the other two approaches perform consistently poorly. Moreover, the impact of the overflow rate on all performance measures is quite insignificant: all curves are nearly flat.

To ensure that these numerical results are robust, we conducted numerical experiments for asymmetric games as well as for other demand distributions (triangular and truncated normal).
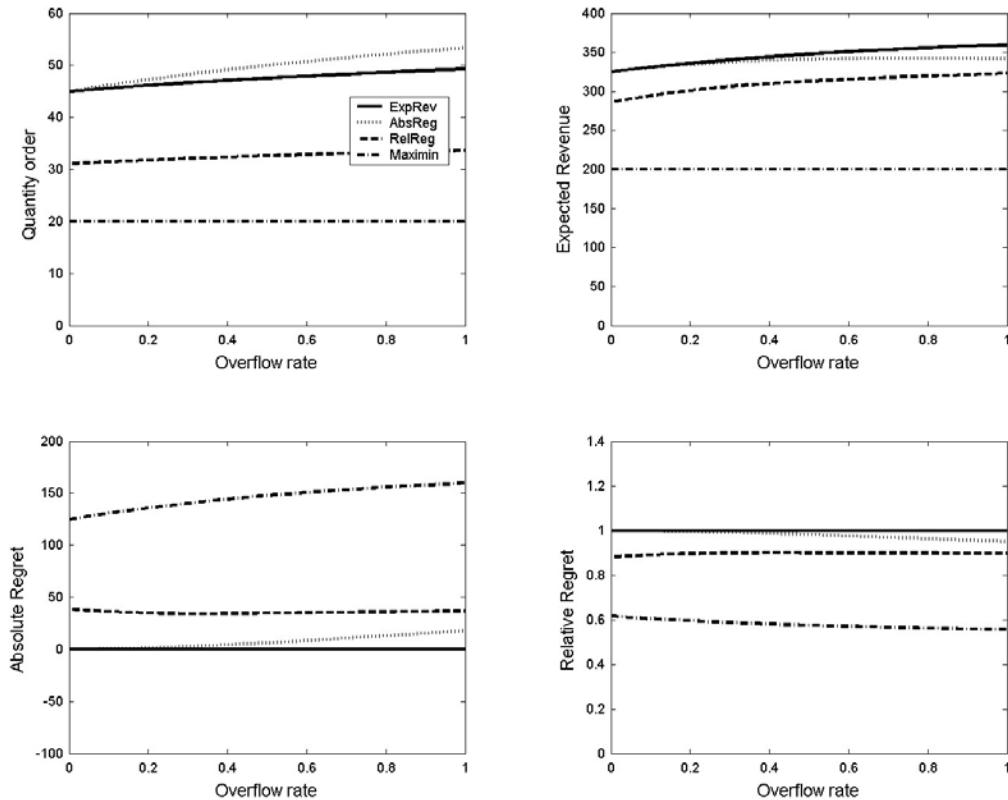
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Figure 3: Demand overflow rate sensitivity analysis for a symmetric game.

We report that our observations were consistent with the numerical experiments presented above: the `Abs-Reg` approach provided consistently good performance even for other demand distribution and asymmetric problem parameters. In the Appendix, we illustrate these observations using the truncated Normal demand distribution. We make only three observations about these results here. First, both newsvendors have slightly higher order quantities under the `Abs-Reg` approach and significantly lower order quantities under the `Rel-Reg` approach than under the `Exp-Rev` approach. Second, as our theoretical results indicate, one newsvendor may stock more than the upper bound of the support of his/her demand, which never occurs in a monopoly or symmetric game setting. Third, in a monopoly setting, the absolute regret is always non-negative because no method can perform better in terms of the expected profit than the method of maximizing the expected profit. However, due to demand overflow and competition, one newsvendor can generate higher-than-expected profits under the `Rel-Reg` approach than under the `Exp-Rev` approach because the latter approach is known to result in inventory overstocking

relative to the centralized solution (Netessine and Rudi [20]).

8 Conclusions

In this paper, we have studied the newsvendor competition model under several robust optimization criteria: the absolute regret minimization, the relative regret minimization, and the worst-case scenario. We have established the existence results for Nash equilibria for robust newsvendor games and derived closed-form solutions for special cases, particularly for duopoly games. The analytical results and computational experiments allow us to better understand the consequences of robust solutions when limited information about demand is available. An interesting result that we have obtained is that the ex-ante and ex-post versions of the competitive newsvendor problem are equivalent under the worst-case or the absolute regret or the relative regret criterion when the only available information is the demand support. Note that the ex-ante and ex-post newsvendor problems are no longer necessarily equivalent when additional information other than the demand support is available. For example, in Theorem 1 Yue et al. [33] prove that the maximum regret is achieved at a two-point probability distribution. Our results indicate that, from a practical standpoint, the absolute regret minimization approach both is analytically tractable and results in solutions that are very close to the expected profit maximization even though it uses only information about support of demand distribution.

This is the first attempt to analyze newsvendor competition through stock-out-based substitution. We anticipate a lot of new research in the area of robust optimization and robust competition. One possible future research topic is to include more information about demand such as its mean and standard deviation. Recently, Perakis and Roels [25, 24] and Zhu, Zhang and Ye [34] have investigated robust newsvendor optimization with such additional demand information in a monopolistic setting. Another interesting direction would be to extend our analysis by investigating a centralized system that includes all newsvendors under robust performance criteria. Our preliminary results indicate that this is a very hard problem that is unlikely to result in any closed-form solutions.

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Appendix to "Robust Newsvendor Competition"

Proof of Proposition 1.

(a) Follows from Proposition 3 of Netessine and Rudi [20].

(b) Let $f(x)$ be the probability distribution of the demand for both newsvendors. Then

$$P(D_1 \leq Q) - P(D_1 < Q < D_1 + \gamma(D_2 - Q)^+) = \frac{p-c}{p} \tag{A1}$$

is equivalent to

$$P(D_1 \leq Q) - P((D_1 < Q < D_1 + \gamma(D_2 - Q)) \cap D_2 > Q) = \frac{p-c}{p}. \tag{A2}$$

We claim that $Q \leq B$. Otherwise, $P(D_1 \leq Q) = \frac{p-c}{p}$, which implies that $Q \leq B$, a contradiction.

The latest equation is in turn equivalent to one of the following two equations:

$$\int_A^Q f(s)ds - \int_Q^B f(v)\left(\int_{Q-\gamma(v-Q)}^Q f(s)ds\right) dv = \frac{p-c}{p}, \tag{A3}$$

when $Q + \frac{Q-A}{\gamma} \geq B$, and

$$\int_A^Q f(s)ds - \int_A^Q f(s)\left(\int_{Q+\frac{Q-s}{\gamma}}^B f(v)dv\right) ds = \frac{p-c}{p}, \tag{A4}$$

when $Q + \frac{Q-A}{\gamma} < B$.

We first solve (A3). If D_1 has a uniform distribution, then $f(x) = \frac{1}{B-A}$ for $x \in [A, B]$; otherwise, $f(x) = 0$. The equilibrium equation (A3) becomes:

$$
\begin{aligned}
\frac{p-c}{p} &= \frac{Q-A}{B-A} - \int_Q^B \frac{1}{B-A}\left(\frac{Q-(Q-\gamma(s-Q))}{B-A}\right) ds \\
&= \frac{Q-A}{B-A} - \frac{\gamma}{(B-A)^2}\int_0^{B-Q} t \, dt \\
&= \frac{Q-A}{B-A} - \frac{\gamma}{2(B-A)^2}(B-Q)^2,
\end{aligned}
\tag{A5}
$$

which is a quadratic equation: $\frac{\gamma}{2}Y^2 - Y - \frac{c}{p} = 0$, where $Y = \frac{Q-B}{B-A}$. Let

$$\widehat{Y} = \frac{1 + \sqrt{1 + \frac{2\gamma c}{p}}}{\gamma}, \qquad \widetilde{Y} = \frac{1 - \sqrt{1 + \frac{2\gamma c}{p}}}{\gamma}. \tag{A6}$$

Then $\widehat{Y}$ and $\widetilde{Y}$ are two solutions of the quadratic equation. Since $Q \leq B$ according to our earlier claim, Q obtained from $\widehat{Y}$ is not a valid Nash equilibrium. Therefore, $\widetilde{Q} = B + (B-A)\frac{1-\sqrt{1+\frac{2\gamma c}{p}}}{\gamma}$ is the only possible Nash equilibrium when $\widetilde{Q} + \frac{\widetilde{Q}-A}{\gamma} \geq B$ and $\widetilde{Q} \leq B$.

It is easy to prove that $A \leq \widetilde{Q} \leq B$. Let us show that $\widetilde{Q} + \frac{\widetilde{Q}-A}{\gamma} \geq B$ if and only if $(2+3\gamma)p \geq 2c(1+\gamma)^2$ holds. Indeed,

$$
\begin{aligned}
(2+3\gamma)p \geq 2c(1+\gamma)^2 \quad &\Leftrightarrow \quad 2p\gamma + 3p\gamma^2 \geq 2c\gamma + 4c\gamma^2 + 2\gamma^3 \\
&\Leftrightarrow \quad p + 4p\gamma + 4p\gamma^2 \geq 2c\gamma + 4c\gamma^2 + 2\gamma^3 + p + 2p\gamma + p\gamma^2 \\
&\Leftrightarrow \quad p(1+2\gamma)^2 \geq (1+\gamma)^2(p+2c\gamma) \\
&\Leftrightarrow \quad 1 + 2\gamma \geq (1+\gamma)\sqrt{\frac{p+2\gamma c}{p}} \\
&\Leftrightarrow \quad 1 - \sqrt{1+\frac{2\gamma c}{p}} + 1 + \frac{1 - \sqrt{1+\frac{2\gamma c}{p}}}{\gamma} \geq 0 \\
&\Leftrightarrow \quad (B-A)\frac{1 - \sqrt{1+\frac{2\gamma c}{p}}}{\gamma} + (B-A)\frac{1 + \frac{1-\sqrt{1+\frac{2\gamma c}{p}}}{\gamma}}{\gamma} \geq 0 \\
&\Leftrightarrow \quad \widetilde{Q} + \frac{\widetilde{Q}-A}{\gamma} \geq B. \quad\quad\quad\quad\text{(A7)}
\end{aligned}
$$

We next solve (A4).

$$
\begin{aligned}
\frac{p-c}{p} &= \frac{Q-A}{B-A} - \frac{1}{(B-A)^2}\int_A^Q \left(B - Q - \frac{Q-s}{\gamma}\right)ds \\
&= \frac{Q-A}{B-A} - \frac{\left((B-Q-\frac{Q}{\gamma})(Q-A) + \frac{Q^2-A^2}{2\gamma}\right)}{(B-A)^2} \\
&= \frac{(Q-A)^2}{(B-A)^2}\left(1 + \frac{1}{2\gamma}\right). \quad\quad\quad\quad\text{(A8)}
\end{aligned}
$$

The valid solution of the above equation is

$$
\widehat{Q} = A + (B-A)\sqrt{\frac{2\gamma(p-c)}{(1+2\gamma)p}}. \quad\quad\quad\quad\text{(A9)}
$$

We now check that $\widehat{Q} + \frac{\widehat{Q}-A}{\gamma} < B$ holds if and only if the condition $(2+3\gamma)p < 2c(1+\gamma)^2$ is satisfied.

$$
\begin{aligned}
(2+3\gamma)p < 2c(1+\gamma)^2 \quad &\Leftrightarrow \quad 2p + 2p\gamma^2 + 4p\gamma - 2c(1+\gamma)^2 < p\gamma + 2p\gamma^2 \\
&\Leftrightarrow \quad 2(p-c)(1+\gamma)^2 < p\gamma(1+2\gamma) \\
&\Leftrightarrow \quad \frac{2\gamma(p-c)}{(1+2\gamma)p}\frac{(1+\gamma)^2}{\gamma^2} < 1 \\
&\Leftrightarrow \quad \sqrt{\frac{2\gamma(p-c)}{(1+2\gamma)p}}\left(1 + \frac{1}{\gamma}\right) < 1 \\
&\Leftrightarrow \quad (B-A)\sqrt{\frac{2\gamma(p-c)}{(1+2\gamma)p}}\left(1 + \frac{1}{\gamma}\right) < B - A
\end{aligned}
$$

$$\Leftrightarrow \quad A + (B-A)\sqrt{\frac{2\gamma(p-c)}{(1+2\gamma)p}} + \frac{1}{\gamma}(B-A)\sqrt{\frac{2\gamma(p-c)}{(1+2\gamma)p}} < B$$

$$\Leftrightarrow \quad \widehat{Q} + \frac{1}{\gamma}\left(A + (B-A)\sqrt{\frac{2\gamma(p-c)}{(1+2\gamma)p}} - A\right) < B$$

$$\Leftrightarrow \quad \widehat{Q} + \frac{1}{\gamma}(\widehat{Q}-A) < B. \tag{A10}$$

Hence either $(\widetilde{Q},\widetilde{Q})$ or $(\widehat{Q},\widehat{Q})$ must be a symmetric Nash equilibrium, but both cannot be Nash equilibria at the same time, thus proving that the newsvendor game has a unique symmetric Nash equilibrium. ∎

Proof of Proposition 2.

(a) For any fixed $\mathbf{F}$, it is clear that $\Pi_i(\mathbf{Q},\mathbf{F})$ is an increasing and linear affine function over the interval $[0, A_i]$, a concave function over the interval $[A_i, B_i^E]$, and a decreasing and linear affine function over the interval $[B_i^E, +\infty]$. Let $g(Q_i, \mathbf{Q}_{-i}) = \min_{\mathbf{F} \in \mathcal{D}} \Pi_i(\mathbf{Q}, \mathbf{F})$. Then $g(Q_i, \mathbf{Q}_{-i}) \leq g(A_i, \mathbf{Q}_{-i})$ for any $Q_i \leq A_i$ and $g(Q_i, \mathbf{Q}_{-i}) \leq g(B_i^E, \mathbf{Q}_{-i})$ for any $Q_i \geq B_i^E$. The desired result follows.

(b) Note that the support for random variable D_j is $[A_j, B_j]$. It follows that the support for D_i^E is $[A_i, B_i^E]$. For any probability distribution $\mathbf{F}$,

$$-c_iQ_i + p_iE_{\mathbf{F}}\left[\min(D_i^E, Q_i)\right] \geq -c_iQ_i + p_iE_{\mathbf{F}}\left[\min(A_i, Q_i)\right] = -c_iQ_i + p_i\min(A_i, Q_i). \tag{A11}$$

On the other hand, when the demand for newsvendor i is of a discrete probability distribution with a unit impulse at A_i,

$$-c_iQ_i + p_iE_{\mathbf{F}}\left[\min(D_i^E, Q_i)\right] = -c_iQ_i + p_i\min(A_i, Q_i), \tag{A12}$$

which shows that the optimal value for the inner minimization problem is achieved at $\mathbf{F}$ such that for all i, F_i is of a discrete probability distribution with a unit impulse at A_i.

(c) From the result of part (a), the objective function for both (5) and (6) is $-c_iQ_i + p_i\min(A_i, Q_i)$. Obviously, $Q_i^* = A_i$ is the unique optimal solution for both (5) and (6). Therefore, the ex-ante maximin and ex-post maximin problems are equivalent. In particular, both problems have the same Nash equilibrium solutions. ∎

Proof of Lemma 1.

(a) Because the support of D_i is $[A_i, B_i]$, the support of D_i^E is $[A_i, B_i^E]$. Since $p_i > c_i > 0$, it is easy to verify that with respect to $\widehat{Q}_i$, $\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F})$ is an increasing and linear affine function in the interval $[0, A_i]$, concave in $[A_i, B_i^E]$, and a decreasing and linear affine function in $[B_i^E, +\infty)$. Hence the optimal value of the relevant maximization problem can be attained at a point in the interval $[A_i, B_i^E]$, and of course can be attained at a point in a larger interval $[A_i, B_i^E]$.

(b) Let $g(Q_i, \mathbf{Q}_{-i}) = \max_{\mathbf{F} \in \mathcal{D}} \Delta_i^{\text{ea}}(\mathbf{Q}, \mathbf{F})$. We need to prove only that $g(Q_i, \mathbf{Q}_{-i}) \geq g(A_i, \mathbf{Q}_{-i})$ if $0 \leq Q_i \leq A_i$, and $g(Q_i, \mathbf{Q}_{-i}) \geq g(B_i^E, \mathbf{Q}_{-i})$ if $Q_i \geq B_i^E$. It follows from (a) that for any $0 \leq Q_i \leq A_i$, $\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}) \leq \Pi_i(A_i, \mathbf{Q}_{-i}, \mathbf{F})$, which shows that $g(Q_i, \mathbf{Q}_{-i}) \geq g(A_i, \mathbf{Q}_{-i})$. Similarly we can prove that $g(Q_i, \mathbf{Q}_{-i}) \geq g(B_i^E, \mathbf{Q}_{-i})$ if $Q_i \geq B_i^E$. ∎

Proof of Proposition 3.

Let $\Delta = \min(D_i^E, \widehat{Q}_i) - \min(D_i^E, Q_i)$ as a function of D_i^E for any fixed Q_i and $\widehat{Q}_i$. Assume $Q_i \leq \widehat{Q}_i$. It is clear that $\Delta = 0$ if $D_i^E \leq Q_i \leq \widehat{Q}_i$, $\Delta = D_i^E - Q_i$ if $Q_i \leq D_i^E \leq \widehat{Q}_i$, and $\Delta = \widehat{Q}_i - Q_i$ if $Q_i \leq \widehat{Q}_i \leq D_i^E$. It shows that $\Delta \leq \widehat{Q}_i - Q_i$ and Δ attains the maximum $\widehat{Q}_i - Q_i$ when $D_i^E = \widehat{Q}_i$.

Assume $\widehat{Q}_i \leq Q_i$. We have that $\Delta = 0$ if $D_i^E \leq \widehat{Q}_i \leq Q_i$, $\Delta = \widehat{Q}_i - D_i^E$ if $\widehat{Q}_i \leq D_i^E \leq Q_i$, and $\Delta = \widehat{Q}_i - Q_i$ if $\widehat{Q}_i \leq Q_i \leq D_i^E$. It follows that $\Delta \leq 0$ and Δ attains the maximum 0 when $D_i^E = \widehat{Q}_i$.

Let $\mathbf{F}$ be any joint probability distribution for the demand of all newsvendors. Since $A_i \leq \widehat{Q}_i \leq B_i^E$ and for all j, $A_j \leq Q_j \leq B_j^E$, it is easy to choose appropriate values for $\widehat{D}_i$ and $\widehat{D}_j$ for $j \neq i$ for the joint probability distribution $\widehat{\mathbf{F}}$ such that $\widehat{Q}_i = \widehat{D}_i + \sum_{j \neq i} o_{ij}(\widehat{D}_j - Q_j)^+$. Clearly, D_i^E generated from the joint probability distribution $\widehat{\mathbf{F}}$ is of a unit impulse probability distribution with mass at $\widehat{Q}_i$.

By the above arguments on Δ, we show that

$$
\begin{aligned}
E_{\mathbf{F}}\left[\min(D_i^E, \widehat{Q}_i)\right] - E_{\mathbf{F}}\left[\min(D_i^E, Q_i)\right] &= E_{\widehat{\mathbf{F}}}[\Delta] \\
&\leq E_{\widehat{\mathbf{F}}}\left[\widehat{Q}_i - \min(\widehat{Q}_i, Q_i)\right] \\
&= E_{\widehat{\mathbf{F}}}\left[\widehat{Q}_i\right] - E_{\widehat{\mathbf{F}}}\left[\min(\widehat{Q}_i, Q_i)\right] \\
&= E_{\widehat{\mathbf{F}}}\left[\min(D_i^E, \widehat{Q}_i)\right] - E_{\widehat{\mathbf{F}}}\left[\min(D_i^E, Q_i)\right].
\end{aligned}
$$

This implies that $\widehat{\mathbf{F}}$ is an optimal solution for the inner maximization problem of the minimax absolute ex-ante regret problem (10). This completes the proof. ∎

Proof of Proposition 4.

(a) Note that the objective function for the outer maximization problem becomes

$$
c_i(Q_i - \widehat{Q}_i) + p_i(\widehat{Q}_i - \min(\widehat{Q}_i, Q_i)) = \begin{cases} c_i(Q_i - \widehat{Q}_i) & \widehat{Q}_i \leq Q_i, \\ c_i(Q_i - \widehat{Q}_i) + p_i(\widehat{Q}_i - Q_i), & \widehat{Q}_i > Q_i, \end{cases} \tag{A13}
$$

which is a piecewise and convex function with respect to $\widehat{Q}_i$. Therefore either A_i or B_i^E is an optimal solution for the outer maximization problem, and the objective function for the outer minimization problem is

$$
\max\left(c_i(Q_i - A_i), (c_i - p_i)(Q_i - B_i^E)\right), \tag{A14}
$$

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which is a piecewise and convex function with respect to Q_i. The optimal value of the minimization problem can only be obtained at the intersection of two straight lines $y = c_i(Q_i - A_i)$ and $y = (c_i - p_i)(Q_i - B_i^E)$, and this intersection is $Q_i^* = \frac{c_i}{p_i}A_i + \frac{p_i - c_i}{p_i}B_i^E$. Clearly, this optimal solution Q_i^* is unique. The result follows.

(b) According to the proof of part (a), newsvendor i solves the following minimization problem

$$\min_{A_i \leq Q_i \leq B_i^E} \left(\max \left(c_i(Q_i - A_i), (c_i - p_i)(Q_i - B_i^E) \right) \right), \tag{A15}$$

which has a convex and piecewise linear objective function and a compact and convex strategy space $\{Q_i : A_i \leq Q_i \leq B_i^E\}$. By Theorem 1 of Rosen [27], the newsvendor game has a Nash equilibrium. The result in (12) follows from part (a). ∎

Proof of Proposition 5.

(a) We first prove by contradiction that for any i, $Q_i^* \leq B_i$. Assume for some m, $Q_m^* > B_m$. By (12), we have

$$Q_m^* = \frac{c_m}{p_m}A_m + \left(1 - \frac{c_m}{p_m}\right)\left(B_m + \sum_{j \neq m} o_{mj}(B_j - Q_j^*)^+\right). \tag{A16}$$

Consequently,

$$B_m < \frac{c_m}{p_m}A_m + \left(1 - \frac{c_m}{p_m}\right)\left(B_m + \sum_{j \neq m} o_{mj}(B_j - Q_j^*)^+\right), \tag{A17}$$

which is equivalent to

$$\begin{aligned}
\frac{c_m}{p_m}(B_m - A_m) &< \left(1 - \frac{c_m}{p_m}\right)\sum_{j \neq m} o_{mj}(B_j - Q_j^*)^+ \\
&= \left(1 - \frac{c_m}{p_m}\right)\left(\sum_{j \neq m, B_j > Q_j^*} o_{mj}(B_j - Q_j^*) + \sum_{j \neq m, B_j \leq Q_j^*} o_{mj} \times 0\right). \tag{A18}
\end{aligned}$$

Let $Q_i^{NV} = (c_i/p_i)A_i + (1 - c_i/p_i)B_i$. It follows that

$$\begin{aligned}
\frac{c_m}{p_m}(B_m - A_m) &< \left(1 - \frac{c_m}{p_m}\right)\left(\sum_{j \neq m, B_j > Q_j^*} o_{mj}(B_j - Q_j^{NV}) + \sum_{j \neq m, B_j > Q_j^*} o_{mj}\frac{c_j}{p_j}(B_j - A_j)\right) \\
&= \left(1 - \frac{c_m}{p_m}\right)\left(\sum_{j \neq m, B_j > Q_j^*} o_{mj}\frac{c_j}{p_j}(B_j - A_j) + \sum_{j \neq m, B_j > Q_j^*} o_{mj}\frac{c_j}{p_j}(B_j - A_j)\right) \\
&= \left(1 - \frac{c_m}{p_m}\right)\left(\sum_{j \neq m} o_{mj}\frac{c_j}{p_j}(B_j - A_j)\right) \\
&\leq \frac{c_m}{p_m}(B_m - A_m), \tag{A19}
\end{aligned}$$

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where the last inequality follows from the condition of this proposition. This is a contradiction. Hence for any i, $Q_i^* \leq B_i$. Moreover, the result in (14) is a direct consequence of (12).

To prove the uniqueness of a solution for (12), we need to prove only that the coefficient matrix of (12) is strictly diagonally column-dominant and hence non-singular. The strictly diagonal column-dominance property follows from the fact that for all i, $1 - \frac{c_i}{p_i} < 1$ and $\sum_{j \neq i} o_{ji} \leq 1$.

(b) Suppose $Q_i^* > B_i$ holds for all i. Since $(Q_1^*, \cdots, Q_n^*)$ is a Nash equilibrium of the newsvendor game, by (12), we have that for all i,

$$Q_i^* = \frac{c_i}{p_i} A_i + \left(1 - \frac{c_i}{p_i}\right) \left(B_i + \sum_{j \neq i} o_{ij} \times 0\right) \leq B_i, \tag{A20}$$

where the last inequality follows from the fact that for all i, $A_i \leq B_i$ and $c_i \leq p_i$. This is a contradiction so that $Q_i^* > B_i$ for all i does not hold.

(c) Due to the symmetry, the condition in (a) holds. Therefore, there exists a unique Nash equilibrium satisfying (12) which can be easily found after symmetry is assumed. ∎

Proof of Proposition 6.

(a) It follows from Proposition 5 that $Q_1^* \leq B_1$ and $Q_2^* \leq B_2$. By Proposition 5 (a), the unique Nash equilibrium (Q_1^*, Q_2^*) satisfies the equation (12), which leads to (17).

(b) Let us prove one case where $i = 1$ and $j = 2$. Since $o_{12} \leq \bar{o}_{12}$ holds, $Q_1^* \leq B_1$. We now prove that $Q_2^* > B_2$. Indeed, suppose that $Q_2^* \leq B_2$. Then, we have

$$Q_2^* = \frac{\frac{c_2}{p_2} A_2 + \left(1 - \frac{c_2}{p_2}\right) B_2 + \left(1 - \frac{c_2}{p_2}\right) \frac{c_1}{p_1} o_{21}(B_1 - A_1) - \left(1 - \frac{c_1}{p_1}\right)\left(1 - \frac{c_2}{p_2}\right) o_{12} o_{21} B_2}{1 - \left(1 - \frac{c_1}{p_1}\right)\left(1 - \frac{c_2}{p_2}\right) o_{12} o_{21}} \leq B_2, \tag{A21}$$

which is equivalent to

$$\frac{c_2}{p_2} A_2 + \left(1 - \frac{c_2}{p_2}\right) B_2 + \left(1 - \frac{c_2}{p_2}\right) \frac{c_1}{p_1} o_{21}(B_1 - A_1) - \left(1 - \frac{c_1}{p_1}\right)\left(1 - \frac{c_2}{p_2}\right) o_{12} o_{21} B_2$$
$$\leq B_2 - \left(1 - \frac{c_1}{p_1}\right)\left(1 - \frac{c_2}{p_2}\right) o_{12} o_{21} B_2. \tag{A22}$$

The latter is, in turn, equivalent to $o_{21} \leq \bar{o}_{21}$, which contradicts the given condition that $o_{21} > \bar{o}_{21}$. Therefore, we have proved that $Q_2^* > B_2$. In view of the fact that $Q_1^* \leq B_1$ and $Q_2^* > B_2$, the system of non-smooth equations (12) becomes

$$\begin{aligned} Q_1^* &= \frac{c_1}{p_1} A_1 + \frac{p_1 - c_1}{p_1} B_1, \\ Q_2^* &= \frac{c_2}{p_2} A_2 + \frac{p_2 - c_2}{p_2} \left(B_2 + o_{21}(B_1 - Q_1^*)\right), \end{aligned} \tag{A23}$$

which can be solved to establish (18). ∎

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Proof of Proposition 7.

The feasible region for the outer minimizer problem of (20) is $Q_i \geq 0$ but, as in the argument we made in Lemma 1 for the minimax absolute ex-ante regret problem, this feasible region can be replaced by $A_i \leq Q_i \leq B_i^E$ without loss of generality. Furthermore, it is straightforward to show that

$$\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D}) - \Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D}) = \begin{cases} (p_i - c_i)(D_i^E - Q_i), & D_i^E \geq Q_i, \\ c_i(Q_i - D_i^E), & D_i^E < Q_i. \end{cases} \tag{A24}$$

Hence, for any fixed $\mathbf{Q}$, $\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D}) - \Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D})$ is a piecewise linear and convex function with respect to D_i^E, so that the optimal value for the inner maximization problem is achieved at either the minimum or the maximum possible value for D_i^E, which is either A_i or B_i^E, respectively. Given this result, the minimax absolute ex-post regret problem becomes:

$$\min_{A_i \leq Q_i \leq B_i^E} \left(\max \left((p_i - c_i)(B_i^E - Q_i), c_i(Q_i - A_i) \right) \right). \tag{A25}$$

It is easily verified that the minimization problem is a piecewise linear and convex function with respect to Q_i, and the optimal solution corresponds to the intersection of two lines $y = (p_i - c_i)(B_i^E - Q_i)$ and $y = c_i(Q_i - A_i)$ provided that this intersection is within the interval $[A_i, B_i^E]$. Solving these two equations simultaneously, we obtain the optimal solution.

The equivalence between absolute ex-post and ex-ante regret problems (20) and (8) follows directly from (A14) and (A25). ∎

Proof of Proposition 8.

It is straightforward to verify that the quantity of interest can be expressed as follows:

$$\frac{\Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D})}{\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D})} = \begin{cases} \frac{p_i D_i^E - c_i Q_i}{(p_i - c_i)D_i^E}, & D_i^E \leq Q_i, \\ \frac{Q_i}{D_i^E}, & D_i^E > Q_i. \end{cases} \tag{A26}$$

From this expression we observe that $\Omega_i(Q_i, \mathbf{Q}_{-i}, \mathbf{D})/\Omega_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{D})$ is an increasing function of D_i^E when $D_i^E \leq Q_i$ and a decreasing function when $D_i^E > Q_i$. Therefore, the optimal value for the inner minimization problem is achieved by selecting a probability distribution with a single mass-point A_i when $D_i^E \leq Q_i$ or a single mass-point B_i^E when $D_i^E > Q_i$. Consequently, the outer maximization problem simplifies to:

$$\max_{A_i \leq Q_i \leq B_i^E} \left(\min \left(\frac{p_i A_i - c_i Q_i}{(p_i - c_i) A_i}, \frac{Q_i}{B_i^E} \right) \right). \tag{A27}$$

This is a piecewise linear and concave objective function with the optimal solution located at the intersection of two straight lines $y = \frac{p_i A_i - c_i Q_i}{(p_i - c_i) A_i}$ and $y = \frac{Q_i}{B_i^E}$ subject to the intersection point

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being within the interval $[A_i, B_i^E]$. It is straightforward to verify that this optimal solution is

$$Q_i^* = \frac{p_i A_i B_i^E}{(p_i - c_i)A_i + c_i B_i^E},\tag{A28}$$

which indeed lies in the interval $[A_i, B_i^E]$.

Now, according to Theorem 1 of [27], a Nash equilibrium exists if the objective function for each player is concave with respect to her own strategy and continuous with respect to the strategies of all newsvendors and the strategy space for each newsvendor is convex and compact. According to (A27), each newsvendor's problem is concave over a compact and convex strategy space. The result follows. ∎

Proof of Proposition 9.

(a) If $p_i A_i \leq c_i B_i$ holds, then it follows from (23) that

$$Q_i^* \leq \frac{c_i B_i \left(B_i + \sum_{j \neq i} o_{ij}(B_j - Q_j^*)^+ \right)}{(p_i - c_i)A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij}(B_j - Q_j^*)^+ \right)} \leq B_i.\tag{A29}$$

If $p_i A_i \leq c_i B_i$ does not hold, but (24) holds, then by (23), we have

$$
\begin{aligned}
Q_i^* &= \frac{p_i A_i^2(1 - \frac{p_i}{c_i})}{(p_i - c_i)A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij}(B_j - Q_j^*)^+ \right)} + \frac{p_i A_i}{c_i} \\
&\leq \frac{p_i A_i^2(1 - \frac{p_i}{c_i})}{(p_i - c_i)A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij}(B_j - Q_j^{NV}) \right)} + \frac{p_i A_i}{c_i} \\
&= \frac{p_i A_i^2(1 - \frac{p_i}{c_i})}{(p_i - c_i)A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij} \frac{c_j B_j(B_j - A_j)}{(p_j - c_j)A_j + c_j B_j} \right)} + \frac{p_i A_i}{c_i} \\
&\leq \frac{p_i A_i^2(1 - \frac{p_i}{c_i})}{(p_i - c_i)A_i + c_i \left(B_i + \frac{c_i B_i(B_i - A_i)}{p_i A_i - c_i B_i} \right)} + \frac{p_i A_i}{c_i} \\
&= -\frac{p_i A_i - c_i B_i}{c_i} + \frac{p_i A_i}{c_i} = B_i,
\end{aligned}\tag{A30}
$$

where the first inequality is implied by the fact that $Q_i^{NV} \leq Q_i^*$ and the second inequality follows from (24). Therefore $Q_i^* \leq B_i$. After combining the two conditions, we obtain the desired result. To prove the uniqueness of a Nash equilibrium, we only need to prove that the Jacobian matrix of (23) is strictly diagonally row-dominant and hence non-singular. Since for all i, $Q_i^* \leq B_i$, the $+$ operator can be removed from (23). Consequently, (23) is equivalent to the following system:

$$
\begin{aligned}
&\left((p_i - c_i)A_i + c_i B_i + c_i \sum_{j \neq i} o_{ij} B_j \right) Q_i^* + p_i A_i \sum_{j \neq i} o_{ij} Q_j^* - c_i \left(\sum_{j \neq i} o_{ij} Q_j^* \right) Q_i^* \\
&= p_i A_i B_i + p_i A_i \sum_{j \neq i} o_{ij} B_j, \forall i.
\end{aligned}\tag{A31}
$$

Let J be the Jacobian matrix of the above system of nonlinear equations at the Nash equilibrium point $(Q_1^*, \cdots, Q_N^*)$. Then for any i and $j \neq i$,

$$
\begin{aligned}
J_{ii} &= p_i A_i + c_i(B_i - A_i) + c_i \sum_{j \neq i} o_{ij}(B_j - Q_j^*), \\
J_{ij} &= p_i A_i o_{ij} - c_i o_{ij} Q_i^*.
\end{aligned}
\tag{A32}
$$

Obviously, $J_{ii} > 0$ since $Q_i^* \leq B_i$.

$$
\sum_{j \neq i} J_{ij} = p_i A_i \sum_{j \neq i} o_{ij} - c_i \left(\sum_{j \neq i} o_{ij} \right) Q_i^* \leq p_i A_i < J_{ii},
\tag{A33}
$$

where the first inequality follows from the assumption that $\sum_{j \neq i} o_{ij} \leq 1$. Thus, J is strictly diagonally row-dominant at $\mathbf{Q}^*$.

(b) Suppose $Q_i^* > B_i$ $\forall i$. By (23), we show that $Q_i^* = Q_i^{NV}$, which is less than or equal to B_j because of the fact that $Q_i^{NV} \leq Q_i^*$, which is a contradiction.

(c) It is easy to prove that Condition (24) is satisfied. If $(Q_1^*, \cdots, Q_N^*)$ is a Nash equilibrium for the newsvendor game, then $Q_i^* \leq B_i$ for all i by the result in (a).

Suppose there exists a symmetric solution $(Q_1^*, \cdots, Q_N^*)$ such that $Q_i^* = Q^*$ for all i. Proposition 8 shows that Q^* is a solution of the following equation:

$$
Q^* = \frac{pA(B + \gamma(B - Q^*))}{(p - c)A + c\,(B + \gamma(B - Q^*))},
\tag{A34}
$$

which is equivalent to the quadratic equation below:

$$
\gamma c (Q^*)^2 + (cA - (1+\gamma)pA - (1+\gamma)cB)\,Q^* + (1+\gamma)pAB = 0.
\tag{A35}
$$

Let Δ be the discriminant of the above equation. Then

$$
\begin{aligned}
\Delta &= (cA - (1+\gamma)pA - (1+\gamma)cB)^2 - 4\gamma(1+\gamma)pcAB \\
&= ((1+\gamma)pA + (1+\gamma)cB - cA)^2 - 4\gamma(1+\gamma)pcAB \\
&= (\gamma pA + (1+\gamma)cB + pA - cA)^2 - 4\gamma(1+\gamma)pcAB \\
&\geq (\gamma pA + (1+\gamma)cB)^2 - 4\gamma(1+\gamma)pcAB \\
&= (\gamma pA - (1+\gamma)cB)^2 \geq 0.
\end{aligned}
\tag{A36}
$$

Therefore the above quadratic equation has a solution which is given by the quadratic formula:

$$
Q^* = \frac{(1+\gamma)pA + (1+\gamma)cB - cA \pm \sqrt{((1+\gamma)pA + (1+\gamma)cB - cA)^2 - 4\gamma(1+\gamma)pcAB}}{2\gamma c}.
\tag{A37}
$$

Because $p > c$ and $0 \leq \gamma \leq 1$, we have

$$\frac{(1+\gamma)pA + (1+\gamma)cB - cA}{2\gamma c} \geq \frac{(1+\gamma)cB}{2\gamma c} \geq B. \tag{A38}$$

This shows that in the above quadratic formula, the minus sign must be taken because $Q^* \leq B$. The proof is complete. ∎   ∎

Proof of Proposition 10.

First let us prove that $o_{ij} > \bar{o}_{ij}$ and $o_{ji} > \bar{o}_{ji}$ do not hold simultaneously. If we suppose that they do not, then we have

$$
\begin{aligned}
p_1 A_1 \;&>\; c_1 B_1, \\
p_2 A_2 \;&>\; c_2 B_2, \\
o_{12} \frac{c_2 B_2 (B_2 - A_2)}{(p_2 - c_2)A_2 + c_2 B_2} \;&>\; \frac{c_1 B_1 (B_1 - A_1)}{p_1 A_1 - c_1 B_1}, \\
o_{21} \frac{c_1 B_1 (B_1 - A_1)}{(p_1 - c_1)A_1 + c_1 B_1} \;&>\; \frac{c_2 B_2 (B_2 - A_2)}{p_2 A_2 - c_2 B_2}.
\end{aligned}
\tag{A39}
$$

Multiplying both sides of the last two inequalities and canceling some common terms, we obtain

$$o_{12} o_{21}(p_1 A_1 - c_1 B_1)(p_2 A_2 - c_2 B_2) > \left((p_1 - c_1)A_1 + c_1 B_1\right)\left((p_2 - c_2)A_2 + c_2 B_2\right). \tag{A40}$$

Since $o_{12} \leq 1$ and $o_{21} \leq 1$, the above inequality implies that

$$(p_1 A_1 - c_1 B_1)(p_2 A_2 - c_2 B_2) > \left(p_1 A_1 + c_1(B_1 - A_1)\right)\left(p_2 A_2 + c_2(B_2 - A_2)\right), \tag{A41}$$

which in turn shows that

$$p_1 A_1 p_2 A_2 > p_1 A_1 p_2 A_2, \tag{A42}$$

because $p_i A_i > p_i A_i - c_i B_i$ and $c_i(B_i - A_i) > 0$ for $i = 1, 2$. This is a contradiction. Hence, $o_{ij} > \bar{o}_{ij}$ and $o_{ji} > \bar{o}_{ji}$ cannot hold simultaneously.

(a) Suppose (Q_i^*, Q_j^*) is a Nash equilibrium. We first prove that $Q_i^* > B_i$ and $Q_j^* \leq B_j$. The statement that $Q_j^* \leq B_j$ follows from Proposition 9 (a). Suppose $Q_i^* \leq B_i$. Note that $o_{ij} > \bar{o}_{ij}$

implies that $p_i A_i > B_i c_i$. Then it follows from (23) and the fact that $Q_j^* \leq B_j$ that

$$Q_i^* = \frac{p_i A_i^2 (1 - \frac{p_i}{c_i})}{(p_i - c_i) A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij} (B_j - Q_j^*) \right)} + \frac{p_i A_i}{c_i} \leq B_i,$$

$$\frac{p_i A_i^2 (1 - \frac{p_i}{c_i})}{(p_i - c_i) A_i + c_i \left(B_i + \sum_{j \neq i} o_{ij} (B_j - Q_j^*) \right)} \leq \frac{-p_i A_i + c_i B_i}{c_i},$$

$$\frac{p_i A_i^2 (p_i - c_i)}{p_i A_i - c_i B_i} \geq (p_i - c_i) A_i + c_i B_i + o_{ij} c_i (B_j - Q_j),$$

$$\frac{c_i B_i c_i (B_i - A_i)}{p_i A_i - c_i B_i} \geq o_{ij} c_i (B_j - Q_j),$$

$$\frac{c_i B_i (B_i - A_i)}{p_i A_i - c_i B_i} \geq o_{ij} (B_j - Q_j^{NV}),$$

$$\frac{c_i B_i c_i (B_i - A_i)}{p_i A_i - c_i B_i} \geq o_{ij} \frac{c_j B_j (B_j - A_j)}{(p_j - c_j) A_j + c_j B_j},$$

$$\bar{o}_{ij} \geq o_{ij},$$

which is a contradiction.

Because $Q_i^* > B_i$, equation (23) shows that $Q_j^* = Q_j^{NV}$ and $Q_i^* = Q_i^\beta$. That is, (Q_i^β, Q_j^*) is the unique Nash equilibrium.

(b) Suppose (Q_i^*, Q_j^*) is a Nash equilibrium. Proposition 9 (a) proves that $Q_i^* \leq B_i$ and $Q_j^* \leq B_j$. Rearranging (23), we obtain:

$$Q_i^* - \frac{p_i A_i}{c_i} = -\frac{\frac{p_i}{c_i}(\frac{p_i}{c_i} - 1) A_i^2}{(\frac{p_i}{c_i} - 1) A_i + B_i + o_{ij} B_j - o_{ij}(\frac{p_j A_j}{c_j}) - o_{ij}(Q_j^* - \frac{p_j A_j}{c_j})}, \quad \text{(A43)}$$

$$Q_j^* - \frac{p_j A_j}{c_j} = -\frac{\frac{p_j}{c_j}(\frac{p_j}{c_j} - 1) A_j^2}{(\frac{p_j}{c_j} - 1) A_j + B_j + o_{ji} B_1 - o_{ij}(\frac{p_i A_i}{c_i}) - o_{ji}(Q_i^* - \frac{p_i A_i}{c_i})}. \quad \text{(A44)}$$

Using the definitions of E_i and G_i, we transform (A43) into

$$E_i o_{ji} \left(Q_i^* - \frac{p_i A_i}{c_i} \right)^2 + (G_i o_{ji} - G_j o_{ij} - E_i E_j) \left(Q_i^* - \frac{p_i A_i}{c_i} \right) - G_i E_j = 0. \quad \text{(A45)}$$

The roots of (A45) are given by

$$Q_i^* = \frac{p_i A_i}{c_i} + \frac{E_i E_j - G_i o_{ji} + G_j o_{ij} \pm \sqrt{(E_i E_j - G_i o_{ji} + G_j o_{ij})^2 + 4 E_i o_{ji} G_i E_j}}{2 E_i o_{ji}}. \quad \text{(A46)}$$

Therefore, $(Q_1^{\alpha+}, Q_2^{\alpha+})$ and $(Q_1^{\alpha-}, Q_2^{\alpha-})$ are two candidates for (Q_i^*, Q_j^*). For $i = 1, 2$, both $Q_i^{\alpha+} \leq B_i$ and $Q_i^{\alpha+} \leq p_i A_i / c_i$ must hold in order for $(Q_1^{\alpha+}, Q_2^{\alpha+})$ to be a valid Nash equilibrium, and the same applies for $(Q_1^{\alpha-}, Q_2^{\alpha-})$. The results follow.

(c) Suppose (27) holds simultaneously when both $i = 1$ and $j = 2$ and $i = 2$ and $j = 1$. Then direct algebraic calculations show that $E_i \geq 0$ and $E_j \geq 0$, which imply that $4 E_i o_{ji} G_i E_j \geq 0$.

Evidently, we conclude that $\alpha_i^+ > 0$ and $\alpha_j^+ > 0$. Consequently, $Q_i^{\alpha+} > A_j p_i / c_i$ and $Q_j^{\alpha+} > A_j p_j / c_j$. Hence $(Q_i^{\alpha+}, Q_j^{\alpha+})$ is not a Nash equilibrium and $(Q_i^{\alpha-}, Q_j^{\alpha-})$ is the unique Nash equilibrium. ∎

Proof of Proposition 11.

First we will show that the optimal solution to the minimax ex-ante regret problem (29) can be attained at a point in the interval $[A_i, B_i^E]$. Indeed, Lemma 1 proves that the inner maximization problem of (29) is equivalent to the same optimization problem with a smaller feasible region $\widehat{Q}_i \in [A_i, B_i^E]$. Let $g(Q_i, \mathbf{Q}_{-i}) = \max_{\mathbf{F} \in \mathcal{D}} (\delta_i^{\text{ea}}(\mathbf{Q}, \mathbf{F}))$. It is easy to prove that $g(Q_i, \mathbf{Q}_{-i}) \geq g(A_i, \mathbf{Q}_{-i})$ for any $Q_i \in [0, A_i]$. It is also easy to prove that $g(Q_i, \mathbf{Q}_{-i}) \geq g(B_i^E, \mathbf{Q}_{-i})$ for any $Q_i \in [B_i^E, +\infty]$.

To simplify our analysis, we can further narrow down the feasible region of Q in the minimax relative ex-ante regret problem. For convenience, we are presenting the following result in the form of a lemma:

Lemma A1 *Consider the problem:*

$$\min_{A_i \leq Q_i \leq \min\left(B_i^E, \frac{p_i A_i}{c_i}\right)} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\frac{\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F})}{\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})} \right) \right). \tag{A47}$$

Then, problems (29) and (A47) are equivalent.

Proof. Let $Q_i = A_i$. Then for any $\mathbf{F}$, we have $\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}) \geq (p_i - c_i)A_i$, and

$$
\begin{aligned}
\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) \right) &= \max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(-c_i \widehat{Q}_i + p_i E_{\mathbf{F}} \left[\min(D_i^E, \widehat{Q}_i) \right] \right) \\
&\leq \max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(-c_i \widehat{Q}_i + p_i E_{\mathbf{F}} \left[\min(B_i^E, \widehat{Q}_i) \right] \right) \\
&= \max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(-c_i \widehat{Q}_i + p_i \widehat{Q}_i \right) \leq (p_i - c_i) B_i^E. \tag{A48}
\end{aligned}
$$

This shows that when $Q_i = A_i$,

$$\max_{\mathbf{F} \in \mathcal{D}} \left(\delta_i^{\text{ea}}(\mathbf{Q}, \mathbf{F}) \right) \leq \frac{B_i^E}{A_i}. \tag{A49}$$

Hence $\frac{B_i^E}{A_i}$ is a positive upper bound for the optimal objective function value of the minimax relative ex-ante regret problem.

Suppose that Q_i satisfies $\frac{p_i A_i}{c_i} \leq Q_i \leq B_i^E$. Choose a probability distribution $\mathbf{F}$ from $\mathcal{D}$ such that the effect demand for newsvendor i has a unit impulse probability distribution with mass at $D_i^E = \frac{c_i}{p_i} Q_i + \varepsilon$, where ε is a sufficiently small and positive constant. Obviously $A_i \leq D_i^E \leq Q_i \leq B_i^E$. Then

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$$\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}) = -c_i Q_i + p_i D_i^E = -c_i Q_i + p_i (\frac{c_i}{p_i} Q_i + \varepsilon) = p_i \varepsilon, \tag{A50}$$

and

$$\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) \right) = (p_i - c_i) D_i^E. \tag{A51}$$

This shows that

$$\frac{\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) \right)}{\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})} > \frac{B_i^E}{A_i}, \tag{A52}$$

and

$$\max_{\mathbf{F} \in \mathcal{D}} \left(\frac{\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) \right)}{\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})} \right) > \frac{B_i^E}{A_i}. \tag{A53}$$

Since $\frac{B_i^E}{A_i}$ is an upper bound for the minimax relative ex-ante regret problem, any Q_i satisfying $Q_i \geq \frac{p_i A_i}{c_i}$ cannot be its optimal solution. That is, the feasible set of solutions to the minimax relative ex-ante regret problem can be reduced to $A_i \leq Q_i \leq \min(B_i^E, \frac{p_i A_i}{c_i})$. This completes the proof of lemma A1. ■

Now, since $\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F}) \geq -c_i Q_i + p_i A_i > 0$ for any $Q_i \in [A_i, \frac{p_i A_i}{c_i})$, we can also rewrite (A47) into another equivalent problem

$$\min_{A_i \leq Q_i \leq \min\left(B_i^E, \frac{p_i A_i}{c_i}\right)} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\frac{\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F})}{\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})} \right) \right) \right). \tag{A54}$$

Swapping the order of two inner maximization problems in the above formulation, we obtain yet another equivalent optimization problem:

$$\min_{A_i \leq Q_i \leq \min\left(B_i^E, \frac{p_i A_i}{c_i}\right)} \left(\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\frac{\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F})}{\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})} \right) \right) \right). \tag{A55}$$

To simplify the minimax relative ex-ante regret problem, we need an additional result:

Lemma A2 *Suppose* $A_i \leq Q_i \leq \min(B_i^E, \frac{p_i A_i}{c_i})$. *Then*

$$\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F}) \right) \right) = \max \left(\frac{(p_i - c_i) A_i}{-c_i Q_i + p_i A_i}, \frac{B_i^E}{Q_i} \right). \tag{A56}$$

Proof. Let $\delta = \frac{c_i \widehat{Q}_i - p_i \min(D_i^E, \widehat{Q}_i)}{c_i Q_i - p_i \min(D_i^E, Q_i)}$ as a function of D_i^E for any fixed Q_i and $\widehat{Q}_i$.

Assume $Q_i \leq \widehat{Q}_i$. We have that

$$\delta = \begin{cases} \frac{\widehat{Q}_i}{Q_i}, & \text{if } D_i^E \geq \widehat{Q}_i, \\ \frac{-c_i \widehat{Q}_i + p_i D_i^E}{-c_i Q_i + p_i Q_i}, & \text{if } Q_i \leq D_i^E < \widehat{Q}_i, \\ \frac{-c_i \widehat{Q}_i + p_i D_i^E}{-c_i Q_i + p_i D_i^E}, & \text{if } D_i^E < Q_i. \end{cases} \tag{A57}$$

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δ attains the maximum at $D_i^E = \widehat{Q}_i$. Similarly when $\widehat{Q}_i \leq Q_i$, we have that

$$\delta = \begin{cases} \frac{\widehat{Q}_i}{Q_i}, & \text{if } D_i^E \geq Q_i, \\ \frac{-c_i\widehat{Q}_i + p_i\widehat{Q}_i}{-c_iQ_i + p_iD_i^E}, & \text{if } \widehat{Q}_i \leq D_i^E < Q_i, \\ \frac{-c_i\widehat{Q}_i + p_iD_i^E}{-c_iQ_i + p_iD_i^E}, & \text{if } D_i^E < \widehat{Q}_i. \end{cases} \tag{A58}$$

δ attains the maximum at $D_i^E = A_i$.

Let $\mathbf{F}$ be any joint probability distribution for the demand of all newsvendors.

We now assume that $Q \leq \widehat{Q}_i$. Let $\widehat{\mathbf{F}}$ be a particular joint probability distribution for the demand of all newsvendors such that $\widehat{F}_i$ is a unit impulse probability distribution with mass at $\widehat{D}_i$ and for any $j \neq i$, $\widehat{F}_j$ is a unit impulse probability distribution with mass at $\widehat{D}_j$ such that $\widehat{Q}_i = \widehat{D}_i + \sum_{j \neq i} o_{ij}(\widehat{D}_j - Q_j)^+$. Since $A_i \leq \widehat{Q}_i \leq B_i^E$ and for all j, $A_j \leq \widehat{Q}_j \leq B_j^E$, it is easy to choose appropriate values for $\widehat{D}_i$ and $\widehat{D}_j$ for $j \neq i$ for the joint probability distribution $\widehat{\mathbf{F}}$ such that $\widehat{Q}_i = \widehat{D}_i + \sum_{j \neq i} o_{ij}(\widehat{D}_j - Q_j)^+$. By the above arguments on δ, we have,

$$\begin{aligned} \frac{-c_i\widehat{Q}_i + p_iE_{\mathbf{F}}\left[\min(D_i^E, \widehat{Q}_i)\right]}{-c_iQ_i + p_iE_{\mathbf{F}}\left[\min(D_i^E, Q_i)\right]} &= \frac{E_{\mathbf{F}}\left[-c_i\widehat{Q}_i + p_i\min(D_i^E, \widehat{Q}_i)\right]}{E_{\mathbf{F}}\left[-c_iQ_i + p_i\min(D_i^E, Q_i)\right]} \\ &= \frac{E_{\mathbf{F}}\left[\delta(-c_iQ_i + p_i\min(D_i^E, Q_i))\right]}{E_{\mathbf{F}}\left[-c_iQ_i + p_i\min(D_i^E, Q_i)\right]} \\ &\leq \frac{\widehat{Q}_i}{Q_i}\frac{E_{\mathbf{F}}\left[-c_iQ_i + p_i\min(D_i^E, Q_i)\right]}{E_{\mathbf{F}}\left[(-c_iQ_i + p_i\min(D_i^E, Q_i))\right]} \\ &= \frac{\widehat{Q}_i}{Q_i} \\ &= \frac{Q_i(p_i - c_i)}{Q_i(p_i - c_i)} \\ &= \frac{-c_i\widehat{Q}_i + p_iE_{\widehat{\mathbf{F}}}\left[\min(D_i^E, \widehat{Q}_i)\right]}{-c_iQ_i + p_iE_{\widehat{\mathbf{F}}}\left[\min(D_i^E, Q_i)\right]}. \end{aligned}$$

This shows that $\widehat{\mathbf{F}}$ is an optimal solution for the inner maximization problem of (A55) if $Q \leq \widehat{Q}_i$.

We next assume that $Q > \widehat{Q}_i$. Let $\widehat{\mathbf{F}}$ be a particular joint probability distribution for the demand of all newsvendors such that $\widehat{F}_i$ is a unit impulse probability distribution with mass at A_i and for any $j \neq i$, and $\widehat{F}_j$ a unit impulse probability distribution with mass at A_j. Similarly, we can prove that $\widehat{\mathbf{F}}$ is an optimal solution for the inner maximization problem of (A55) if $Q > \widehat{Q}_i$.

Combining the above results and fixing any Q_i, we have

$$\max_{A_i \leq \widehat{Q}_i \leq B_i^E} \left(\max_{\mathbf{F} \in \mathcal{D}} \left(\frac{\Pi_i(\widehat{Q}_i, \mathbf{Q}_{-i}, \mathbf{F})}{\Pi_i(Q_i, \mathbf{Q}_{-i}, \mathbf{F})} \right) \right)$$

$$= \max \left(\max_{A_i \leq \widehat{Q}_i \leq B_i^E, \widehat{Q}_i \geq Q_i} \left(\frac{\widehat{Q}_i}{Q_i} \right), \max_{A_i \leq \widehat{Q}_i \leq B_i^E, \widehat{Q}_i < Q_i} \left(\frac{-c_i \widehat{Q}_i + p_i A_i}{-c_i Q_i + p_i A_i} \right) \right)$$

$$= \max \left(\frac{B_i^E}{Q_i}, \frac{(p_i - c_i) A_i}{-c_i Q_i + p_i A_i} \right).$$

This completes the proof of Lemma A2. ∎

By Lemma A2, the minimax relative ex-ante regret problem is converted into a simpler minimization problem:

$$\min_{A_i \leq Q_i \leq \min\left(B_i^E, \frac{p_i A_i}{c_i}\right)} \left(\max \left(\frac{(p_i - c_i) A_i}{-c_i Q_i + p_i A_i}, \frac{B_i^E}{Q_i} \right) \right). \tag{A59}$$

The objective function of (A59) takes the minimum of two pieces. The first piece is an increasing function and the second piece is a decreasing function. Therefore, the optimal solution of (A59) lies at the intersection of two curves $y = \frac{(p_i - c_i) A_i}{-c_i Q_i + p_i A_i}$ and $y = \frac{B_i^E}{Q_i}$ if this intersection is within the interval $[A_i, \min(B_i^E, \frac{p_i A_i}{c_i})]$, and at either A_i or $\min(B_i^E, \frac{p_i A_i}{c_i})$ if this intersection is outside of this interval. The intersection is

$$\frac{p_i A_i B_i^E}{(p_i - c_i) A_i + B_i^E c_i}, \tag{A60}$$

which lies within the interval $[A_i, \min(B_i^E, \frac{p_i A_i}{c_i})]$ following some simple calculations.

Similar to Proposition 7, we can prove that the maximin relative ex-post problem (22) and the minimax ex-ante regret problem (29) are equivalent in the sense that the optimal solution sets of both problems coincide. This follows from (A27), (A56), and the fact that the maximin relative ex-post problem (22) can be equivalently formulated as the following minimax problem:

$$\min_{A_i \leq Q_i \leq B_i^E} \max \left(\frac{(p_i - c_i) A_i}{-c_i Q_i + p_i A_i}, \frac{B_i^E}{Q_i} \right). \tag*{∎}$$

Additional numerical results.

By varying values for problem parameters in our base example (Example 1), we first generated five additional test examples as shown in Table 1. The price and demand support are chosen to vary in Examples 2 and 3 respectively. In Examples 4 and 5, the overflow rate is made either very small or very large. Example 6 was selected to verify the result in Proposition 1, which states that there are two possible types of Nash equilibria when maximizing the expected profit for a known probability distribution of demand. Differences in problem parameters between Example 1 and Examples 2-6 are highlighted in bold in Table 1.

Example	A	B	p	c	γ
1	20	70	20	10	0.5
2	20	70	**40**	10	0.5
3	20	**120**	20	10	0.5
4	20	70	20	10	**0.1**
5	20	70	20	10	**1.0**
6	20	70	**15**	10	**0.95**

Table 1: Data for test examples.

Computational results for Examples 1-6 are shown in Table 2, where each mini-table corresponds to one of the examples 1-6. In Table 2, for each of four criteria, we also report three additional performance measures: the worst possible profit from all possible demand realizations (`WorstRev`), the worst possible absolute regret from all possible demand realizations (`WorstAbsReg`), and the worst possible relative regret from all possible demand realizations (`WorstRelReg`).

We make several observations about the results in Table 2. It is not surprising to see that the newsvendor can achieve the highest expected profit among all criteria if he/she orders based on `Exp-Rev`. We observe that the `Abs-Reg` approach performs exceptionally well given that the Nash equilibrium order quantities for this method are obtained using only demand support information. The performance of the `Rel-Reg` approach drops below the level of `Abs-Reg`, but can still be called satisfactory. In contrast, `Maximin` performed poorly due to its overly conservative nature. Nevertheless, this conservative attitude allows a newsvendor to avoid the worst results as can be seen from column `WorstRev`. Similarly, in terms of the worst absolute (relative) regret, the `Abs-Reg` (`Rel-Reg`) approach is the best as shown in column `WorstAbsReg` (`WorstRelReg`).

Results for Examples 2 and 3 demonstrate that the effects of the width of the demand support and the change of the price are quite significant. On the other hand, the changes in the overflow rate have a smaller impact as shown in the results for examples 4 and 5. The limited influence of the overflow rate is due to the fact that the total number of "overflow" customers is not very large, since each newsvendor sets his/her order quantity so as to ensure that most of his primary customers get served by this very newsvendor. Example 6 differs from the first five examples in that the Nash equilibrium order quantities are obtained using the second formula rather than the first formula of (4), as defined in Proposition 1. We also note that in all examples, there is a unique Nash equilibrium based on the criterion of maximizing the expected profits.

For asymmetric games, we generated three sets of 9 examples each that are variations of our basic example. Here we chose to report the results based on the truncated normal distribution

Method	Solution	ExpRev	AbsReg	RelReg	WorstRev	WorstAbsReg	WorstRelReg
Exp-Rev	47.53	347.75	0.00	1.00	-75.26	337.12	-0.38
Abs-Reg	50.00	341.33	6.41	0.98	-100.00	300.00	-0.50
Rel-Reg	32.64	312.93	34.82	0.90	73.61	560.42	0.37
Maximin	20.00	200.00	147.75	0.58	200.00	750.00	0.21
Exp-Rev	58.20	1182.50	0.00	1.00	218.03	531.15	0.36
Abs-Reg	60.91	1170.87	11.62	0.99	190.91	409.09	0.32
Rel-Reg	46.18	1161.62	20.87	0.98	338.24	1072.07	0.56
Maximin	20.00	600.00	582.50	0.51	600.00	2250.00	0.21
Exp-Rev	75.05	495.49	0.00	1.00	-350.51	674.23	-1.75
Abs-Reg	80.00	482.67	12.83	0.97	-400.00	600.00	-2.00
Rel-Reg	35.61	349.75	145.74	0.71	43.91	1265.86	0.22
Maximin	20.00	200.00	295.49	0.40	200.00	1500.00	0.12
Exp-Rev	45.60	330.83	0.00	1.00	-55.96	268.45	-0.28
Abs-Reg	46.19	330.48	0.36	1.00	-61.90	261.90	-0.31
Rel-Reg	31.48	294.47	36.36	0.89	85.24	423.76	0.43
Maximin	20.00	200.00	130.83	0.60	200.00	550.00	0.27
Exp-Rev	49.29	359.73	0.00	1.00	-92.89	414.21	-0.46
Abs-Reg	53.33	341.98	17.75	0.95	-133.33	333.33	-0.67
Rel-Reg	33.67	323.06	36.67	0.90	63.32	726.65	0.32
Maximin	20.00	200.00	159.73	0.56	200.00	1000.00	0.17
Exp-Rev	43.37	165.13	0.00	1.00	-133.66	259.68	-1.34
Abs-Reg	44.68	162.47	2.66	0.98	-146.84	246.84	-1.47
Rel-Reg	27.51	135.82	29.31	0.82	24.92	414.30	0.25
Maximin	20.00	100.00	65.13	0.61	100.00	487.50	0.17

Table 2: Computational results for six test examples.

because similar results were obtained based for either the uniform or the triangular distributions. In each of our tested examples, the mean of the demand distribution was set to be equal to the middle value of the demand support, and the standard deviation was equal to the mean multiplied by 0.4. In the first set of experiments we used parameters identical to Example 1 above but varied the product price for newsvendor 2 from 12 to 28. In the second set of experiments we varied the upper bound of the demand support for newsvendor 2 from 30 to 110. In the third set of experiments we varied the overflow rate from newsvendor 2 to newsvendor 1 from 0.1 to 0.9.

Numerical results for the three sets of experiments are shown in Figures 4, 5, and 6, respectively. In each of these 12 charts, three solid lines describe the performance measures of the three

methods `Exp-Rev`, `Abs-Reg`, and `Rel-Reg` for newsvendor 1, and three dotted lines depict the performance measures of the three methods `Exp-Rev`, `Abs-Reg`, and `Rel-Reg` for newsvendor 2. To simplify presentation, we do not include numerical results for the `Maximin` approach because of its inferior performance.
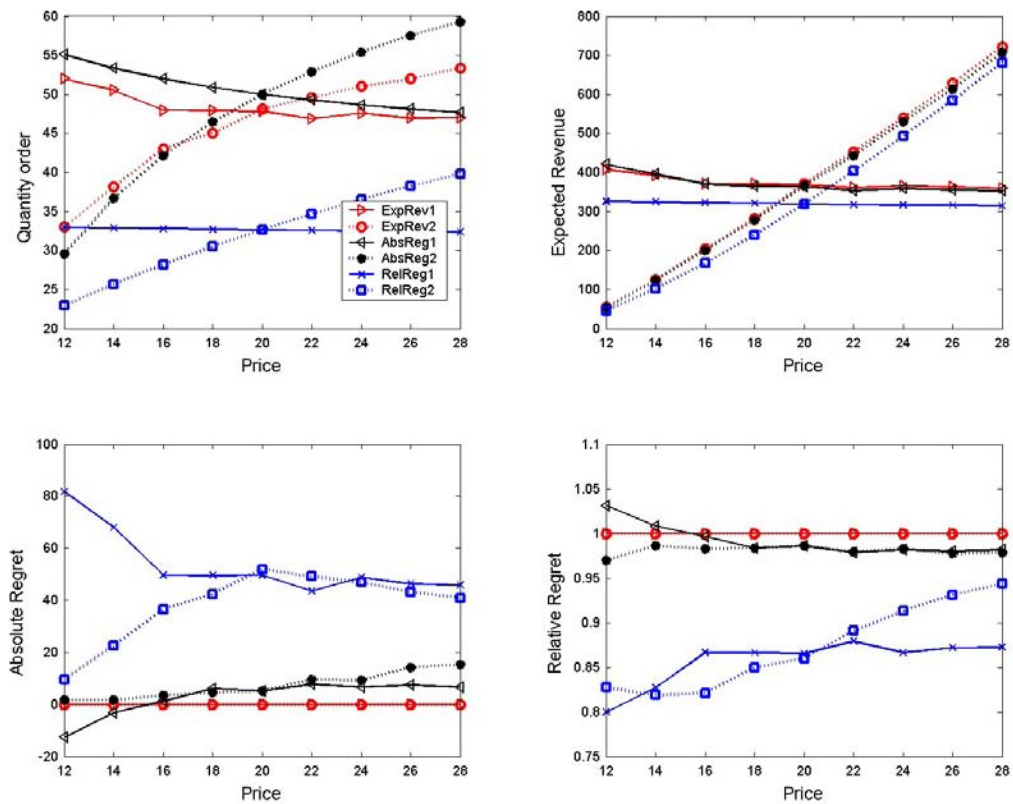

Figure 4: Price sensitivity analysis for an asymmetric game.

A close look at the top-left chart in Figure 4, where the Nash equilibria (or optimal order quantities for both players) based on the three methods `Exp-Rev`, `Abs-Reg`, and `Rel-Reg` are displayed as functions of the product price for the newsvendor 2, yields several observations. First, as expected, newsvendor 2 exhibits lower order quantities than newsvendor 1 when $p_2 < p_1 = 20$, and higher order quantities than newsvendor 1 when $p_2 > p_1 = 20$. Second, both newsvendors have slightly higher order quantities under the `Abs-Reg` approach and significantly lower order quantities under the `Rel-Reg` approach than under the `Exp-Rev` approach. Third, newsvendor 1, as expected, changes his/her order quantity at a slower rate than newsvendor 2, when the price of newsvendor 2 is changed. Fourth, the Nash equilibrium based on the
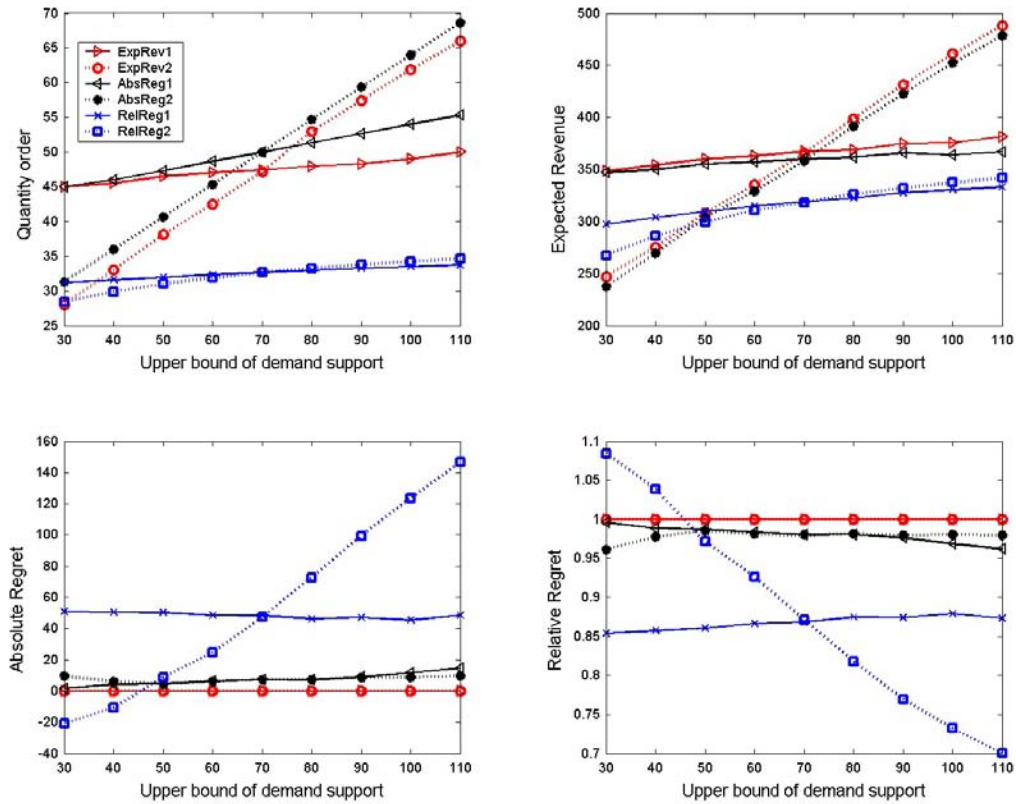
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Figure 5: Demand support sensitivity analysis for an asymmetric game.

`Abs-Reg` approach provides a very good approximation for the Nash equilibrium under the `ExpRev` approach, while the Nash equilibria under `Rel-Reg` are very different from those under `ExpRev`. The observations made in the previous paragraph are further supported by the other three charts in Figure 4. That is, in terms of either `ExpRev` or `AbsReg` or `RelReg`, the performance of the `Rel-Reg` approach is very poor, and the performance of the `Abs-Reg` approach is excellent.

Most observations that we can make from Figures 5 and 6 are similar to those from Figure 4. Here we reiterate two important facts. First, in terms of either `ExpRev` or `AbsReg` or `RelReg`, method `Abs-Reg` performs very well and `Rel-Reg` performs badly compared with `Exp-Rev`. Second, `Abs-Reg` results in slightly higher stocking quantities than the levels under `Exp-Rev`, and `Rel-Reg` results in significantly lower stocking quantities than the levels under `Exp-Rev`. A further interesting observation can be drawn from Figure 5 at $B_2 = 30$. In this case, under the `Exp-Rev` approach, newsvendors 1 and 2 stock approximately 45.42 and 27.65 units, respectively; under the `Abs-Reg` approach, newsvendors 1 and 2 stock 45.00 and 31.25 units, respectively;
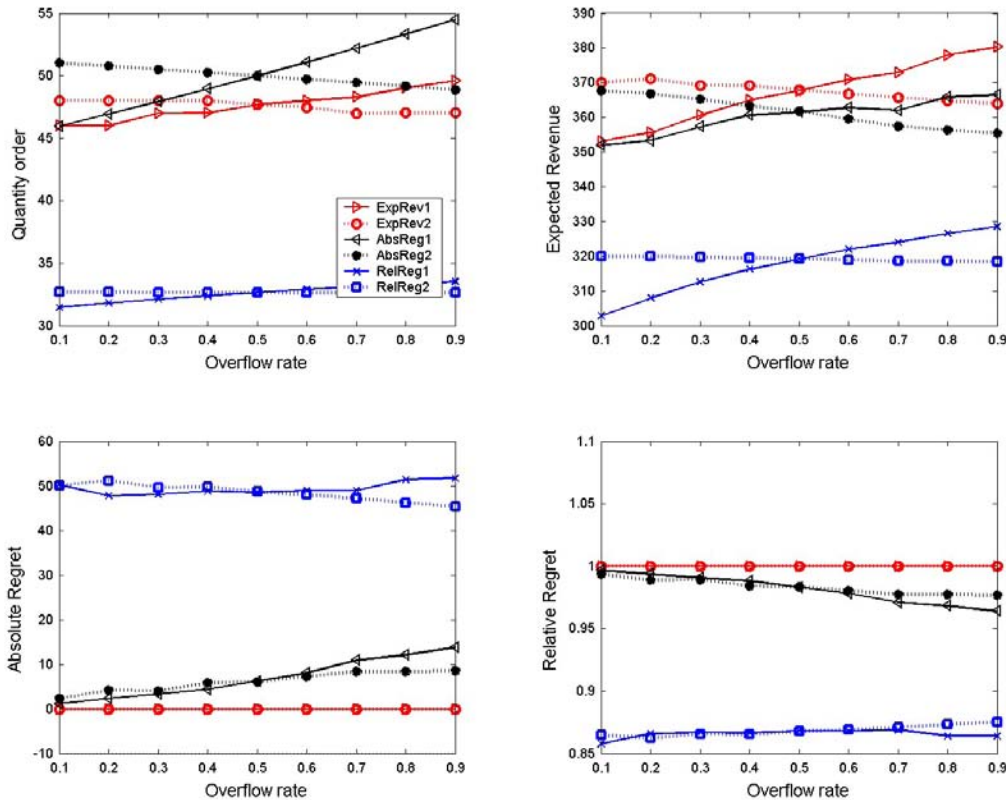
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Figure 6: Demand overflow rate sensitivity analysis for an asymmetric game.

and under the `Rel-Reg` approach, newsvendors 1 and 2 stock 31.19 and 28.47 units, respectively. It is interesting to note that newsvendor 2 stocks more than the upper bound of the support of his/her demand, which never occurs in a monopoly setting. It is even more interesting to see that the absolute regrets for newsvendors 1 and 2 under `Rel-Reg` are 39.25 and -19.78, respectively. In a monopoly setting, the absolute regret is always non-negative because no method can perform better in terms of maximizing the expected profit than `ExpRev`. However, due to the demand overflow and competition, newsvendor 2 can generate higher expected profits using the equilibrium order quantity under the `Rel-Reg` approach than under the `Exp-Rev` approach. The reason is that, as previous literature indicates, under competition newsvendors are likely to overstock inventory. Since the `Rel-Reg` method results in lower stocking quantities, it may lead to higher expected profits. This counterintuitive phenomenon is also reflected in the chart of relative regret in Figure 5, where the relative regret for method `Rel-Reg` is 1.08 at $B_2 = 30$. On the other hand, it is well known that the relative regret is always less than or equal to 1 in

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a monopoly setting. In conclusion, unlike in a monopoly setting, in competition some newsvendors may achieve higher profits under a robust optimization criterion than under the criterion of maximizing the expected profit.