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Congestion Pricing, Bertrand Oligopoly, and Forward Contracts for Bandwidth

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Abstract

We develop a pricing game modelling a monopoly and an oligopoly of Internet Service Providers selling bandwidth on two complementary segments of a multi-provider communication network. We consider pricing behavior when the oligopolists have previously sold part of their capacity by means of forward contracts, assuming all prices are set simultaneously. We find the equilibria in pure strategies where they exist. Where they do not exist, we find an equilibrium allowing the oligopolists to use mixed strategies. This requires solving an extension of the Bertrand-Edgeworth game with symmetric capacities and asymmetric contracting levels. Although providers have an incentive to sell forward contracts to insure against demand uncertainty, contracting also commits them to lower prices in general. We find that any equilibrium with contracting levels is asymmetric with a unique provider choosing the lowest level of contracting. By refraining from signing too many contracts, this provider guarantees a high general downstream price level at a private cost. An increase in the lowest contracting level results in negative marginal externalities on all other oligopolists. On the other hand, an increase in any other contracting level causes positive marginal externalities.

1 Introduction

We develop a model of network operators selling bandwidth on two complementary segments of a multi-provider communication network by means of congestion pricing (a spot market). The "upstream" segment is provided by a monopoly, and the "downstream" segment is provided by an oligopoly. The upstream network provider can be thought of as a large Internet Service Provider (ISP) connecting the oligopoly of small downstream ISPs to the Internet backbone. Customers must purchase the same amount of bandwidth from both the upstream provider and the oligopoly in order to use the network services. We assume this high level of concentration, since the provision of bandwidth exhibits significant economies of scale and it is unlikely that perfect competition is sustainable in the long run.[1]

We model the downstream oligopoly as Bertrand-Edgeworth price competition with capacity constraints. Edgeworth (1925) showed that the duopoly case might not have an equilibrium in prices. Levitan and Shubik (1972) analyzed the same problem specifying the rationing rule that we consider here. They found that prices are competed down to the perfectly competitive level equal to marginal cost when demand is low; and there is a pure-strategy Nash equilibrium, a pair of prices such that neither firm can increase his profit by changing his price, coinciding with the Cournot quantity strategy equilibrium when demand is high. For the intermediate region of demand, they derived a Nash equilibrium in mixed (random) strategies. Vives (1986) established the mixed-strategy equilibrium for the general case of oligopoly with more than two competitors and proved convergence to the perfectly competitive price as the number of competitors increases. We extend this model to the case where the oligopolists have sold forward contracts for diverse fractions of their bandwidth capacities.

The monopoly would be able to appropriate the entire margin from a perfectly competitive downstream industry producing a complementary good. This may not be the case in the scenario we consider here, where the downstream industry is a Bertrand-Edgeworth oligopoly. Depending on the market potential, we find a unique outcome or a bargaining game with multiple outcomes.

We apply the results of the pricing analysis to a multi-provider data network using congestion pricing, where the downstream ISPs fund part of their investment into bandwidth by forward contracts on the congestion price. The purchase of forward contracts by the network users has been proposed in Anderson et al. (2006) as a "Contract and Balancing Mechanism." Our model differs by treating the fraction of an ISP's capacity to be funded by long-term contracts as a strategic variable.

Congestion pricing in communication networks is a relatively new idea. An important early paper is that of Courcoubetis et al. (2000) who consider congestion pricing as a mechanism to recover costs and make a profit. They describe methods of computing usage charges from simple measurements on the network. A more recent paper is that of Ganesh et al. (2007) who consider congestion pricing as a mechanism for sharing bandwidth. They model the interaction among the users as a game and propose a decentralized algorithm. For further reading on the topic, see the survey paper of Steinberg (2003), and the books by Courcoubetis and Weber (2003) and Srikant (2004).

The pricing interaction between the downstream and upstream firms in our model resembles the analysis by Tyagi (1999) where a strategic upstream supplier's possible reactions to entry in the downstream industry are classified based on the derivative of the upstream supplier's price with respect to the number of downstream oligopolists. The upstream provider's price response to a change in a downstream firm's contracting level in our model causes a similar "input cost effect".

This paper is organized as follows. We first describe a pricing game in §2 to model the interaction between the upstream monopolist and the capacity-constrained downstream oligopolists competing in prices with each other. We show in §3 that, for sufficiently low market potential, downstream prices are competed down to marginal cost, while for sufficiently high market potential, there may be multiple pure-strategy Nash equilibrium outcomes, with different divisions of the total industry profit between the upstream and downstream providers. We assume the large upstream monopolist has all the bargaining power and can choose which equilibrium will arise. In the region of intermediate market potential, we find an equilibrium point using mixed strategies for the downstream oligopolists in §4.

In §5 we use the pricing analysis to investigate the downstream oligopolists' incentives for using forward contracts to fund their bandwidth. We find that an oligopolist choosing a low level of contracting is able to raise the general downstream price level, allowing its competitors to contract more. A pure-strategy Nash equilibrium of contracting levels, if it exists, must have a unique lowest level of contracting. We further prove that an increase in this lowest level has a negative marginal externality on other oligopolists' utility, whereas an increase in any other contracting level creates positive marginal externalities. In §6, we present conclusions. In order to aid readability, we have relegated the more technical aspects of the proofs of the first two theorems to three lemmas; the proofs of the lemmas are provided in the Appendix.

2 Pricing Model

We consider the pricing game played by $n+1$ firms supplying bandwidth on two perfectly complementary network segments: an upstream monopolist $\mathcal{M}$ supplying one segment, and n downstream oligopolistic firms $\mathcal{O}_1$, $\mathcal{O}_2$, $\ldots$, $\mathcal{O}_n$, $n \geq 1$, supplying the other segment in Bertrand (price) competition. We are assuming that $\mathcal{M}$ is a large ISP connecting the $\mathcal{O}_i$ to the Internet backbone and has all the bargaining power. Thus, where the pricing game has multiple equilibria, the equilibrium with largest $p_{\mathcal{M}}$ arises. In the special case of $n = 1$, $\mathcal{O}_1$ is another monopolist and our game describes a bilateral monopoly.

The firms simultaneously choose prices $p_{\mathcal{M}}, p_1, p_2, \ldots, p_n$. Where the pricing game has no pure-strategy Nash equilibrium and prices fluctuate, a realistic analysis needs to take into account the timescales over which providers are likely to adjust their prices.

This in turn depends on the technologies used for price updates. While the downstream providers can directly broadcast their prices to local users connected to their networks every few seconds, this approach does not scale to a large multi-provider network such as the Internet. The monopolistic transit provider is more likely to make use of a general pricing system. A natural suggestion for implementing inter-provider pricing by including a price path attribute in the Border Gateway Protocol (defined in Rekhter and Li (1995), Rekhter and Gross (1995)) has been made by Mortier (2001). This system would propagate price changes over the BGP convergence timescale of several minutes. For this reason, we assume that the downstream oligopolists' prices in our pricing game are updated on a shorter timescale than the monopolist's price.

The output $D_{\mathcal{M}}$ of the monopoly $\mathcal{M}$ is the sum of the outputs D_i of each oligopolist $\mathcal{O}_i$ serving the complementary market, i.e.,

$$D_{\mathcal{M}} = \sum_{i=1}^{n} D_i.$$

We assume the costs of building the firms' infrastructure are sunk, and zero marginal costs are incurred during operation of the network. This is a good approximation for Internet provision. On the other hand, any constant marginal costs can be normalized to zero by redefining the prices, provided the marginal costs incurred by the competing oligopolists are equal. Let the firm $\mathcal{M}$'s payoff be

$$\pi_{\mathcal{M}} = p_{\mathcal{M}} D_{\mathcal{M}}.$$

Suppose each oligopolist $\mathcal{O}_i$ has previously sold capacity f_i by means of forward contracts, so his payoff is

$$\pi_i = p_i(D_i - f_i).$$

We assume a linear market demand function for simplicity, as used by Levitan and Shubik (1972),

$$D_{\text{market}}(p_{\mathcal{M}} + p_i) = \alpha - \beta(p_{\mathcal{M}} + p_i),$$

rationed by capacity constraints of k on each oligopolist $\mathcal{O}_i$. The oligopolists' incentives for choosing their *contracting levels* f_i under demand uncertainty are to be discussed in § 5. For the pricing model, we suppose simply that the *market potential* α and the *price sensitivity* β are given non-negative constants, and the contracting levels are given constants with $0 \leq f_i < k$ for some k.

Assume the monopolist $\mathcal{M}$ is not subject to any capacity constraint, other than the total capacity nk resulting from the market for the complementary good. We consider the rationing rule maximizing consumer surplus and chosen by Kreps and Scheinkman (1983), Levitan and Shubik (1972): demand fills the cheapest oligopolists' capacities first and there is no income effect on consumption.

Formally, let us order the oligopolists so $p_1 \leq p_2 \leq \cdots \leq p_n$, and write

$$i^* = \max\{1 \leq i \leq n : \ \alpha - \beta(p_{\mathcal{M}} + p_i) > k(i-1)\} \tag{1}$$

so p_{i^*} is the highest price yielding a positive market share. Then the output of each $\mathcal{O}_i$ is

$$D_i = \begin{cases} k & \text{for } p_i < p_{i^*} \\ 0 & \text{for } p_i > p_{i^*} \\ \frac{\alpha - \beta(p_{\mathcal{M}} + p_{i^*}) - k(j^*-1)}{m} & \text{for } p_i = p_{i^*}, \end{cases} \tag{2}$$

where there are m downstream providers $p_{j^*}, \ \ldots, \ p_{j^*+m-1}$ pricing at p_{i^*}. In the special case when the set in the Definition (1) of i^* is empty, we define $D_i = 0$ for every $\mathcal{O}_i$, since market demand is zero even at the lowest total price $p_1 + p_{\mathcal{M}}$.

The output of the monopolist $\mathcal{M}$ is then given by

$$D_{\mathcal{M}} = \sum_{i=1}^{n} D_i = \max\left\{\alpha - \beta(p_{\mathcal{M}} + p_{i^*}), \ k(j^* + m - 1)\right\}. \tag{3}$$

Without loss of generality, we shall assume that the contracting levels are ordered as

$$0 \leq f_1 \leq f_2 \leq \cdots \leq f_n < k.$$

3 Pure-Strategy Equilibrium Analysis

The equilibrium outcome of the pricing game depends on the available production capacity and the market potential α. The following definition divides the set of possible levels of market potential into three regions.

Definition 1 (high, low, intermediate market potential). Let $0 \leq f < k$. Consider the thresholds

$$\alpha_l(f) = 2(n-1)k + 2f,$$
$$\alpha_h(f) = (2n+1)k - f.$$

We say that market potential is *f-high* if

$$\alpha \geq \alpha_h(f); \tag{4}$$

that market potential is *f-low* if

$$\alpha \leq \alpha_l(f); \tag{5}$$

and that market potential is *f-intermediate* if

$$\alpha_l(f) < \alpha < \alpha_h(f). \tag{6}$$

As we will now show, in the region of f_1-high market potential network capacity is exhausted, so the total upstream and downstream price $p_1 + p_{\mathcal{M}}$ is the *congestion price*, the lowest price at which demand can be satisfied. In the region of f_1-low market potential, competition forces the downstream market price p_1 down to marginal cost, which is normalized to zero. In the region of f_1-intermediate market potential, oscillatory price behavior follows, as will be explored in the next section. The following theorem characterizes the pure-strategy Nash equilibria in the three regions.

Theorem 1. *Pure-strategy equilibria are characterized as follows:*

(i) If market potential is f_1-high in the pricing game, then there is a range of

pure-strategy equilibria given by

$$p_1 = p_2 = \cdots = p_n \tag{7}$$

$$\beta(p_1 + p_{\mathcal{M}}) = \alpha - kn \tag{8}$$

$$\forall i, \quad f_i(p_1) \geq k - \beta p_1 \tag{9}$$

$$\beta p_{\mathcal{M}} \geq kn, \tag{10}$$

moreover, any f_1-high pure-strategy equilibrium is of this form.

(ii) If market potential is f_1-low, then there is a pure-strategy equilibrium such that every oligopolist sets a zero price ($p_i = 0$ for each i) and $\mathcal{M}$ sets $p_{\mathcal{M}} = \frac{\alpha}{2\beta}$; moreover, in any f_1-low pure-strategy equilibrium, each $\mathcal{O}_i$ either sets zero price or produces zero output ($p_i D_i = 0$ for each i).

(iii) If market potential is f_1-intermediate and $n = 1$, then there is a unique pure-strategy equilibrium given by

$$p_1 = \frac{\alpha - 2f_1}{3\beta}, \qquad p_{\mathcal{M}} = \frac{\alpha + f_1}{3\beta}. \tag{11}$$

If market potential is f_1-intermediate and $n \geq 2$, then there is no pure-strategy equilibrium.

Some observations may be in order. To begin, note that the general form of the result only differs between the bilateral monopoly and the true downstream oligopoly case ($n \geq 2$) when market potential is f_1-intermediate and competition results in the non-existence of any pure-strategy equilibrium. However, the boundaries between the regions depend on the number n of downstream firms. In the bilateral monopoly case ($n = 1$), for example, the equilibrium with $p_1 = 0$ arises only if $f \geq \frac{\alpha}{2}$. In the absence of competition to force the downstream price to zero, this will only happen when market potential is so low that, given the contracting level f_1, provider $\mathcal{O}_1$ cannot obtain a positive profit by setting $p_1 > 0$.

Observe that none of the results stated in Theorem 1 depend on $f_2, \ldots, f_n$, but only on the lowest contracting level f_1. In general, any contracting weakens a downstream provider $\mathcal{O}_i$'s incentive to set a high price in the pricing game, and the provider $\mathcal{O}_1$ with the lowest contracting level will have the strongest incentive to do so. Under our assumptions, including the condition that $\mathcal{M}$ holds all the bargaining power and, when market potential is f_1-high, the equilibrium with the highest $p_{\mathcal{M}}$ arises, the equilibrium price levels are determined by $\mathcal{O}_1$ and $\mathcal{M}$, the other oligopolists being able to follow $\mathcal{O}_1$'s price p_1.

Proof of Theorem 1. We first show that the prices specified for f_1-high and f_1-low market potential are in equilibrium. We then proceed by proving two non-existence results in the regions where market potential is not f_1-high and not f_1-low, respectively, allowing us to deduce the unique characterization of the stated pure-strategy equilibria in the extremal regions and non-existence for f_1-intermediate market potential. Special attention is paid to the case of bilateral monopoly $n = 1$.

If market potential is f_1-high, this allows the choice of $p_1, p_{\mathcal{M}}$ satisfying the outlined conditions. We verify that these moves do indeed form a pure-strategy Nash equilibrium. Here $\mathcal{M}$ serves a market of maximal size nk and he can do no better by cutting his price. The effect on $\mathcal{M}$'s profit of a rise in $p_{\mathcal{M}}$ is

$$\left.\frac{\partial \pi_{\mathcal{M}}}{\partial p_{\mathcal{M}}}\right|_{+} = \alpha - \beta(p_1 + p_{\mathcal{M}}) - \beta p_{\mathcal{M}} \leq nk - \beta p_{\mathcal{M}} \leq 0,$$

at the chosen point as well as for any higher value of $p_{\mathcal{M}}$. Therefore, $\mathcal{M}$ has no incentive to change his strategy.

Firm $\mathcal{O}_1$'s profit, on the other hand, is

$$\pi_1 = p_1 \max\{k;\ \alpha - \beta(p_{\mathcal{M}} + p_1) - (n-1)k\} - f_1 p_1.$$

Since the market share $\alpha - \beta(p_{\mathcal{M}} + p_1) - (n-1)k$ is equal to k at our chosen point, and $f_1 \leq k$, $\mathcal{O}_1$ cannot gain by cutting his price. Here $\mathcal{O}_1$ cannot increase his profit by raising his price either. We have shown that the chosen point is indeed a pure-strategy Nash equilibrium.

If market potential is f_1-low, consider the set of strategies $p_i = 0\ \forall i$, $\beta p_{\mathcal{M}} = \frac{\alpha}{2}$. The price $p_{\mathcal{M}}$ is clearly player $\mathcal{M}$'s best response to the zero strategy played by the $\mathcal{O}_i$: it is the monopolistic price. Observe that the total market served is $D_{\mathcal{M}} = \frac{\alpha}{2} \leq (n-1)k + f_1$. Therefore, if $\mathcal{O}_i$ chooses any other price $p_i > 0$, his profit is negative. We have established that this set of strategies is indeed a Nash equilibrium.

The unique characterization for the previous two equilibria follows from the following lemma, proved in the appendix.

Lemma 1. *Let* $(p_{\mathcal{M}};\ p_1,\ p_2,\ \ldots,\ p_n)$ *be a pure-strategy Nash equilibrium in prices.*

(i) Suppose market potential is not f_1*-low. Then there exists* $1 \leq i \leq n$ *such that player* $\mathcal{O}_i$ *has* $p_i D_i > 0$ *in equilibrium.*

(ii) Suppose market potential is not f_1*-high. If* $n \geq 2$ *then every* $\mathcal{O}_i$ *has* $p_i D_i = 0$ *in equilibrium.*

When market potential is f_1-high, consider any pure-strategy equilibrium given by the tuple of prices $(p_{\mathcal{M}};\ p_1,\ p_2,\ \ldots,\ p_n)$. We will show that the equilibrium satisfies (7) to (10). Let i^* be as specified in (1), which satisfies $p_{i^*} > 0$, $D_{i^*} > 0$ by the lemma. Suppose there was some j such that $p_j > p_{i^*}$. Then we would have $D_j = 0$ by the definition of i^*, so player $\mathcal{O}_j$ would have an incentive to set p_j equal to p_{i^*}. Suppose now that there was some j such that $p_j < p_{i^*}$. Then player $\mathcal{O}_j$ would be able to set any $p_j < p_{i^*}$ while retaining a market share of k. Since $f_j \leq k$, he would not decrease his profit by doing so. Therefore, we have shown that all prices are equal in our equilibrium (7).

Suppose we had $D_1 < k$. Then $\mathcal{O}_1$ would be able to increase his market share to k by cutting his price by any small amount. Hence we must have $D_1 = k$ at equilibrium and the total market served is nk (8).

Our previous argument shows that (9) and (10) must hold at equilibrium, so the players $\mathcal{O}_1$ and $\mathcal{M}$ respectively have no incentive to increase their price. We have therefore shown that every non-trivial pure-strategy equilibrium is of the given form.

If market potential is f_1-low, $n \geq 2$, Lemma 1 shows that every $\mathcal{O}_i$ has $p_i D_i = 0$. If market potential is f_1-low and $n = 1$, it is easy to see that the unique pure-strategy equilibrium is given by

$$p_1 = 0, \quad p_{\mathcal{M}} = \frac{\alpha}{2\beta}.$$

Lemma 1 shows that there can be no pure-strategy equilibrium if market potential is f_1-intermediate and $n \geq 2$. Finally, if market potential is f_1-intermediate and $n = 1$, it is easy to see that the unique pure-strategy equilibrium is given by (11) This completes the proof of the theorem. □

4 Mixed-Strategy Equilibrium Analysis

From Theorem 1, we know that for f_1-intermediate market potential there is no pure-strategy Nash equilibrium when the downstream market is a true oligopoly ($n \geq 2$). Since the oligopolists $\mathcal{O}_i$ set their prices on a shorter timescale than the monopolist $\mathcal{M}$, we assume they use mixed strategies, interpreted as distributions of fluctuating prices following Levitan and Shubik (1972). The pricing game can be shown to have an equilibrium point.

Theorem 2. *Suppose $n \geq 2$ and market potential is f_1-intermediate in the pricing game. Then there exists a unique equilibrium point ($p_{\mathcal{M}}$; p_1, ..., p_n) where the price $p_{\mathcal{M}}$ is a pure strategy for $\mathcal{M}$ and the prices p_i are mixed strategies for each $\mathcal{O}_i$, respectively, such that $p_{\mathcal{M}}$ is locally optimal and each p_i is optimal given the other players' strategies.*

Local optimality of the upstream equilibrium price $p_{\mathcal{M}}$ means that the player $\mathcal{M}$ has no incentive to make small-scale deviations. The question of global optimality of $p_{\mathcal{M}}$ is of little importance, since the other players can in any case not be expected to maintain their strategies if $\mathcal{M}$ makes large-scale deviations. This argument for the stability of local equilibria is made in Rothschild and Stiglitz (1976). However, an interesting question that remains is whether allowing $\mathcal{M}$ to play a mixed strategy leads to a different equilibrium point. We will consider this in Theorem 3.

Proof of Theorem 2. The proof of this theorem makes use of a generalization of the solution of the Bertrand-Edgeworth oligopoly in Levitan and Shubik (1972), Vives (1986) taking forward contracting into account.

Preliminaries: Reduced Pricing Game. We start by considering the *reduced pricing game* arising between the $\mathcal{O}_i$ if $\mathcal{M}$ has precommitted to a fixed price $p_{\mathcal{M}}$. In analogy with Definition 1, the following regions turn out to be useful.

Definition 2. Let $0 \leq f \leq k$. We say that market potential is $(f,\ p_{\mathcal{M}})$-*high* if

$$\beta p_{\mathcal{M}} \leq \alpha - k(n+1) + f; \tag{12}$$

that market potential is $(f,\ p_{\mathcal{M}})$-*low* if

$$\beta p_{\mathcal{M}} \geq \alpha - k(n-1) - f; \tag{13}$$

and that market potential is $(f,\ p_{\mathcal{M}})$-*intermediate* if

$$\alpha - k(n+1) + f < \beta p_{\mathcal{M}} < \alpha - k(n-1) - f. \tag{14}$$

The form of the equilibrium depends on the level of market potential.

Lemma 2. *The reduced pricing game has the following Nash equilibria.*

(i) If market potential is $(f_1,\ p_{\mathcal{M}})$*-high then there is a unique pure-strategy equilibrium, in which each* $\mathcal{O}_i$ *names almost surely*

$$p_i = \frac{\alpha - \beta p_{\mathcal{M}} - kn}{\beta}. \tag{15}$$

(ii) If market potential is $(f_1,\ p_{\mathcal{M}})$*-low then there is a pure-strategy equilibrium, in which each* $\mathcal{O}_i$ *names almost surely*

$$p_i = 0. \tag{16}$$

In every pure-strategy equilibrium, $D_i p_i = 0$ *for every player* $\mathcal{O}_i$*.*

(iii) If market potential is $(f_1,\ p_{\mathcal{M}})$*-intermediate then the reduced pricing game has the following unique mixed-strategy equilibrium.*

Let

$$p_1^1 \equiv \frac{\alpha - \beta p_{\mathcal{M}} - k(n-1) - f_1}{2\beta} \tag{17}$$

$$p_0 \equiv \frac{\beta (p_1^1)^2}{k - f_1} \tag{18}$$

$$h(p) = \frac{p - p_0}{p(kn - \alpha + \beta(p + p_{\mathcal{M}}))} \tag{19}$$

$$H_j(p) = (k - f_j) h(p). \tag{20}$$

We define $p_1^{i+1} \in [0,\ p_1^1]$ *to be the unique value satisfying*

$$h(p_1^{i+1}) \equiv \frac{(k - f_{i+1})^{i-1}}{\prod_{j=1}^{i} (k - f_j)} \qquad \textit{for } 2 \leq (i+1) \leq n \tag{21}$$

$$p_1^{n+1} \equiv p_0. \tag{22}$$

For each $1 \leq j \leq n$*, define the function* $G_j(p)$ *on* $[p_0,\ p_1^j]$ *piecewise as*

$$G_j(p) \equiv \left(\frac{\prod_{k \leq i,\, k \neq j} H_k(p)}{(H_j(p))^{i-2}} \right)^{\frac{1}{i-1}} \qquad \textit{for } p_1^{i+1} \leq p \leq p_1^i,\ j \leq i. \tag{23}$$

Then the reduced pricing game has a unique mixed-strategy Nash equilibrium, in which each $\mathcal{O}_j$ plays a random $p_j \in [p_0\ p_1^j)$ according to the cumulative density function G_j, and, moreover, $\mathcal{O}_1$ plays the value $p_1 = p_1^1$ with positive probability $1 - \frac{k-f_2}{k-f_1}$.

The mixed strategies p_i (as random variables) almost surely satisfy

$$\max\left\{0,\ \frac{\alpha - kn}{\beta} - p_{\mathcal{M}}\right\} < p_i < \frac{\alpha - k(n-1)}{\beta} - p_{\mathcal{M}}, \tag{24}$$

and player $\mathcal{O}_i$'s expected payoff over every mixed strategy p_j is

$$\mathbb{E}_p \pi_i = p_0(k - f_i). \tag{25}$$

Moreover, $\mathbb{E}p_{max} = \mathbb{E}\max_i\{p_i\}$ is everywhere a continuous function of $p_{\mathcal{M}}$. It is continuously differentiable in the region of $(f_1,\ p_{\mathcal{M}})$-intermediate market potential, but it is not differentiable at the boundary points $\beta p_{\mathcal{M}} = \alpha - k(n+1) + f_1$ and $\beta p_{\mathcal{M}} = \alpha - k(n-1) - f_1$ towards $(f_1,\ p_{\mathcal{M}})$-low and $(f_1,\ p_{\mathcal{M}})$-high market potential.

Existence. We now prove existence of the equilibrium point. Let $p_{\mathcal{M}}$ be such that

$$\max\{k(n-1),\ \alpha - k(n+1) + f_1\} \le \beta p_{\mathcal{M}} \le \min\left\{kn,\ \frac{\alpha}{2}\right\}.$$

It follows that

$$\beta p_{\mathcal{M}} \le \frac{\alpha}{2} = \alpha - \frac{\alpha}{2} < \alpha - k(n-1) - f_1,$$

since $\alpha > 2(n-1)k + 2f_1$.

Let $\{p_i\}_i$ be the mixed-strategy equilibrium of Lemma 2. Then the mixed strategy p_i maximizes $\mathcal{O}_i$'s profit. To prove our theorem, we just need to show that $\mathcal{M}$'s expected profit is at a local maximum at some $p_{\mathcal{M}}$ in this range.

First, suppose that $\beta p_{\mathcal{M}} > \alpha - k(n+1) + f_1$. Then, almost surely,

$$\frac{\alpha - kn}{\beta} - p_{\mathcal{M}} < \min\{p_i\} \le \max\{p_i\} < \frac{\alpha - k(n-1)}{\beta} - p_{\mathcal{M}},$$

so $\mathcal{M}$'s market share $D_{\mathcal{M}}$ satisfies

$$k(n-1) < D_{\mathcal{M}} = \alpha - \beta(p_{\mathcal{M}} + p_{max}) < kn,$$

and $\mathcal{M}$'s expected profit is

$$\mathbb{E}\pi_{\mathcal{M}} = p_{\mathcal{M}}(\alpha - \beta(p_{\mathcal{M}} + \mathbb{E}p_{max})),$$

which is locally maximizes by $p_{\mathcal{M}}$ if and only if

$$p_{\mathcal{M}} = \frac{\alpha - \beta\mathbb{E}p_{max}}{2\beta}$$
$$\Leftrightarrow \mathbb{E}p_{max} = \frac{\alpha}{\beta} - 2p_{\mathcal{M}}. \tag{26}$$

At the upper bound of the allowed range for $p_{\mathcal{M}}$, if $\beta p_{\mathcal{M}} = kn < \frac{\alpha}{2}$, then

$$\mathbb{E}p_{max} \geq \frac{\alpha - kn}{\beta} - p_{\mathcal{M}} = \frac{\alpha}{\beta} - 2p_{\mathcal{M}},$$

whereas if $\beta p_{\mathcal{M}} = \frac{\alpha}{2} \leq kn$, then

$$\mathbb{E}p_{max} \geq 0 = \frac{\alpha}{\beta} - 2p_{\mathcal{M}}.$$

At the lower bound of the allowed range, if $\beta p_{\mathcal{M}} = k(n-1) > \alpha - k(n+1) + f_1$, then

$$\mathbb{E}p_{max} \leq \frac{\alpha - k(n-1)}{\beta} - p_{\mathcal{M}} = \frac{\alpha}{\beta} - 2p_{\mathcal{M}}.$$

Since $\mathbb{E}p_{max}$ is continuous in $p_{\mathcal{M}}$, the Intermediate Value Theorem shows that there exists a value $p^*_{\mathcal{M}} \in \left[k(n-1),\ \min\left\{kn,\ \frac{\alpha}{2}\right\}\right]$ such that (26) holds.

On the other hand, at the lower bound $\beta p_{\mathcal{M}} = \alpha - k(n+1) + f_1 \geq k(n-1)$ the mixed-strategy equilibrium of p_i turns out to be the pure-strategy equilibrium given by $p_i = \mathbb{E}p_{max} = \frac{\alpha - \beta p_{\mathcal{M}} - kn}{\beta} < \frac{\alpha}{\beta} - 2p_{\mathcal{M}}$ since $\beta p_{\mathcal{M}} < kn$. Thus, by the Intermediate Value Theorem, there exists a value $p^*_{\mathcal{M}} \in \left(\alpha - k(n+1) + f_1,\ \min\left\{kn,\ \frac{\alpha}{2}\right\}\right]$ such that (26) holds. Since $\beta p^*_{\mathcal{M}} > \alpha - k(n+$

$1)+f_1$, the total demand served by $\mathcal{M}$ retains its functional form in some neighborhood of $p^*_{\mathcal{M}}$, and $p^*_{\mathcal{M}}$ does indeed locally maximize $\mathcal{M}$'s profit.

Uniqueness. To prove that there is only one equilibrium point with the given properties, we first need a technical lemma on the variation with the constant price $p_{\mathcal{M}}$ of the expected maximum price named by an $\mathcal{O}_i$.

Lemma 3. *Suppose market potential is f_1-intermediate. Let the expected maximum downstream price be $\mathbb{E}p_{max} = \mathbb{E}p_{max}\left(p_{\mathcal{M}}, (f_i)_{i=1}^n\right)$ as specified in Lemma 2. Let $p^*_{\mathcal{M}}$ be the pure strategy followed by $\mathcal{M}$ at the equilibrium point constructed above. Then, at $p^*_{\mathcal{M}}$, the function $\mathbb{E}p_{max}$ satisfies*

$$\left.\frac{\partial \mathbb{E}p_{max}}{\partial p_{\mathcal{M}}}\right|_{p_{\mathcal{M}}=p^*_{\mathcal{M}}} > -2. \tag{27}$$

Consider any equilibrium point $(p_{\mathcal{M}};\ p_1,\ \ldots,\ p_n)$, where $p_{\mathcal{M}}$ is a locally optimal pure strategy and each p_i is an optimal mixed strategy. Suppose, for a contradiction, that market potential is not $(f_1, p_{\mathcal{M}})$-intermediate. Then, by Lemma 2, $p_1,\ \ldots,\ p_n$ are pure strategies. It is easy to see that the price $p_{\mathcal{M}}$ must in fact be a globally optimal strategy for $\mathcal{M}$, so the equilibrium point is a pure-strategy Nash equilibrium. This contradicts Theorem 1, so we have proved that market potential is $(f_1, p_{\mathcal{M}})$-intermediate in any equilibrium point with the stated properties.

Consider the function

$$f(p_{\mathcal{M}}) = \alpha - 2\beta p_{\mathcal{M}} - \beta \mathbb{E}p_{max}(p_{\mathcal{M}}).$$

At any equilibrium point satisfying our assumptions, we have $f(p_{\mathcal{M}}) = 0$. We have already shown the existence of such a point $p_{\mathcal{M}} = p^{(1)}_{\mathcal{M}}$. It follows from Lemma 2 that f is continuously differentiable. By Lemma 3

$$f'(p_{\mathcal{M}}) = -2\beta - \beta\frac{\partial \mathbb{E}p_{max}}{\partial p_{\mathcal{M}}} < 0.$$

Suppose, for a contradiction, that there exists $p_{\mathcal{M}}^{(2)} \neq p_{\mathcal{M}}^{(1)}$ with the same properties. Without loss of generality $p_{\mathcal{M}}^{(1)} < p_{\mathcal{M}}^{(2)}$. It follows from the sign of the derivative of f that we can find $0 < \epsilon_1,\ \epsilon_2 < \frac{1}{2}(p_{\mathcal{M}}^{(2)} - p_{\mathcal{M}}^{(1)})$ such that $f(p_{\mathcal{M}}^{(1)} + \epsilon_1) < 0$ and $f(p_{\mathcal{M}}^{(2)} - \epsilon_2) > 0$. Since f is a continuous function, the Intermediate Value Theorem gives $p_{\mathcal{M}}^{(3)} \in (p_{\mathcal{M}}^{(1)} + \epsilon_1,\ p_{\mathcal{M}}^{(2)} - \epsilon_2)$ such that

$$f(p_{\mathcal{M}}^{(3)}) = 0.$$

Inductively, we obtain an infinite sequence $p_{\mathcal{M}}^{(1)},\ p_{\mathcal{M}}^{(2)},\ p_{\mathcal{M}}^{(3)},\ \ldots$ of distinct points in $[p_{\mathcal{M}}^{(1)},\ p_{\mathcal{M}}^{(2)}]$ such that $f(p_{\mathcal{M}}^{(1)}) = f(p_{\mathcal{M}}^{(2)}) = \cdots = 0$. By the Bolzano-Weierstrass Theorem, this sequence must have an accumulation point $\overline{p_{\mathcal{M}}}$. Clearly then $f(\overline{p_{\mathcal{M}}}) = 0$ and $f'(\overline{p_{\mathcal{M}}}) = 0$, which contradicts Lemma 3. We have therefore established uniqueness of $\mathcal{M}$'s equilibrium price $p_{\mathcal{M}}^{(1)}$. By Lemma 2, the equilibrium point is unique. □

One remaining question is whether allowing the monopolist $\mathcal{M}$ to play any mixed strategy gives rise to a different equilibrium. It turns out that this is not the case for mixed-strategy Nash equilibria where demand can be served completely and is sufficient to fill all but one oligopolists' networks *almost surely* (with probability one). When the mixed strategies have no point weights, the condition that demand fills all but one network is equivalent to specifying that almost surely no oligopolist has zero output.

Theorem 3. *Let market potential be f_1-intermediate. Suppose there exists a mixed-strategy Nash equilibrium in the pricing game such that almost surely*

$$k(n-1) \le \alpha - \beta(p_{\mathcal{M}} + p_i) \le kn. \tag{28}$$

Then $p_{\mathcal{M}}$ is a pure strategy and the equilibrium is the equilibrium point given in Theorem 2.

Proof of Theorem 3. Consider a mixed-strategy equilibrium. Suppose that almost surely

$$k(n-1) \leq \alpha - \beta p_{\mathcal{M}} - \beta p_{max} \leq kn. \tag{29}$$

Let

$$\underline{p_{\mathcal{M}}} = \sup\{p : \mathbb{P}\{p_{\mathcal{M}} < p\} = 0\},$$

$$\overline{p_{\mathcal{M}}} = \inf\{p : \mathbb{P}\{p_{\mathcal{M}} > p\} = 0\}.$$

But (29) must still hold almost surely if $\mathcal{M}$ plays any pure strategy $p_{\mathcal{M}} \in [\underline{p_{\mathcal{M}}}, \overline{p_{\mathcal{M}}}]$. For any such pure strategy, $\mathcal{M}$'s expected profit is

$$\mathbb{E}\pi_{\mathcal{M}}(p_{\mathcal{M}}) = p_{\mathcal{M}}(\alpha - \beta p_{\mathcal{M}} - \beta \mathbb{E}p_{max}).$$

This is a quadratic function with a unique maximum on the domain $p_{\mathcal{M}} \in [\underline{p_{\mathcal{M}}}, \overline{p_{\mathcal{M}}}]$. Therefore, $\mathcal{M}$ plays a pure strategy. □

5 Forward Contracting

We now consider contracting under demand uncertainty in the following two-stage game, whose second stage subgame is the pricing game described so far. Suppose the parameter β describing the market's price sensitivity to pay for bandwidth is random. In the first stage, the oligopolists simultaneously choose to sell capacities $0 \leq f_i \leq k$ by means of forward contracts.

When some $f_i = k$, we assume the outcome of the second-stage pricing game is the continuous extension of the pure-strategy equilibrium of Theorem 1 or the equilibrium of Theorem 2, as appropriate.[2]

Having analyzed the second-stage pricing subgame, by backward induction we can now turn our attention to the first stage choice of forward contracting. Between the two stages, the true value of β is revealed. Assume $\mathcal{O}_i$ has the mixed pricing strategy p_i in the second stage and can sell forward contracts for bandwidth over its network segment at the risk-neutral expected price, obtaining first-stage income from the contracts of

$$I_i = f_i \mathbb{E}_\beta \mathbb{E}_p p_i, \tag{30}$$

where $\mathbb{E}_p$ denotes expectation over the players' mixed pricing strategies and $\mathbb{E}_\beta$ denotes expectation over the random parameter β.

If $\mathcal{O}_i$ is risk-averse with a utility function $\mathscr{U}$ which is increasing and strictly concave, its total payoff is

$$\Pi_i = \mathbb{E}_\beta \mathscr{U}(I_i + \mathbb{E}_p \pi_i). \tag{31}$$

Firm $\mathcal{M}$'s payoff is

$$\Pi_\mathcal{M} = \mathbb{E}_\beta \mathbb{E}_p \pi_\mathcal{M}. \tag{32}$$

In general, we do not know if there is a pure-strategy Nash equilibrium in the first-stage choice of contracting levels. However, any such equilibrium must satisfy the following result.

Theorem 4. *Suppose market potential is not* 0*-low (in the sense of Definition 1 with* $f = 0$*) and the players' second-stage moves are the ones predicted by Theorems 1 and 2, assuming the greatest* $p_\mathcal{M}$ *when there are multiple equilibria. Suppose there is a pure-strategy equilibrium of positive first-stage contracting levels, so without loss of generality*

$$0 < f_1 \leq f_i \qquad \text{for every } i. \tag{33}$$

Then the lowest contracting level is unique

$$f_1 < f_i \qquad \text{for every } i > 1. \tag{34}$$

When market potential is not 0-low, any contracting equilibrium where the oligopolists obtain positive payoffs must be asymmetric. In the special case of

0-high market potential, it is easy to show that a pure-strategy Nash equilibrium of contracting levels exists and all but one contracting levels are maximal $f_2 = f_3 = \cdots = f_n = k$ in equilibrium.

Proof of Theorem 4. Clearly market potential is not f_1-low, since $\mathcal{O}_1$ can achieve a positive profit by choosing a sufficiently low tariff schedule $f_1 > 0$, subject to market potential not being 0-low.

Suppose first that market potential is f_1-high. The second-stage subgame has a pure-strategy equilibrium, which is independent of f_j, for $j \geq 1$. Since $\mathcal{O}_j$, $j > 1$, is strictly risk-averse, he has an incentive to choose $f_j > f_1$.

Suppose that market potential is f_1-intermediate instead. Suppose, for a contradiction, that $f_2 = f_1$. We will show that, if $\mathcal{O}_1$ has no incentive to choose a lower tariff schedule, then he must have an incentive to choose a higher one. For each β, $\mathcal{O}_1$'s profit varies with $f_1 = f_2$ according to

$$\begin{aligned}
\left.\frac{d}{df_1}\right|_{\pm} (\pi_1 + I_1) &= (p_0(\beta)(k - f_1) + f_1 \mathbb{E}_\beta \mathbb{E} p_1) \\
&= -p_0 + (k - f_1) \left.\frac{dp_0}{df_1}\right|_{\pm} + \mathbb{E}_\beta \mathbb{E} p_1 + f_1 \mathbb{E}_\beta \left.\frac{d\mathbb{E}p_1}{df_1}\right|_{\pm} \\
&= -p_0 + (k - f_1) \left(\left.\frac{\partial p_0}{\partial f_1}\right|_{\pm} + \frac{\partial p_0}{\partial p_{\mathcal{M}}} \left.\frac{d\mathbb{E}p^*_{\mathcal{M}}}{df_1}\right|_{\pm} \right) \\
&+ \mathbb{E}_\beta \mathbb{E} p_1 + f_1 \mathbb{E}_\beta \left(\left.\frac{\partial \mathbb{E}p_1}{\partial f_1}\right|_{\pm} + \frac{\partial \mathbb{E}p_1}{\partial p_{\mathcal{M}}} \left.\frac{d\mathbb{E}p^*_{\mathcal{M}}}{df_1}\right|_{\pm} \right),
\end{aligned}$$

where

$$\left.\frac{d\mathbb{E}p^*_{\mathcal{M}}}{df_1}\right|_{\pm} = - \left.\frac{\partial \mathbb{E}p_{max}}{\partial f_1}\right|_{\pm} \left(2 + \frac{\partial \mathbb{E}p_{max}}{\partial p_{\mathcal{M}}}\right)^{-1}.$$

It is easy to check that

$$\left.\frac{\partial \mathbb{E}p_1}{\partial f_1}\right|_{-} \leq \left.\frac{\partial \mathbb{E}p_1}{\partial f_1}\right|_{+},$$

and

$$\left.\frac{\partial \mathbb{E}p_{max}}{\partial f_1}\right|_{-} \leq \left.\frac{\partial \mathbb{E}p_{max}}{\partial f_1}\right|_{+}.$$

Trivially

$$\left.\frac{\partial p_0}{\partial f_1}\right|_- < 0 = \left.\frac{\partial p_0}{\partial f_1}\right|_+ .$$

Since $\frac{\partial \mathbb{E} p_1}{\partial p_{\mathcal{M}}} < 0$ and $\frac{\partial p_0}{\partial p_{\mathcal{M}}} < 0$, clearly

$$\left.\frac{d}{df_1}\right|_- (\pi_1 + I_1) < \left.\frac{d}{df_1}\right|_+ (\pi_1 + I_1).$$

This gives

$$\begin{aligned}\left.\frac{\partial}{\partial f_1}\right|_+ \mathbb{E}_\beta \mathscr{U}(\pi_1 + I_1) &= \mathbb{E}_\beta \left(\mathscr{U}'(\pi_1 + I_1) \left.\frac{\partial}{\partial f_1}\right|_+ (\pi_1 + I_1) \right) \\ &> \mathbb{E}_\beta \left(\mathscr{U}'(\pi_1 + I_1) \left.\frac{\partial}{\partial f_1}\right|_- (\pi_1 + I_1) \right) \\ &= \left.\frac{\partial}{\partial f_1}\right|_- \mathbb{E}_\beta \mathscr{U}(\pi_1 + I_1).\end{aligned}$$

The right-hand side must be non-negative since $\mathcal{O}_1$ has no incentive to decrease his tariff schedule. Hence the left-hand side is positive, and $\mathcal{O}_1$ can increase his expected utility by raising his tariff schedule slightly. This is a contradiction, so $f_2 \neq f_1$ as required. □

Theorem 5. *Suppose*

$$0 \leq f_1 < f_2 \leq \cdots \leq f_n < k, \tag{35}$$

and the players' second-stage moves are the ones predicted by Theorems 1 and 2, assuming the greatest $p_{\mathcal{M}}$ when there are multiple equilibria.

If market potential is f_1-intermediate, an increase of f_1 by $\mathcal{O}_1$ results in a negative marginal externality on the other oligopolists' payoffs; and an increase of f_j by $\mathcal{O}_j$, for any $j > 1$, results in a positive marginal externality on the other oligopolists' payoffs.

If market potential is f_1-high, an increase of f_1 by $\mathcal{O}_1$ results in a negative marginal externality on the other oligopolists' payoffs; and an increase of f_j by

$\mathcal{O}_j$*, for any* $j > 1$*, results in zero marginal externality on the other oligopolists' payoffs.*

Choosing a low level of contracting f_1 is like providing a "public good"[3] to the oligopoly, by raising the general price level, but doing so is privately costly to $\mathcal{O}_1$, as it implies a low level of insurance against demand uncertainty. In the case of f_1-intermediate market potential, the choices of the contracting levels $f_2, \ldots, f_n$ result in externalities with the opposite sign, so greater levels of contracting benefit other oligopolists.

Proof of Theorem 5. If market potential is f_1-high, every $\mathcal{O}_i$ charges price

$$p_1 = \frac{k - f_1}{\beta}$$

in the second stage. The theorem is trivial in this case.

If market potential is f_1-intermediate, write $p^*_{\mathcal{M}}$ for the equilibrium value of $p_{\mathcal{M}}$, and let

$$p^*_0 = p_0(p^*_{\mathcal{M}}),$$
$$\mathbb{E}_p p^*_i = (\mathbb{E}_p p_i)(p^*_{\mathcal{M}}).$$

It is easy to show that

$$\begin{aligned} \frac{dp^*_0}{df_i} &= \frac{\partial p_0}{\partial f_i} + \frac{\partial p_0}{\partial p_{\mathcal{M}}} \frac{dp^*_{\mathcal{M}}}{df_i} \\ &= \frac{\partial p_0}{\partial f_i} - \frac{\partial p_0}{\partial p_{\mathcal{M}}} \frac{\partial \mathbb{E} p_{max}}{\partial f_i} \left(2 + \frac{\partial \mathbb{E} p_{max}}{\partial p_{\mathcal{M}}}\right)^{-1}. \end{aligned}$$

When $f_i > f_1$, $\frac{\partial p_0}{\partial f_i} = 0$, $\frac{\partial p_0}{\partial p_{\mathcal{M}}} < 0$ and $\frac{dp^*_{\mathcal{M}}}{df_i} > 0$. Hence $\frac{dp^*_0}{df_i} > 0$. On the other hand, $\frac{\partial p_0}{\partial f_1} < 0$ and $\frac{dp^*_{\mathcal{M}}}{df_i} > 0$, so $\frac{dp^*_0}{df_1} < 0$.

It is also easy to show that

$$\frac{d\mathbb{E}p_j^*}{df_i} = \frac{\partial \mathbb{E}p_j}{\partial f_i} + \frac{\partial \mathbb{E}p_j}{\partial p_{\mathcal{M}}} \frac{dp_{\mathcal{M}}^*}{df_i}$$
$$= \frac{\partial \mathbb{E}p_j}{\partial f_i} - \frac{\partial \mathbb{E}p_j}{\partial p_{\mathcal{M}}} \frac{\partial \mathbb{E}p_{max}}{\partial f_i} \left(2 + \frac{\partial \mathbb{E}p_{max}}{\partial p_{\mathcal{M}}}\right)^{-1}.$$

In particular, when $1 < i \neq j$, $\frac{\partial \mathbb{E}p_j}{\partial f_i} \geq 0$ and $\frac{\partial \mathbb{E}p_j}{\partial p_{\mathcal{M}}} < 0$, so we have $\frac{d\mathbb{E}p_j^*}{df_i} > 0$. On the other hand, if $j > 1$, $\frac{\partial \mathbb{E}p_j}{\partial f_1} \leq 0$, so $\frac{d\mathbb{E}p_j^*}{df_1} < 0$.

Since $\mathcal{O}_j$'s profit is the stochastic quantity $I_j + \pi_j$ where $I_j = f_j \mathbb{E}_\beta \mathbb{E}_p p_j^*$ and $\pi_j = p_0^*(k - f_j)$, the result follows immediately. □

6 Conclusions

We have analyzed a pricing game with an upstream monopoly and a downstream oligopoly providing bandwidth on two complementary segments of a multi-provider communication network, where the oligopolists have previously sold different proportions of their bandwidth by forward contracts. Our results are as follows.

When market potential is low, there is a pure-strategy Nash equilibrium with downstream prices equal to zero or marginal cost. The downstream oligopolists compete the price down in this case, or, for a single downstream firm operating as part of a bilateral monopoly, the capacity sold by forward contracts absorbs all demand.

When market potential is high, there is a range of pure-strategy Nash equilibria with different divisions of the same total network price between the upstream and downstream industries. Output attains the level of available capacity and the total price is the *congestion price*, representing the value of a marginal unit of capacity. The balance of bargaining power between the firms determines which equilibrium arises. When the upstream monopolist $\mathcal{M}$ has all the bargaining power, the fraction of the total income obtained by the downstream industry is a decreasing function of the lowest level of contracting f_1, but is independent of all other levels of contracting.

For intermediate market potential, there is a pure-strategy Nash equilibrium only in the case of a bilateral monopoly (and capacity is not exhausted in this case). For a downstream oligopoly ($n \geq 2$), there exists an equilibrium point consisting of optimal mixed strategies for each oligopolist and a locally optimal pure strategy for the upstream monopolist.

We can draw some conclusions on the choice of forward contracts in a two-stage game assuming the market's price-sensitivity is random and the downstream firms are risk-averse. We assume that market potential is not so low that downstream prices are competed down to zero. We prove that any pure-strategy Nash equilibrium of positive contracting levels must be asymmetric and have a unique lowest contracting level, although the existence of such an equilibrium is only clear for high market potential when a pure-strategy price equilibrium exists in the second-stage subgame.

Under the same assumptions, we prove that the choice of contracting levels causes externalities. An increase in the lowest contracting level has a negative marginal externality on other oligopolists. An increase in any other contracting level has no externality for high market potential, but a positive marginal externality for intermediate market potential. We can think of the oligopolist with the least forward contracting as providing a public good to the oligopoly.

Finally, while our model was developed for a wholesale bandwidth market, it can more generally be seen as an abstract economic model, with conclusions applicable to the pricing of complementary goods where firms sell part of their capacities by forward contracting.

Proofs of the Lemmas

Proof of Lemma 1. (i) Suppose that there exists an equilibrium where every $\mathcal{O}_i$ has $p_i D_i = 0$. Suppose $\mathcal{O}_1$ deviates from his equilibrium strategy by choosing a low price p_1. Then

$$D_1 \geq \alpha - \beta p_{\mathcal{M}} - k(n-1) + O(p_1).$$

Using $\beta p_{\mathcal{M}} \leq \frac{\alpha}{2}$, we get

$$\pi_1 \geq \left(\frac{\alpha}{2} - k(n-1) - f_1\right) p_1 + O((p_1)^2).$$

Since $\alpha > 2(n-1)k + f_1$, this is positive for $p_1 > 0$ small enough, allowing $\mathcal{O}_1$ to achieve a positive profit. This contradicts our assumption that the strategies form an equilibrium.

(ii) Consider any pure-strategy equilibrium given by the tuple of prices $(p_{\mathcal{M}};\ p_1,\ p_2,\ \ldots,\ p_n)$. Suppose some $\mathcal{O}_i$ has $p_i D_i > 0$ in equilibrium. We will show that the equilibrium satisfies (7) to (10). Let i^* be as specified in (1), which satisfies $p_{i^*} > 0$, $D_{i^*} > 0$ (by our assumptions). Suppose there was some j such that $p_j > p_{i^*}$. Then we would have $D_j = 0$ by the definition of i^*, so player $\mathcal{O}_j$ would have an incentive to set p_j equal to p_{i^*}. Suppose now that there was some j such that $p_j < p_{i^*}$. Then player $\mathcal{O}_j$ would be able to set any $p_j < p_{i^*}$ while retaining a market share of k. Since $f_j \leq k$, he would not decrease his profit by doing so. Therefore, we have shown that all prices are equal in our equilibrium (7).

Suppose we had $D_1 < k$. Then $\mathcal{O}_1$ would be able to increase his market share to k by cutting his price by any small amount. Hence we must have $D_1 = k$ at equilibrium and the total market served is nk (8).

Our previous argument shows that (9) and (10) must hold at equilibrium, so the players $\mathcal{O}_1$ and $\mathcal{M}$ respectively have no incentive to increase their price. We have therefore shown that the pure-strategy equilibrium satisfies (7) to (10). But this system of equations has no solution when market potential is not f_1-high, so we have a contradiction and every $\mathcal{O}_i$ must have $p_i D_i = 0$.

□

Proof of Lemma 2. (i) At the given prices, capacity is exhausted, and no $\mathcal{O}_i$ has an incentive to lower his price. Since $\beta p_i \geq k - f_1 \geq k - f_i$, no $\mathcal{O}_i$ has an incentive to raise his price either, so this point is indeed a pure-strategy equilibrium.

To establish uniqueness, consider any pure-strategy equilibrium. Since each $\mathcal{O}_i$ can achieve a positive profit by playing the strategy above, we must have one price $p_i = p_j$ for every $\mathcal{O}_i$, $\mathcal{O}_j$. If p_i is larger than the value given above, then each market share is less than k, and $\mathcal{O}_i$ has an incentive to cut his price by an arbitrarily small amount. If, on the other hand, p_i is smaller than the value given above, then the entire demand is not served. Therefore, $\mathcal{O}_i$ has an incentive to increase his price slightly; he does not lose any market share. This establishes uniqueness.

(ii) If every downstream network names price zero, then the total demand is

$$D_{\mathcal{M}} = \alpha - \beta p_{\mathcal{M}} \leq k(n-1) + f_1.$$

Given that every other $\mathcal{O}_j$ names price zero, $\mathcal{O}_i$'s second-stage profit when naming $p_i > 0$ is

$$\pi_i(p_i) = (k(n-1) + f_1 - \beta p_i - k(n-1))p_i - f_i p_i = (f_1 - f_i - \beta p_i)p_i < 0,$$

so no $\mathcal{O}_i$ has an incentive to play a positive price. This point is indeed a pure-strategy equilibrium, with $\mathbb{E}p_{max} = 0$ almost surely.

Consider any pure-strategy equilibrium. Suppose some $\mathcal{O}_i$ has $p_i D_i > 0$ Without loss of generality, let $\mathcal{O}_i$ be the network playing the highest price who still has a positive market share. If there is any $\mathcal{O}_j$ playing $p_j > p_i$, then $\mathcal{O}_j$ receives zero market share and $\mathcal{O}_j$ has an incentive to just undercut $\mathcal{O}_i$. Hence at equilibrium every $p_j \le p_i$. Suppose there are $m > 1$ downstream providers playing price p_i. The market share obtained by each is

$$\frac{\alpha - \beta p_{\mathcal{M}} - \beta p_i - k(n-m)}{m} < \frac{(m-1)k + f_1}{m} < k,$$

so every such provider has an incentive to just undercut all other providers playing price p_i, and we must have $m = 1$. Therefore, $\mathcal{O}_i$'s positive market share is at most $\alpha - \beta p_{\mathcal{M}} - \beta p_i - k(n-1) < f_1 \le f_i$. Hence $\mathcal{O}_i$ is better off playing $p_i = 0$, which is a contradiction.

(iii) Let market potential be $(f_1, p_{\mathcal{M}})$-intermediate. The following results are direct consequences of the definitions stated in the lemma.

(a) We have

$$0 < p_0 < p_1^1 < \frac{k - f_1}{\beta}.$$

(b) For $p_0 \le p \le p_1^1$, we have

$$k(n-1) < \alpha - \beta(p + p_{\mathcal{M}}) < kn.$$

(c) The function $H_1(p)$ is continuous and strictly increasing on $[p_0,\ p_1^1]$, with $H_1(p_0) = 0$, $H_1(p_1^1) = 1$.

(d) We have

$$p_0 = p_1^{n+1} \le p_1^n \le \cdots \le p_1^2 = p_1^1,$$

where, for $1 < i < n$, $p_1^i = p_1^{i+1}$ if and only if $f_i = f_{i+1}$. Also, $p_1^i = p_0$ if and only if $f_i = k$.

(e) For any j, G_j is a continuous and strictly increasing function on

$[p_0, p_1^j]$. For any j, it satisfies $G_j(p_0) = 0$. For $j > 1$, $G_j(p_1^j) = 1$. $G_1(p_1^1) = \frac{k-f_2}{k-f_1}$.

(f) Finally, for $p_0 \leq p \leq p_1^i$, the cumulative density function of $\max_{j \neq i}\{p_j\}$ satisfies

$$G_{-i}(p) \equiv \prod_{j \neq i,\, p_j > p} G_j(p) = H_i(p).$$

Given the above facts, we can show that the strategies defined in the lemma form a Nash equilibrium. Note that $\mathcal{O}_i$'s market share depends on $\max_{j \neq i}\{p_j\}$. If this is greater than p_i, then $\mathcal{O}_i$'s capacity k is exhausted. If it is less than p_i, then $\mathcal{O}_i$'s market share is the residual after all other $(n-1)$ downstream networks $\mathcal{O}_j$ have been served. Since the probability distributions have no point weights at any $p_0 < p < p_1^1$, and at least one $\mathcal{O}_j$ with $j \neq i$ has $p_j > p_0$ almost surely, the event that $\max_{j \neq i}\{p_j\} = p$ has zero probability for any $p < p_1^1$.

Thus $\mathcal{O}_i$'s profit, when playing some $p_0 \leq p < p_1^i$, is

$$\begin{aligned}
\mathbb{E}\pi_i(p) &= (1 - G_{-i}(p))p(k - f_i) + G_{-i}(p)p(\alpha - \beta(p + p_{\mathcal{M}}) - k(n-1) - f_i) \\
&= p(k - f_i) + G_{-i}(p)p(\alpha - \beta(p + p_{\mathcal{M}}) - kn) \\
&= p_0(k - f_i) = \mathbb{E}\pi_i(p_0).
\end{aligned}$$

Moreover, for $\mathcal{O}_1$,

$$\mathbb{E}\pi_1(p_1^1) = p_0(k - f_1) = \mathbb{E}\pi_1(p_0).$$

To establish the equilibrium, we just need to prove that no $\mathcal{O}_i$ can do any better than this, conditional on the other players' strategies.

Since each $\mathcal{O}_i$ can set price p_0 and be sure of market share k, setting a lower price $p < p_0$ leads to lower profits

$$\mathbb{E}\pi_i(p) = p(k - f_i) < p_0(k - f_i) = \pi_i(p_0).$$

For $\mathcal{O}_1$, p_1^1 is the optimal price to set, if the other players' market shares are served first. Thus $\mathcal{O}_1$ cannot have an incentive to play any price $p > p_1^1$. Since for $i > 1$, $f_i \geq f_1$, no $\mathcal{O}_i$ has an incentive to play a price $p > p_1^1$ either. Moreover

$$\lim_{p \to p_1^{1-}} \mathbb{E}\pi_i(p) \geq \mathbb{E}\pi_i(p_1^1),$$

so it is sufficient to show that $\mathcal{O}_i$ cannot increase his expected profit by playing any p with $p_1^i \leq p < p_1^1$.

Suppose $p_1^i \leq p_1^{j+1} \leq p \leq p_1^j \leq p_1^1$. Observe that

$$\begin{aligned}
G_{max}(p) &= \prod_{l=1}^{j} G_l(p) \\
&= \left(h(p) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} h(p) \\
&\geq \left(h(p_1^{j+1}) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} h(p) \\
&= (k - f_{j+1}) h(p) = H_{j+1}(p) \\
&\geq H_i(p).
\end{aligned}$$

Thus $\mathcal{O}_i$'s profit playing p satisfies

$$\begin{aligned}
\mathbb{E}\pi_i(p) &= p(k - f_i) + G_{max}(p) p(\alpha - \beta(p + p_{\mathcal{M}}) - kn) \\
&\leq p(k - f_i) + H_i(p) p(\alpha - \beta(p + p_{\mathcal{M}}) - kn) = \mathbb{E}\pi_i(p_0),
\end{aligned}$$

since $\alpha - \beta(p + p_{\mathcal{M}}) - kn < 0$. This establishes that the given probability distributions form a mixed-strategy Nash equilibrium.

Conversely, to prove uniqueness, consider any mixed-strategy Nash equilibrium given by cumulative density functions

$$G_j(p) = \mathbb{P}\{p_j < p\}.$$

There are well-defined low- and high-price thresholds for each player

$$\overline{p_0^j} = \sup\{p : G_j(p) = 0\},$$
$$\overline{p_1^j} = \inf\{p : G_j(p) = 1\}.$$

Note that

(a) Every $\mathcal{O}_j$ obtains a positive expected profit $\mathbb{E}\pi_j$ in equilibrium.

(b) In equilibrium, there is enough capacity for the total demand at each low-price threshold, and each $\mathcal{O}_j$'s market share is positive even at his high-price threshold.

$$\alpha - \beta(\overline{p_0^j} + p_{\mathcal{M}}) \leq kn,$$
$$\alpha - \beta(\overline{p_1^j} + p_{\mathcal{M}}) > k(n-1).$$

It follows that every $\mathcal{O}_j$ has the same low-price threshold p_0 as in the equilibrium we have constructed

$$\overline{p_0^j} = p_0 \quad \forall j;$$

every $\mathcal{O}_j$ has the same expected profit $\mathbb{E}\pi_j$ as in the equilibrium we have constructed

$$\mathbb{E}\pi_j = p_0(k - f_j);$$

and $\mathcal{O}_1$ has the same high-price threshold p_1^1 as in the equilibrium we have constructed, and no high-price threshold exceeds it

$$\overline{p_1^1} = p_1^1 \geq \overline{p_1^j} \quad \forall j.$$

Define cumulative density functions for $p_{max}^{-j} = \max_{i \neq j}\{p_i\}$ as before

$$G_{-j}(p) = \prod_{i \neq j} G_i(p).$$

For any $\mathcal{O}_j$, it can be shown that there exists an open interval U of pure strategies containing $[p_0, p_1^1]$ such that whenever $p \in U$, we have

$$G_{-j}(p) \geq H_j(p);$$

and, whenever

$$G_{-j}(p) > H_j(p),$$

there exists $\epsilon > 0$ such that

$$G_j(p - \epsilon) = G_j(p + \epsilon).$$

(This follows from the equilibrium requirement that $\mathcal{O}_j$ should have no incentive to change his mixed strategy.)

The following is further easily shown.

(a) Each G_j is continuous on $(p_0,\ p_1^1]$. (So the players' mixed strategies have no point weights, except possibly at p_0 and p_1^1.)

(b) If $f_i < f_j$, then $\overline{p_1^j} \leq \overline{p_1^i}$.

(c) We have

$$\overline{p_1^2} = p_1^1 = p_1^2.$$

We are ready to prove that the mixed strategies employed by the players are indeed those of our constructed equilibrium. As before, let

$$\overline{p_1^{n+1}} = p_1^{n+1} = p_0.$$

Define $\widetilde{G^i_j}(p)$ for $p_1^{i+1} \leq p \leq p_1^i$, $i \geq j$, $p > p_0$, as

$$\widetilde{G^i_j}(p) = \frac{\left(\prod_{l \leq i,\, l \neq j} H_l(p)\right)^{\frac{1}{i-1}}}{(H_j(p))^{\frac{i-2}{i-1}}}.$$

The proof of uniqueness follows by induction, with the hypothesis, for each $1 \leq i \leq n$. We have $\overline{p_1^i} = p_1^i$, and for $p \in [p_1^{i+1},\ p_1^i]$, $j \leq i$, $p > p_0$, we have $G_j(p) = \widetilde{G^i_j}(p)$. We have already proved this for $i = 1$.

Assume the inductive hypothesis holds for some $(i-1) < n$. We show that it still holds for i. We first show that the second part holds. If $\overline{p_1^{i+1}} = p_1^i$ then every $G_j(p) = G_j(p_1^i) = \widetilde{G^{i-1}_j}(p_1^i) = \widetilde{G^i_j}(p_1^i)$ by the inductive hypothesis and the definition of p_1^i. Suppose $\overline{p_1^{i+1}} < p_1^i$. For every $\overline{p_1^{i+1}} \leq p \leq p_1^i$, if $p > p_0$,

$$G_j(p) = \frac{\left(\prod_{l \leq i,\, l \neq j} G_{-l}(p)\right)^{\frac{1}{i-1}}}{(G_{-j}(p))^{\frac{i-2}{i-1}}}.$$

If $G_{-l}(p) = H_l(p)$ for every $l \leq i$ then $G_j(p) = \widetilde{G^i_j}(p)$.

Suppose, on the other hand, that there exists some $l \leq i$, such that $G_{-l}(p) > H_l(p)$. Then for every j such that $G_{-j}(p) = H_j(p)$, we have $G_j(p) > \widetilde{G^i_j}(p)$. For every l satisfying $G_{-l}(p) > H_l(p)$, define

$$\overline{p_l} = \sup\{q : G_{-l}(q') > H_l(q')\ \forall p \leq q' \leq q\}.$$

By the inductive hypothesis, the supremum exists and $\overline{p_l} \leq p_1^i$. If $\overline{p_l} < p_1^i$,

$$G_{-l}(\overline{p_l}) = H_l(\overline{p_l})$$

follows by continuity; while if $\overline{p_l} = p_1^i$, the same follows by the inductive hypothesis for i. However, for every $p \leq p' < \overline{p_l}$,

$$G_{-l}(p') > H_l(p').$$

Therefore G_l is constant on $(p, \overline{p_l})$, so by continuity at p and left-continuity at $\overline{p_l}$,

$$G_l(p) = G_l(\overline{p_l}) \geq \widetilde{G_l^i}(\overline{p_l}) > \widetilde{G_l^i}(p).$$

We have shown that $G_j(p) > \widetilde{G_j^i}(p)$ for every $j \leq i$, whence

$$G_{-j}(p) = \prod_{l \leq i,\, l \neq j} G_l(p) > H_j(p),$$

for every $j \leq i$. Note that the set

$$\{p' \in [\overline{p_1^{i+1}},\ p] :\ G_l(p'') > \widetilde{G_l^i}(p'')\ \forall p' \leq p'' \leq p,\ l \leq i\}$$

is open in $[\overline{p_1^{i+1}},\ p]$, since each G_l is locally constant at every point inside it, and each $\widetilde{G_l^i}$ is increasing. It is also equal to the set

$$\{p' \in [\overline{p_1^{i+1}},\ p] :\ G_l(p') = G_l(p)\},$$

which is closed by continuity. But since this set is non-empty, open and closed, it must be the entire interval $[\overline{p_1^{i+1}},\ p]$. Hence, $G_l(\overline{p_1^{i+1}}) > \widetilde{G_l^i}(\overline{p_1^{i+1}})$ for every $l \leq i$.

If $i = n$, this contradicts the definition of p_0. We deduce that $G_l(p) = \widetilde{G_l^i}(p)$ for $l \leq i$, $\overline{p_1^{i+1}} < p \leq p_1^i$. If $i < n$, then

$$G_{-(i+1)}(\overline{p_1^{i+1}}) = G_{-(i+1)}(p) \geq H_{i+1}(p) > H_{i+1}(\overline{p_1^{i+1}}),$$

which contradicts the definition of p_1^{i+1}. As before, we deduce that $G_l(p) = \widetilde{G_l^i}(p)$ for $l \leq i$, $p > \overline{p_1^{i+1}}$. This extends to $\overline{p_1^{i+1}}$ by continuity. We have proved the second part of the inductive hypothesis.

As to the first part, we again use the fact that by the definition of $\overline{p_1^{i+1}}$, we must have

$$G_{-(i+1)}(\overline{p_1^{i+1}}) = \prod_{j=1}^{i} G_j(\overline{p_1^{i+1}}) = H_{i+1}(\overline{p_1^{i+1}}),$$

which rearranges as

$$h(\overline{p_1^{i+1}}) = \frac{(k - f_{i+1})^{i-1}}{\prod_{j=1}^{i}(k - f_j)}.$$

The unique solution of this equation is $\overline{p_1^{i+1}} = p_1^{i+1}$. This completes the inductive argument. Since $G_j(p) = 0$ for $p \leq p_0$ and any $j \leq n$, we have proved that the cumulative density functions specifying the mixed strategies employed by the $\mathcal{O}_j$ in any equilibrium coincide with those in the equilibrium we have constructed. Hence the mixed-strategy equilibrium of our game is unique.

Continuous differentiability of $\mathbb{E}p_{max}$ as a function of $p_{\mathcal{M}}$ is trivial inside the regions of $(f_1,\ p_{\mathcal{M}})$-high and -low market potential. For $(f_1,\ p_{\mathcal{M}})$-intermediate market potential, it can be easily shown that p_1^1, p_0 and p_1^i are continuously differentiable functions of $p_{\mathcal{M}}$. We can write

$$\begin{aligned} \mathbb{E}p_{max} &= \int_0^{\infty} (1 - G_{max}(p))dp \\ &= p_0 + \sum_{i=2}^{n} \int_{p_1^{i+1}}^{p_1^i} \left(1 - \left(\prod_{l=1}^{i} H_l(p)\right)^{\frac{1}{i-1}}\right) dp. \end{aligned}$$

Each integral term is continuously differentiable with respect to $p_{\mathcal{M}}$, since the limits are continuously differentiable with respect to $p_{\mathcal{M}}$, the integrands are continuously differentiable with respect to $p_{\mathcal{M}}$ and with respect to p, and the derivative with respect to $p_{\mathcal{M}}$ of each integrand can be bounded above by an integrable function independently of $p_{\mathcal{M}}$, for values of $p_{\mathcal{M}}$ in some sufficiently small interval. Therefore $\mathbb{E}p_{max}$ is continuously differentiable for $(f_1,\ p_{\mathcal{M}})$-intermediate market potential. Continuity and lack of differentiability is easy to verify at the boundary points, completing the proof of the lemma. □

Proof of Lemma 3. As we have seen in the proof of Lemma 2, the function $\mathbb{E}p_{max}$ satisfies the assumptions required for the existence of a continuous derivative which can be found by differentiating under the integral sign. Using the symbols defined there, we obtain

$$\begin{aligned}
\frac{\partial \mathbb{E}p_{max}}{\partial p_{\mathcal{M}}} &= \frac{\partial}{\partial p_{\mathcal{M}}} \int_0^\infty (1 - G_{max}(p))\, dp \\
&= \frac{\partial}{\partial p_{\mathcal{M}}} \left(p_1^1 - \sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} (h(p))^{\frac{j}{j-1}} \left(\prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} dp \right) \\
&= -\frac{1}{2} \left(1 - \frac{k - f_2}{k - f_1} \right) \\
&\quad - \sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} \frac{j}{j-1} \left(h(p) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} \frac{\partial h(p)}{\partial p_{\mathcal{M}}}\, dp. \qquad (36)
\end{aligned}$$

The proof that this expression is always greater than -2 uses different approximations for the overlapping regions given by $\delta < \frac{12}{5}(k - f_1)$ and $\delta > \frac{20}{9}(k - f_1)$, where $\delta = \alpha - 2(n-1)k - 2f_1$.

Region of Lower Market Potential. For $\delta < \frac{12}{5}(k - f_1)$, we show that

$$Q \equiv \sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} \frac{j}{j-1} \left(h(p) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} \frac{\partial h(p)}{\partial p_{\mathcal{M}}}\, dp \;<\; \frac{3}{2}. \qquad (37)$$

To establish this equation note that, for $j \geq 2$,

$$\frac{j}{j-1} \leq 2,$$

and

$$\begin{aligned}
\left(h(p) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} &\leq \left(h(p_1^j) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} \\
&= k - f_j \leq k - f_1.
\end{aligned}$$

We calculate the partial derivative of $h(p)$ with respect to p_M directly

$$
\begin{aligned}
\frac{\partial h(p)}{\partial p_M} &= \frac{\partial}{\partial p_M} \frac{p - p_0}{p(kn - \alpha + \beta(p + p_M))} \\
&= \frac{-\frac{\partial p_0}{p_M} p(kn - \alpha + \beta(p + p_M)) - (p - p_0)\beta p}{p^2(kn - \alpha + \beta(p + p_M))^2} \\
&= \frac{\beta\left(1 - \frac{p_0}{p_1^1}\right)(p_1^1 - p)}{p\left(\frac{p_1^1}{p_0}\beta(p_1^1 - p_0) - \beta(p_1^1 - p)\right)^2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Q &\leq 2(k - f_1) \sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} \frac{\partial h(p)}{\partial p_{\mathcal{M}}} \, dp \\
&= 2(k - f_1) \int_{p_0}^{p_1^1} \frac{\partial h(p)}{\partial p_{\mathcal{M}}} \, dp \\
&= 2(k - f_1) \int_{p_0}^{p_1^1} \frac{\beta\left(1 - \frac{p_0}{p_1^1}\right)(p_1^1 - p)}{p\left(\frac{p_1^1}{p_0}\beta(p_1^1 - p_0) - \beta(p_1^1 - p)\right)^2} \, dp \equiv \overline{Q}.
\end{aligned}
$$

The substitution $t = \frac{p - p_0}{p_1^1 - p_0}$ yields

$$
Q \leq \overline{Q} \equiv \int_0^1 \frac{2\gamma(1-\gamma)(1-t)}{(\gamma + t(1-\gamma))(1 - \gamma(1-t))^2} \, dt,
$$

where $\gamma = \frac{p_0}{p_1^1}$. This can easily be bounded above for appropriate values of γ as follows.

In equilibrium

$$
p_M = \frac{\alpha - \beta \mathbb{E} p_{max}}{2\beta}. \tag{38}
$$

We use $\mathbb{E} p_{max} \leq p_1^1$, so

$$
p_M \geq \frac{\alpha - \beta p_1^1}{2\beta},
$$

and

$$
p_1^1 \leq \frac{\delta + \beta p_1^1}{4\beta},
$$

yielding

$$p_1^1 \leq \frac{\delta}{3\beta}.$$

Thus

$$\gamma = \frac{p_0}{p_1^1} = \frac{\beta p_1^1}{k - f_1} \leq \frac{\delta}{3(k - f_1)} \leq \frac{4}{5}.$$

Since $0 < \gamma < 1$, we can evaluate the integral

$$\overline{Q} = 2\gamma \frac{(\gamma - 1)(\log \gamma - \log(1 - \gamma)) + (2\gamma - 1)}{(2\gamma - 1)^2}.$$

Clearly the integral expression exists everywhere on $0 < \gamma < 1$. Although the evaluated integral is undefined for $\gamma = \frac{1}{2}$, an application of L'Hôpital's Rule shows that it can be extended to this point giving a continuous function of γ on $(0,\ 1)$. We show that $\overline{Q}$ is an increasing function of γ by differentiating it

$$\frac{d\overline{Q}}{d\gamma} = \frac{2(\log \gamma - 4\gamma - \log(1 - \gamma) + 2)}{(2\gamma - 1)^3}.$$

For $\gamma > \frac{1}{2}$, this has the same sign as

$$f(\gamma) = \log \gamma - 4\gamma - \log(1 - \gamma) + 2,$$

which has derivative

$$f'(\gamma) = -4 + \frac{1}{\gamma(1 - \gamma)}.$$

Since $f'(\gamma) > 0$ for $\gamma \neq \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = 0$, we deduce $f(\gamma) > 0$ for $\gamma > \frac{1}{2}$. Hence $\overline{Q}$ is increasing there.

For $\gamma < \frac{1}{2}$, $\frac{d\overline{Q}}{d\gamma}$ has the opposite sign to $f(\gamma)$. As before, $f(\gamma)$ is increasing, so we must have $f(\gamma) < 0$ for $\gamma < \frac{1}{2}$. Then $\overline{Q}$ is increasing there.

Hence $\overline{Q}$ is increasing everywhere on $0 < \gamma < 1$. Since $\overline{Q}(\frac{4}{5}) < \frac{3}{2}$, $\overline{Q}(\gamma) < \frac{3}{2}$ for any $0 < \gamma \leq \frac{4}{5}$. The lemma follows immediately for $\delta < \frac{12}{5}(k - f_1)$.

Region of Higher Market Potential. For $\delta > \frac{20}{9}(k - f_1)$, the following approximation is straightforward to verify:

$$\frac{\partial h(p)}{\partial p_{\mathcal{M}}} \leq \frac{\partial h(p)}{\partial p} + \frac{h(p)}{p}. \tag{39}$$

The first term can be integrated exactly:

$$\begin{aligned}
&\sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} \frac{j}{j-1} \left(h(p) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} \frac{\partial h(p)}{\partial p} \, dp \\
&= \sum_{j=2}^{n} \left(\prod_{l=1}^{j} (k - f_l)^{\frac{1}{j-1}} \right) \left[(h(p))^{\frac{j}{j-1}} \right]_{p_1^{j+1}}^{p_1^j} \\
&= \sum_{j=2}^{n} \frac{(k - f_j)^j - (k - f_{j+1})^j}{\prod_{l=1}^{j} (k - f_l)} \\
&= \frac{k - f_2}{k - f_1}.
\end{aligned}$$

The second term can be approximated as

$$\begin{aligned}
&\sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} \frac{j}{j-1} \left(h(p) \prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} \frac{h(p)}{p} \, dp \\
&\leq \frac{2}{p_0} \sum_{j=2}^{n} \int_{p_1^{j+1}}^{p_1^j} (h(p))^{\frac{j}{j-1}} \left(\prod_{l=1}^{j} (k - f_l) \right)^{\frac{1}{j-1}} dp \\
&= \frac{2}{p_0} (p_1^1 - \mathbb{E} p_{max}) \\
&\leq 2 \left(\frac{k - f_1}{\beta p_1^1} - 1 \right) < 1.
\end{aligned}$$

This last inequality follows for $\delta(\alpha) > \frac{20}{9}(k - f_1)$ because we have

$$\beta p_1^1 \geq 2(k - f_1) \left(1 - \sqrt{1 - \frac{\delta(\alpha)}{4(k - f_1)}} \right) > \frac{2}{3}(k - f_1). \tag{40}$$

Indeed, substituting $\mathbb{E} p_{max} \geq p_0$ into (38) yields

$$p_M \leq \frac{\alpha - \beta p_0}{2\beta},$$

whence

$$p_1^1 \geq \frac{\alpha + \beta p_0 - 2k(n-1) - 2f_1}{4\beta}.$$

Using $p_0(k - f_1) = \beta(p_1^1)^2$, we obtain

$$(p_1^1)^2 - \frac{4(k-f_1)}{\beta} p_1^1 + \frac{\delta(k-f_1)}{\beta^2} \leq 0.$$

Since the coefficient of $(p_1^1)^2$ is positive, this quadratic may be non-positive only if p_1^1 is at least as large as the smaller root of the quadratic, yielding immediately (40).

Substituting our approximations into (36), we get

$$\begin{aligned} \frac{\partial \mathbb{E} p_{max}}{\partial p_M} &> -\frac{1}{2}\left(1 - \frac{k-f_2}{k-f_1}\right) - \frac{k-f_2}{k-f_1} - 1 \\ &= -\frac{3}{2} - \frac{1}{2}\left(\frac{k-f_2}{k-f_1}\right) \geq -2. \end{aligned}$$

This establishes the lemma for $\delta > \frac{20}{9}(k - f_1)$. □

Notes

[1] Several empirical studies of telecommunications cost structures reviewed in Sharkey (1982) conclude for the most part that economies of both scale and scope are significant.

[2] It is easy to show that this extension is well-defined and constitutes an equilibrium. However, when $f_i = k$ the equilibrium may no longer be uniquely characterized as above. The player $\mathcal{O}_i$ is indifferent between two prices if even the higher one guarantees full network utilization. This leads to the emergence of equilibria where $\mathcal{O}_i$ can raise his price without any loss of second-stage income, violating the law of one price. To exclude such unrealistic equilibria, our construction explicitly restricts attention to equilibria which are the limit of equilibria arising when every $f_j < k$.

[3]A *public good* is a good which is non-excludable, non-rivalrous and often non-rejectable. These assumptions mean respectively that it is not possible to exclude someone from using the good, that one individual's usage does not prevent another's, and that an individual cannot refrain from using it.

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