Information Cascades and Threshold Implementation: An Application to Crowdfunding*

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Abstract

Economic activities such as crowdfunding often involve sequential interactions, observational learning, and project implementation contingent on achieving certain thresholds of support. We incorporate endogenous all-or-nothing thresholds in a classic model of information cascade. We find that early supporters tap the wisdom of a later "gate-keeper" and effectively delegate their decisions, leading to uni-directional cascades and preventing agents' herding on rejections. Consequently, entrepreneurs or project proposers can charge supporters higher fees, and proposal feasibility, project selection, and information aggregation all improve, even when agents have the option to wait. Novel to the literature, equilibrium outcomes depend on the crowd size, and in the limit, efficient project implementation and full information aggregation ensue. Our findings are robust to introducing contribution and information acquisition costs, thresholds based on dollar amounts, or selection of alternative equilibria.

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1 Introduction

Financing activities and business processes for support-gathering often involve sequential contributors, observational learning, and project implementation contingent on achieving certain thresholds of support. Crowd-based fundraising, which includes equity or reward crowdfunding, peer-to-peer lending, and initial coin offerings, constitutes arguably the most salient example.[1] Such economic interactions with sequential actions from privately informed agents are prone to information cascades that create incomplete information aggregation and suboptimal financing. Standard models (e.g., Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992) focus on the case of pure informational externalities with each agent's payoff structure independent of others' actions. We incorporate into a model of dynamic contribution game the fact that many projects or proposals are only implemented with a sufficient level of support—an "all-or-nothing" (AoN) threshold, and show that threshold implementation drastically alters the informational environments and economic outcomes, with implications for financing projects and aggregating information, the two most important functions of modern financial markets.[2]

We find that early supporters tap the wisdom of a later "gate-keeper" and effectively delegate their contribution decisions, leading to uni-directional cascades. As the first dynamic model of crowdfunding incorporating observational learning and AoN thresholds, our theory

[1]Since its inception in the arts and creativity-based industries (e.g., recorded music, film, video games), crowdfunding has quickly become a mainstream source of capital for entrepreneurs, partially fueled by the change of financial climate following the 2008 financial crisis has given rise to the culture of decentralized finance. In the span of a few years, its total volume has reached a whopping 35 billion USD globally in 2017. It has surpassed the market size for angel funds in 2015, and the World Bank Report estimates that global investment through crowdfunding will reach \$93 billion in 2025 ($www.infodev.org/infodev-files/wb_crowdfundingreport-v12.pdf$). Statista similarly projects a compound annual growth rate of 14.7% for the next four years ($www.statista.com/outlook/335/100/crowdfunding/worldwidemarket-revenue$). One well-known market leader, Kickstarter, has helped fund almost 200,000 campaigns, raising over 5.6 billion dollars from 19.36 million people ($//www.kickstarter.com/help/stats$ accessed on March 15, 2021). President Obama also signed into law the Jumpstart Our Business Startups (JOBS) Act in April 2012, whose Title III legalized crowdfunding for equity by relaxing various requirements concerning the sale of securities in May 2016. What is more, with the rise of initial coin offerings, corporate crowdfunding using tokens is becoming a new norm, with over ten billion USD raised in the United States alone in 2017 and 2018 and the market cap for tokens exceeding 1.5 trillion USD at the dawn of 2021 (Cong and Xiao, 2020; Cong, Li, Tang, and Yang, 2020). All these took place against the backdrop of a thriving P2P lending market globally of a capitalization over 20 billion USD (Cong, Tang, Xie, and Miao, 2020).

[2]AoN threshold is predominant on crowdfunding platforms and in venture financing; super-majority rule or q-rule is a common practice in many voting procedures; assurance contract or crowdaction in public goods provision is also characterized by sequential decisions and implementation thresholds (e.g., Bagnoli and Lipman, 1989); charitable projects need a minimum level of funding-raised to proceed (e.g., Andreoni, 1998).

constitutes an initial step in understanding crowdfunding dynamics and other sequential contribution games, especially concerning how threshold implementation and large crowds can improve proposal feasibility, project selection, and information aggregation. We also contribute to the theory of observational learning by demonstrating that a simple addition of threshold implementation can generate asymmetric information cascades and that project implementation and information aggregation are crowd-size dependent and can achieve full efficiency in the large-market limit, results hitherto unobtainable in the literature.

Specifically, we introduce threshold implementation in a standard framework of information cascade à la Bikhchandani, Hirshleifer, and Welch (1992), allowing potentially endogenous AoN thresholds and pricing. A project proposer sequentially approaches N agents who choose to support or reject the project. Each supporter pays a pre-determined price, and then gets a payoff normalized to one if the project is good. All agents are risk-neutral and have a common prior on the project's quality. They each receive a private, informative signal, and observe the actions of preceding agents, before deciding whether to make a contribution/support. Deviating from the literature, supporters only pay the price and receive the project payoff if the number of supporters reaches an AoN threshold, potentially pre-specified by the proposer.

AoN thresholds lead to uni-directional cascades in which agents never rationally ignore positive private signals to reject the project (DOWN cascade), but may rationally ignore negative private signals to support the project (UP cascade). Information aggregation (and its costly production) also become more efficient, especially with a large crowd of agents, leading to more successes of good projects and weeding out bad projects. When the implementation threshold and price for supporting are endogenous, the proposer no longer under-price the issuance as seen in Welch (1992). Consequently, proposal feasibility, project selection, and information aggregation improve. In particular, when the number of agents grows large, equilibrium project implementation and information aggregation approach full efficiency, in stark contrast to the literature's previous findings (Banerjee, 1992; Lee, 1993; Bikhchandani, Hirshleifer, and Welch, 1998; Ali and Kartik, 2012).

To derive these, we first take the AoN threshold and price as given, and show that before reaching the threshold, the aggregation of private information only stops upon an UP cascade. The intuition follows from that the threshold links an agent's payoff to subsequent agents' actions, making her partially internalize the informational externalities of her action.[3] Interestingly, such forward-looking considerations lead to asymmetric outcomes: agents with positive private signals always support because they essentially delegate their decisions to a subsequent "gate-keeping" agent whose supporting decision brings the total support to the threshold. Delegation hedges against supporting a bad project because the "gate-keeping" agent, having observed a longer history of actions by the time she makes the decision, evaluates the value of supporting with better information than previous agents. Meanwhile, before an UP cascade, agents with negative private signals are reluctant to support before the threshold is reached, because in equilibrium their supporting actions may mislead subsequent agents and cause either a too-early UP cascade or the support's reaching the AoN threshold without enough number of positive signals, both implying a negative expected payoff for her. Therefore, DOWN cascades are always interrupted by agents with positive signals before the threshold is reached.

We then allow the entrepreneur or proposer to endogenously design the AoN threshold (in addition to setting the contribution price) to maximize the proceeds or the level of support. A higher AoN threshold delays potential DOWN cascades (after AoN threshold being reached) but is also less likely to be reached. In other words, the proposer trades off increasing price to increase the proceed from every supporter with lowering price to boost the probability of winning more supporters and implementing the project. Consequently, in equilibrium there is no DOWN cascade except for a special scenario in which a DOWN cascade starts at the last agent and the project would not be implemented anyway even if all private signals become public.

AoN thresholds (especially when it is endogenous) and uni-directional cascades have three important implications. First, they allow good projects with costly production to be supported. Unlike the case in (Welch, 1992), AoN threshold provides the proposer an additional tool to expand the feasible pricing range to potentially finance all positive NPV projects no matter what the production cost is. Second, in standard models of financial markets with information cascades, the proposer may underprice contributions to avoid DOWN cascade. AoN thresholds hedges against implementation failure, and subsequently

[3] AoN thresholds are just one of many mechanisms that would cause agents to internalize the effects of their decisions on other agents. For example, a combination of feedback effect and alternative utility function also constitutes an internalization channel (Garcia and Strobl, 2011). Instead of a general discussion about internalization channels, we focus on AoN thresholds because of their prevalence in economics and properties when large crowds of agents.

allows a better harnessing of the wisdom of the crowd to distinguish good projects from bad ones. Third, AoN thresholds produce more information whose benefits go beyond project implementation and may facilitate entrepreneurial entry and innovation (Manso, 2016), as well as future decision-making (Chemla and Tinn, 2018; Xu, 2017). A proposer facing a large number of potential agents can utilize threshold implementation to guard against DOWN cascades and charge a high price for contributions to delay UP cascades, therefore aggregating more information regardless whether the project is not implemented eventually.

While outcomes in standard models of information cascades typically are independent of the size of agent base, the case with AoN thresholds differs: the errors in mis-supporting or mis-rejecting decrease with the crowd size, as does the convergence of the endogenous price to the level at which the proposer extracts full surplus. In the limit, projects are implemented if and only if they are of high quality. The public knowledge about the project's true type also becomes perfect. We therefore obtain socially efficient project implementation and full information aggregation with a large crowd, hitherto unachievable in most models of information cascades.

Finally, we demonstrate that our key insights apply even when agents have the option to postpone their decisions, and are thus less subject to the usual critiques of exogenous action timing in information-cascade models. We also show that our findings are not driven by knife-edge cases and are robust to introducing small contribution or information acquisition costs, or investor heterogeneity and thresholds based on dollar amounts. For completeness, we discuss how AoN thresholds induce in sequential interactions strategic complementarity of agents' actions, a phenomenon novel to models of information cascades. We analyze all resulting equilibria and show that their limiting behaviors converge in terms of project implementation and information aggregation to the aforementioned equilibrium outcomes.

The theoretical insights we derive apply to general sequential contribution games such as venture financing or syndicated loans.[4] That said, we focus on its application to crowdfund-

[4]In an angel or A round of financing, entrepreneurs seek financing from multiple agents who face strategic risk: the firm can only implement its project with sufficient funding from them (Halac, Kremer, and Winter, 2020). Investors approached later often learn which others indicate support for the project, and many condition their contributions on the fundraising reaching the threshold for implementing the project. For example, the blockchain startup String Labs (predecessor of Dfinity) approached multiple agents such as IDG capital and Zhenfund sequentially, many of whom decided to invest after observing Amino Capital's investment decision, and conditioned the pledge on the founders' "successfully fundraising" in the round (meeting the AoN threshold). Syndicates involving both incumbent agents from earlier rounds and new agents are also common. Another related example involves initial public offerings (IPOs): late investors learn from observing the behavior of early investors, and IPOs with high institutional demand in the first

ing for several reasons: First, as described earlier, crowdfunding is a financial innovation that has grown tremendously and makes up a market too big to ignore; second, it presents a setting where the technology allows the outreach to large crowds, which renders the large-crowd limits relevant and important; third, the sequential nature of agents' game and threshold setting are salient, which differs from other settings such as auctions.[5] Decentralized individuals often chance upon a project, for example, through social media, but lack the expertise to fully evaluate a startup's prospect or a product's quality (due diligence is too costly when their investment is limited), e.g., in recent blockchain-based initial coin offerings, leading to high uncertainty and collective-action problems (Ritter, 2013). Yet the observation of funding targets and supports up-to-date allow them to learn and act in a Bayesian manner (Agrawal, Catalini, and Goldfarb, 2011; Zhang and Liu, 2012; Burtch, Ghose, and Wattal, 2013).[6] Other forms of entrepreneurial finance also feature investors frequently inquiring about preceding investments as well as threshold implementation written as clauses in the contingency offering contracts, subscription money-back guarantees, or private placement memoranda.[7] Therefore, they can be analyzed through our conceptual lens as well. Our study not only adds to the theory of observational learning but also highlights the practical importance of threshold implementation design and outreach to broad supporter base in a considerable variety of economic interactions and financing situations.

Literature — Our paper adds to the theory of informational cascades, sequential decisions, and observational learning.[8] The insights from prior dynamic informational models can

days of book-building also see high levels of bids from retail investors in later days (e.g., Welch, 1992; Amihud, Hauser, and Kirsh, 2003). The issuer faces an unknown demand for its stock and aggregates information from sequential agents about the demand curve (e.g., Ritter and Welch, 2002), therefore the issuer may choose to withdraw the offering if the market reaction is lukewarm.

[5]An average crowdfunding campaign lasts 9 weeks long, and many stretch even longer ($https://blog.fundly.com/crowdfunding-statistics/$). Empirical evidence points to sequential arrivals of agents (Deb, Oery, and Williams, 2021). Even the JOBS Act mandates that crowdfunding platforms need to "ensure that all offering proceeds are only provided to the issuer when the aggregate capital raised from all agents is equal to or greater than a threshold offering amount"(Sec. 4A.a.7). See $http://beta.congress.gov/bill/112th-congress/senate-bill/2190/text$.

[6]Peloton, a recent market darling, reached over 40 billion valuation with a revenue over 600 million by the end of 2020, according to the company's shareholder letter. But before Peloton became a household name with a cult-like following, investors were unsure about its quality even after the prototype showed promises and the demand was not certain enough to sway venture capitalists. It went through a Kickstarter campaign to finance the early stages of its bike manufacturing and to aggregate information about market demand. Fortunately, the campaign saw 297 backers pledge $307,332 on a $250,000 goal in less than a month and the rest is history. The dynamic learning and interaction by the startup and investors are believed to be integral to both the campaign success and the company's subsequent operations (Canal, 2020).

[7]We thank Steve Kaplan for pointing this out.

[8]Most notably Banerjee (1992); Bikhchandani, Hirshleifer, and Welch (1992) and their subsequent gener-

be best summarized along two dimensions: the signal structure and the learning bias. First, when the signal is discrete and bounded, which means that each individual cannot be arbitrarily informed, informational cascade and consequently incomplete learning are inevitable (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992; Welch, 1992; Bikhchandani, Hirshleifer, and Welch, 1998; Chamley, 2004; Callander, 2007). In contrast, if the signal structure is continuous, information cascade may not arise once the signal is unbounded or the increasing hazard ratio property is satisfied (Herrera and Hörner, 2013). Second, a learning bias can lead to asymmetric information cascade. Informational cascade is asymmetric or even uni-directional when some of the actions are not observable (Chari and Kehoe, 2004; Guarino, Harmgart, and Huck, 2011; Herrera and Hörner, 2013).

Our contributions here are two-folded. First, we obtain asymmetric informational cascades endogenously due to threshold implementation even with all actions observable. Second, we show that full learning can be achieved with bounded signals once we allow for payoff externality/interdependence via threshold implementation. Information aggregation is a key measure of the efficacy of financial markets (Wilson, 1977; Pesendorfer and Swinkels, 1997; Kremer, 2002), and the full learning result hinges on the implicit "coordination" among agents. The setup with externality/interdependence is in stark contrast with those of Dekel and Piccione (2000), Ali and Kartik (2006), and Ali and Kartik (2012), in which either economic agents consuming their choice regardless of the choices of others or voters consuming the group selection independent of their own choice. We obtain perfect information aggregation in large markets, which is typically unachievable in settings with information cascades (Ali and Kartik, 2012).[9] Our model therefore describes a new set of equilibrium behavior

alization by Smith and Sørensen (2000). Studies such as Anderson and Holt (1997), Çelen and Kariv (2004), and Hung and Plott (2001) provide experimental evidence for information cascades.

[9]Information aggregation with externality is discussed extensively in the context of sequential voting, Ali and Kartik (2012) explore the optimality of collective choice problems; Dekel and Piccione (2000) extend Feddersen and Pesendorfer (1997) and demonstrate the equivalence for simultaneous and sequential elections; by restricting off-equilibrium beliefs, Wit (1997) and Fey (2000) show that a signaling motive can always halt cascades; Callander (2007) introduces voters' desire to conform to show the sequential nature matters and cascades occur with probability one in the limit of large voter crowd. In our setting, sequential action matters because it reveals more information than is contained in the event that the voter is pivotal. More fundamentally, the payoff structure in our setting is closer to the classic models of information cascades in that it not only depends on whether the project is implemented, but also depends on whether the agent supports the project. For example, unlike the case of voting wherein an elected candidate or bill passed affect all agents regardless whether they voted in favor or against, in many situations such as crowdfunding, venture investment, and campaign contributions intended to buy favors, the implementation of the project only affects agents taking a particular action. Moreover, AoN thresholds in extant models are typically taken as exogenous, yet entrepreneurs or campaign leaders frequently set contribution amounts and target thresholds for implementation.

by large crowds and adds to the understanding of how the latest technologies such as the Internet and blockchains democratize investment opportunities through initial coin offerings, crowdfunding, online IPO auctions, etc. (Ritter, 2013) and consequently impact the social efficiency of information aggregation and financing.

Second, the paper also adds to an emerging literature on AoN design in the context of crowdfunding. Strausz (2017) and Ellman and Hurkens (2015) find that AoN is crucial for mitigating moral hazard and price discrimination. Chemla and Tinn (2018) share the concern for moral hazard as in Strausz (2017), but in addition emphasize the real option of learning through crowdfunding; they demonstrate that learning is important and can generate different predictions from those generated with moral hazard alone, in addition to showing that the AoN design Pareto-dominates the alternative "keep-it-all" mechanism. Chang (2016) shows that in simultaneous move games as in Chemla and Tinn (2018), AoN also generates more profit under common-value assumptions by making the expected payments positively correlated with values. As a cautionary tale, Brown and Davies (2017) show in a static setting that an exogenous AoN threshold can reduce the financing efficiency. Hakenes and Schlegel (2014) argue that endogenous loan rates and AoN thresholds encourage information acquisition by individual households in lending-based crowdfunding.[10] Instead of introducing moral hazard or financial constraint, or deriving static optimal designs, we focus on pricing and learning under both exogenous and *endogenous* AoN thresholds in a *dynamic* environment. Our focus on sequential actions with observational learning distinguishes our study from and complement studies such as Kremer (2002); García and Urošević (2013).

The rest of the paper is organized as follows: Section 2 sets up the model and derives agents' belief dynamics; Section 3 characterizes the equilibrium, starting with exogenous AoN threshold and issuance price to highlight the main mechanism of uni-directional cascades, before endogenizing them; Section 4 discusses model implications and demonstrates how AoN better utilizes the wisdom of the crowd to improve proposal feasibility, project selection, and information aggregation, as well as the equilibrium outcomes in the large-crowd limit; Section 5 extends the model to all agents' option to wait, contribution or information acquisition costs, budget heterogeneity and thresholds in dollar amounts, before characterizing all other equilibria; finally, Section 6 concludes. The appendix contains all the proofs and an discussion

[10] Most theoretical studies on crowdfunding (whether with AoN or not) only consider simultaneous-move games. Astebro, Fernández Sierra, Lovo, and Vulkan (2017) is another exception that considers risk-averse agents who fully reveal their private signal through the investment quantity.

of alternative specification on the private signal structure.

2 A Dynamic Model of Crowd-based Investment

2.1 Setup

Consider a project proposal presented to agents $i = 1, 2, \ldots, N$ who sequentially take actions $a_i \in \{-1, 1\}$ to either to support ($a_i = 1$) or reject ($a_i = -1$) it.[11] In crowdfunding, supporting means contributing financially. More broadly, supporting can be interpreted as adopting or advocating certain behavior by incurring a personal cost. If the proposal is implemented, then the proposer collects from every supporting agent a pre-specified "contribution" p, and each agent receives a payoff V from the project, which is either 0 or 1.[12] Given that crowdfunding serves a demand discovery function in many cases, V can be interpreted as a proxy for the true but uncertain market demand, which would affect how easy the project would progress (on the legal side, upstream contractors, etc).[13]

Threshold implmentation. We depart from the prior literature by incorporating the "all-or-nothing" (AoN) thresholds commonly observed in practice: the proposer receives "all" contributions if the campaign reaches a pre-specified threshold support, or "nothing" if it fails to do so. In other words, the project is implemented if and only if at least T agents support it. T could be exogenous in the case of voting thresholds inherited from earlier institutions or in IPOs with extreme economies of scale (Welch, 1992). In many situations, it is driven by the need to cover a minimum scale of the project that is outside the entrepreneur's control. In many other cases such as crowdfunding, T is typically endogenous.

[11] In applications such as crowdfunding agents are typically restricted to a small set of choices regarding the quantity of investment, which we model as a unit contribution for simplicity. We consider an extension with variable investment amount in Section 5.3.

[12] A separate literature studies herding and financial markets that allows price to dynamically change and focuses on asset pricing implications (Avery and Zemsky, 1998; Brunnermeier, 2001; Vives, 2010; Park and Sabourian, 2011). We follow the standard cascade models to fix the price for taking an action ex ante, which closely matches applications in crowdfunding and entrepreneurial finance, in which p is the amount of funding that each investor commits and is returned if the fundraising target is not achieved. In other activities such as political petitions, p can be interpreted as the supporting effort or reputation cost if the petition goes through and becomes public.

[13] Consistent with Strausz (2017), $V = 1$ just means true demand is sufficiently high that the entrepreneur would work on the project; $V = 0$ means that true demand is low and that entrepreneur would run away with the money (moral hazard), yielding zero payoff to investors. We shall demonstrate that the information aggregated through crowdfunding is informative about V.

Note that supporters incur p only when the project is implemented, i.e., when target T is reached. Threshold implementations are an important feature of crowdfunding markets and entrepreneurial finance, and our contribution centers around providing insights on their informational effects, especially concerning financing and informational efficiencies.

Agents' information and decision. All agents including the proposer are rational, risk-neutral, and share the common prior that the project pays $V = 0$ and $V = 1$ with equal probability.[14] Our specification is fitting for equity-based crowdfunding and Peer-to-peer lending, which constitutes 80% of the entire crowdfunding market as of 2020. It also applies to token-based fundraising if we interpret $V = 1$ as successful launch of many blockchain-based platforms. Even in reward-based crowdfunding whereby agents have private valuations and idiosyncratic preferences, there is a common value corresponding to the basic quality of the product. We recognize that it does not fully capture the cases such as sales of art piece or music where private value dominates. We assume common value also to make unambiguous comparisons concerning the project implementation and information aggregation efficiency (Fey, 1996; Wit, 1997).

Each agent i observes one conditionally independent private signal $x_i \in \{1, -1\}$, which is informative:

$$Pr(x_i = 1|V = 1) = Pr(x_i = -1|V = 0) = q \in \left(\frac{1}{2}, 1\right); \tag{1}$$

$$Pr(x_i = -1|V = 1) = Pr(x_i = 1|V = 0) = 1 - q \in \left(0, \frac{1}{2}\right). \tag{2}$$

We represent the sequence of private signals by $x = (x_1, \ldots, x_N)$ and the set of all such sequences by $X = \{1, -1\}^N$.

The order of agents' decision-making is exogenous and known to all.[15] When agent i

[14] The binary information and action structure here are the canonical focus in both the information cascade literature (Bikhchandani, Hirshleifer, and Welch, 1992) and the voting literature (Feddersen and Pesendorfer, 1996; McLennan, 1998). We show that the main results and intuition are robust when signals are asymmetrically distributed in Appendix A.16.

[15] While real world examples such as crowdfunding may involve endogenous orders of agents, our setup allows us to relate and compare to the large literature on information cascades which typically assumes exogenous orders of agents (Kremer, Mansour, and Perry, 2014). Moreoever, because agents in practice update their beliefs based on the passage of campaign time (also seen in Herrera and Hörner, 2013) and use contribution information alone to predict final funding outcomes (Dasgupta, Fan, Li, and Xiao, 2020), our setup can capture the case in which the agents roughly know their position in line by referencing the usual accumulation and rejection with the passage of calendar time. Indeed, Deb, Oery, and Williams (2021) document that contributions occur throughout the campaigns which typically last for weeks. In

makes her decision, she observes x_i and the history of actions $\mathcal{H}_{i-1} \equiv (a_1, a_2 \ldots, a_{i-1}) \in \{-1, 1\}^{i-1}$. Her strategy can thus be represented as $a_i(\cdot, \cdot) : \{1, -1\} \times \{-1, 1\}^{i-1} \to \Delta(\{-1, 1\})$, which includes mixed strategies as probability distributions of the action set $\{-1, 1\}$. To simplify exposition, we define $A_i = \sum_{j=1}^{i} a_j \mathbb{1}_{\{a_j=1\}}$, for $1 \leq i \leq N$. When $1 \leq i' < i \leq N$ and $\mathcal{H}_{i'}$ has the same first i' elements as $\mathcal{H}_i$ does, we say $\mathcal{H}_i \in \{-1, 1\}^i$ ***nests*** $\mathcal{H}_{i'} \in \{-1, 1\}^{i'}$ and write $\mathcal{H}_{i'} \prec \mathcal{H}_i$. Agent i's optimization is:

$$\max_{a_i \in \{-1,1\}} \mathbb{1}_{\{a_i=1\}} \mathbb{E}\left[(V - p)\mathbb{1}_{\{A_N \geq T\}} \mid x_i, \mathcal{H}_{i-1}, a_i = 1, a^*_{-i}\right], \tag{3}$$

where $\mathbb{1}_{\{A_N \geq T\}}$ is the indicator function for project implementation, and a^*_{-i} are equilibrium strategies of other agents as defined later in Definition 2. Agent i gets zero payoff from rejecting ($a_i = -1$) and gets $(V - p)\mathbb{1}_{\{A_N \geq T\}}$ from supporting ($a_i = 1$) the proposal. $a_i = 1$ appears in the conditioning term because given a^*_{-i}, subsequent agents' decisions and thus project implementation generally depend on agent i's action.

Following common practices in the literature (e.g., Banerjee, 1992; Bose, Orosel, Ottaviani, and Vesterlund, 2008), we introduce a tie-breaking rule for agents.

Assumption 1 (Tie-breaking). *When indifferent between supporting and rejecting, an agent supports if and only if either (i) the AoN threshold can be reached with all remaining agents supporting regardless of their signals or (ii) the AoN threshold is impossible to reach but the agent has a positive private signal.*

As discussed later in Section 5, the strategic complementarity of agents' actions becomes important with AoN thresholds. The first part of Assumption 1 rules out trivial equilibria where everyone believes that there would not be enough supporting agents and therefore rejects. The second part of Assumption 1 specifies agent's strategy when it is impossible to reach the AoN target. Here the action strategies could differ, but are irrelevant for agent payoffs and project implementation outcomes. The assumption can thus be viewed as a equilibrium refinement to avoid discussing extreme forms of coordination and redundant equilibria (essentially the same as the one we analyze). The assumption is also natural in

almost every 12-hour period, more than half of the eventually successful campaigns on Kickstarter receive a contribution. They concluded that strategic waiting is not a first-order concern and model contributor arrival as a Poisson process, consistent with our description of sequential contributors. We show in Section 5.1 that our fundamental result is robust when agents have the option to wait. Related are Herrera and Hörner (2013) that analyzes the position inference problem when agents observe supporting actions but not rejection actions, as well as Liu (2018) that studies how AoN affects the timing of investor moves.

that in practice, when implementation is not completely infeasible, the proposer can always provide an infinitesimal subsidy contingent on implementation to break agents' indifference to induce more support.

Proposer's optimization. Let $0 \leq \nu < \frac{q^N}{q^N+(1-q)^N}$ be the cost per supporter incurred by the proposer. ν can be the production cost of each product in reward-based crowdfunding or private valuation (outside option) of issuer's shares when the project is funded without an equity-based crowdfunding or IPO. To a social planner, varying ν is essentially varying the prior on the project's NPV. As we show in Lemma 1 shortly, $\frac{q^N}{q^N+(1-q)^N}$ simply corresponds to the posterior expected investment payoff if all agents observes positive signals and support. If ν exceeds this upper bound, the problem becomes trivial and the project would not be implemented for sure. Given the campaign length N, the proposer chooses price p and AoN threshold T to solve:

$$\max_{p,T} \; \pi(p,T,N) = E\left[(p-\nu)A_N \mathbb{1}_{\{A_N \geq T\}} \mid \{a_i^*\}_{i=1,2\ldots,N}\right]. \tag{4}$$

Again, $\{a_i^*\}_{i=1,2\ldots,N}$ are investor agents' equilibrium strategies. For the remainder of the paper, we drop a_i^* and a_{-i}^* in (3) and (4) for notational simplicity. In fundraising, the proposer maximizes his expected profit; in non-financial scenarios, the proposer solicits the maximum amount of support, with p interpreted as each agents' additive amount of support.

2.2 Belief Dynamics and Information Cascade

We first analyze the dynamics of the common posterior belief after observing the action history. In particular, suppose the agents update their beliefs according to Bayes' rule, then the common posteriors, $\mathbb{E}[V|\mathcal{H}_i, \{a_i^*\}_{i=1,2\ldots,N}]$, only takes values from a countable set.

Lemma 1. *(a) For every $1 \leq i \leq N$ and every history of actions $\mathcal{H}_i \in \{-1,1\}^i$, there exists an integer $k(\mathcal{H}_i, \{a_i^*\}_{i=1,2\ldots,N})$ such that $\mathbb{E}[V|\mathcal{H}_i, \{a_i^*\}_{i=1,2\ldots,N}] = V_{k(\mathcal{H}_i,\{a_i^*\}_{i=1,2\ldots,N})}$, where*

$$V_k = \frac{q^k}{q^k + (1-q)^k}, \qquad k \in \mathbb{Z}. \tag{5}$$

(b) If $\mathcal{H}_i = (\mathcal{H}_{i-1}, a_i)$ for some $a_i \in \{-1,1\}$, then the update, $k(\mathcal{H}_i, \{a_i^*\}_{i=1,2\ldots,N}) - k(\mathcal{H}_{i-1}, \{a_i^*\}_{i=1,2\ldots,N}) \in \{-1,0,1\}$, depends on whether agent i's action is informative in

equilibrium (no update if the strategy is pooling), and if it is, whether she supports or rejects (corresponding to positive and negative updates respectively under separating strategy).

Here V_k is the posterior valuation of the project from an agent's perspective right after her decision. Lemma 1 states that the posterior belief on project type only depends on k, the difference between the numbers of inferred high and low signals so far, a convenient property also in Bikhchandani, Hirshleifer, and Welch (1992). Note that Assumption 1 rules out partial-pooling strategies in equilibrium. Given Lemma 1, it is easy to verify that agent i's expected project value conditional on $\mathcal{H}_{i-1}$ and her private signal x_i is

$$\mathbb{E}_i(V|\mathcal{H}_{i-1}, x_i, a^*_{-i}) = V_{k(\mathcal{H}_{i-1},\{a^*_i\}_{i=1,2\dots,N})+x_i} \tag{6}$$

It should be understood that the expectation is on an agent i's information set (her own signal and actions up till her decision-making, given action strategies of other agents). But for notational simplicity, we drop the subscript i in the expectation sign unless otherwise stated.

When an agent's action does not reflect her private signal, the market fails to aggregate dispersed information. Our notion of informational cascade is standard (e.g., Bikhchandani, Hirshleifer, and Welch, 1992):

Definition 1 (Information Cascade). *An* UP *cascade occurs following a history of actions $\mathcal{H}_n$ ($1 \le n < N$) if along the equilibrium path, all subsequent agents support the proposal, regardless of their private signal, while agent n herself is not part of any cascade. We denote the set of such histories by $\mathbb{H}^{\mathbb{U}}$. A* DOWN *cascade is similarly defined, replacing "support" with "reject," and $\mathbb{H}^{\mathbb{U}}$ with $\mathbb{H}^{D}$.*

Standard models feature both UP and DOWN cascades. If a few early agents observe high signals, their support may push the posterior so high that the project remains attractive even with a private low signal. Similarly, a series of low signals may doom the offering. An early preponderance towards supporting or rejecting causes all subsequent individuals to ignore their private signals, which are then never reflected in the public pool of knowledge.

2.3 Equilibrium Definition

We use the concept of perfect Bayesian Nash equilibrium (PBNE).

Definition 2 (Equilibrium)**.** *An equilibrium consists of the proposer's proposal choice* $\{p^*, T^*\}$*, agents' action strategies* $\{a_i^*(x_i, \mathcal{H}_{i-1}, p^*, T^*)\}_{i=1,2,\dots,N}$*, and their beliefs such that:*

1. *For each agent* i*, given the price* p^**, implementation threshold* T^**, and other agents' strategies* $a_{-i}^* = \{a_j^*\}_{j=1,2,\dots,i-1,i+1,\dots,N}$*,* a_i^* *solves her optimization problem in (3).*

2. *Given agents' strategies* $\{a_i^*\}_{i=1,2\dots,N}$*,* p^* *and* T^* *solve the proposer's optimization problem in (4).*

3. *Agents' belief dynamics are formed according to Bayes rule whenever an action history is reached with positive probability in equilibrium.*

In our baseline model we focus on equilibria in which all actions are informative outside a cascade, as formalized here:

Definition 3 (Informer Equilibrium)**.** *An equilibrium is called an "informer equilibrium" if for every* i *and history* $\mathcal{H}_{i-1} \in \{-1, 1\}^{i-1}$ *that does not nest any history in* $\mathbb{H}^U$ *or* $\mathbb{H}^D$*, agent* i*'s action differs for different* x_i*, i.e.* $a_i(1, \mathcal{H}_{i-1}) \neq a_i(-1, \mathcal{H}_{i-1})$*.*

In other words, agents' actions are informative before an information cascade, making them "informers." Subsequent agents Bayesian-update their beliefs. Informer equilibria are intuitive and clearly illustrate our economic mechanisms. We analyze all other PBNE in Section 5.4 and show that they can be viewed as variants of the "informer equilibrium" and asymptotically converge ($N \to \infty$) to the "informer equilibrium" in terms of pricing, project implementation, and information aggregation.

2.4 Benchmark without Threshold Implementation

Let us first consider a benchmark without threshold implementation (equivalently, $T = 1$). Every agent knows that the project is implemented for sure if she supports and her payoff does not depend on subsequent agents' actions. Then our model reduces to those in Bikhchandani, Hirshleifer, and Welch (1992) and Welch (1992). Agent i simply chooses to support if and only if

$$\mathbb{E}[V|x_i, \mathcal{H}_{i-1}] \geq p. \tag{7}$$

With exogenous p, both UP and DOWN cascades can occur, which halt the information aggregation. With endogenous p, imprecise signals can cause "underpricing":

Lemma 2. *The proposer always charges $p \leq q$. In particular, when $\nu = 0$ and $q \leq \frac{3}{4} + \frac{1}{4}\left(3^{\frac{1}{3}} - 3^{\frac{2}{3}}\right)$, the optimal price is $p^* = 1 - q < \frac{1}{2} = \mathbb{E}[V]$.*

The lemma generalizes the underpricing results in Welch (1992) (in which $\nu = 0$), particularly Theorem 5. The general pricing upper bound q is not tight but serves to illustrate the concern for DOWN cascades from the start. If $p > q$, then even with a positive signal $x_1 = 1$, the first agent rejects and so does every subsequent agent, yielding zero payoff for the proposer. The second part of the lemma concerns the optimal pricing when agents' signals are imprecise. UP and DOWN cascades affect the proposer's payoff asymmetrically because he benefits from UP cascades by attracting support from future agents with negative signals whereas DOWN cascades means a few early rejections may doom his offering. Consequently, he optimally underprices $p = 1 - q < \frac{1}{2}$ (which is less costly when signals are imprecise) to ensure an UP cascade at the very first agent (even when the agent has bad signal).

3 Equilibrium Characterization

We now solve the equilibrium in several steps. First, we examine agents' supporting/rejection decisions taking the price p and AoN threshold T as exogenous. We then derive the proposer's endogenous p and T and compare the equilibrium outcomes to the benchmark outcomes without implementation thresholds.

3.1 Exogenous Price and AoN Threshold

For a given price $p \in (0, 1)$, define $\bar{k}(p)$ as the smallest integer such that

$$p \leq \frac{q^{\bar{k}(p)}}{q^{\bar{k}(p)} + (1-q)^{\bar{k}(p)}}. \tag{8}$$

The first main result entails a precise characterization of the equilibrium once p and T are given. We find that only UP cascades may exist before the AoN threshold is reached.

Proposition 1. *For any given pair of (p, T), there exists a unique informer equilibrium with the agents' strategies and belief dynamics recursively defined by two variables: A, the number of supporters in a history $\mathcal{H}$ and k, the difference between the numbers of inferred positive private signals and of negative private signals in $\mathcal{H}$. In other words, $a_i^*(x_i, \mathcal{H}_{i-1}) \equiv a^*(x_i, k(\mathcal{H}_{i-1}), A(\mathcal{H}_{i-1}))$ and posteriors $P(V = 1|\mathcal{H}_i) = V_{k(\mathcal{H}_i)}$.*

An UP cascade starts whenever the history has $k(\mathcal{H}_{i-1}) > \bar{k}(p)$; a DOWN cascade starts only when $A(\mathcal{H}_{i-1}) \geq T-1$ and $k(\mathcal{H}_{i-1}) < \bar{k}(p)-1$. Before any cascades, an agent supports if and only if the private signal is positive.

We provide the detailed expressions for the strategies, belief dynamics, and the evolution of the state variables in the proof in Appendix A.3. Recall that $A(\mathcal{H}_i) \equiv A_i = \sum_{j=1}^{i} a_j \mathbb{1}_{\{a_j=1\}}$ is the level of support. Proposition 1 shows that there is no DOWN cascade before approaching the AoN threshold ($A_{i-1} < T-1$) and an UP cascade starts once the posterior belief $k_i(\mathcal{H}_i)$ exceeds $\bar{k}(p)$. In equilibrium agents with positive signals always support regardless of the history they observe while agents with bad signals support only when there is an UP-cascade. Once $A_{i-1} = T-1$, agent i and subsequent ones face exactly the same decision as in standard cascade model, and beliefs update correspondingly.

Intuitively, uni-directional cascades occur because an agent with a positive signal is protected in that she does not pay if the project turns out to be bad. The agent observing $T-1$ preceding supporters would be the "gate-keeper" for her because their interests are aligned yet the gate-keeper observes a longer history and makes a more informed decision.

Observing a longer history is helpful only when actions reflect private information. So to complete the argument, we need to show that when there is no UP cascade yet and before the AoN threshold is approached, agents with negative signals reject the proposal. If an agent with a negative signal deviates and supports, then all subsequent agents would misinterpret her action and form wrong posterior beliefs. The over-optimistic belief implies that subsequent agents either start an UP cascade too early or reach the AoN threshold when the true posterior is not high enough. Taking that into account, agents with negative signals find deviations unattractive.

In Appendix A.4, we discuss how the equilibrium varies when N varies and show that as long as the exogenous AoN target is not too small related to the crowd size, a good project ($V=1$) is implemented with an UP cascade with probability 1, as $N \to \infty$. In a sense, a large agent base improves the implementation of good projects. The gain in implementation efficiency becomes more salient with endogenous AoNs, as we demonstrate later.

3.2 Endogenous Price and AoN Threshold

In real life, especially in financial settings such as crowdfunding, the proposer endogenously sets the price and the AoN threshold, which we now model.

Notice that we assume $0 \leq \nu < V_N$ to avoid the trivial case in which the proposal fails due to the production cost exceeding the highest possible valuation. With the AoN threshold, there exists an informer equilibrium such that DOWN cascade is impossible except for some special scenarios.

Proposition 2. *For each N, given the agents' subgame equilibrium strategies in Proposition 1, a duplet (p_N^*, T_N^*) exists that maximizes the proposer's expected revenue in (4) and satisfies $p_N^* = V_{k_N^*}$ and $T_N^* = \lfloor \frac{N+k_N^*}{2} \rfloor$, for some $k_N^* \in \{-1, 0, \ldots, N\}$. Moreover, there exist constants $\gamma_1, \gamma_2 > 0$ such that $N^{\gamma_1}(1 - p_N^*) < \gamma_2$ for every N. In particular, $\lim_{N\to\infty} p_N^* = 1$.*

Proposition 2 states the existence of an equilibrium and characterizes the proposer's endogenous optimal price and AoN target (i.e., proposal design) in the equilibrium. Since for any $p \in (V_{k-1}, V_k]$, all agents make the same supporting decisions, the proposer can always charge $p = V_k$ and receives the highest profit. We therefore can focus our analysis on $p \in \{V_{-1}, V_0, \ldots, V_N\}$. We exclude $k < -1$ because $V_{-1} = 1 - q$ is already sufficiently low to induce an UP cascade from the very beginning.

Recall that there is no DOWN cascade before approaching the threshold. For a given equilibrium price p, a higher AoN threshold reduces the burden of using underpricing to exclude DOWN cascade once the threshold is reached. Yet a higher threshold itself is more difficult to reach. In the equlibrium, the proposer finds the optimal AoN threshold the one that is the smallest to increase the chance of capital formation but still large enough to help exclude all relevant DOWN cascades, as described in the following corollary.

Corollary 1 (Uni-directional Cascades). *The sufficient and necessary condition for a DOWN cascade entails only the last agent ($i = N$) herding and an implementation failure.*

For all practical purposes, DOWN cascades are of no concern here because they rarely happen (as we show in the appendix) and only start from the last agent. Moreover, the project would not be implemented anyway in those scenarios even if all private signals were aggregated. Consequently, a DOWN cascade, when exists, does not affect project implementation and has almost no impact on information aggregation with at most one agent herding. Cascades with endogenous threshold implementation are basically uni-directional.

Next, we examine the properties of UP cascades and their impact on pricing. In appendix A.5, we characterize the distribution of UP-cascade arrival time using well-established results on hitting times (Van der Hofstad and Keane, 2008). Given the optimal AoN target T characterized in Proposition 2, once an UP cascade starts in equilibrium, all following investors will contribute and the project is implemented for sure. But for any agent $i \leq N - 2$, if the UP cascade has not started yet, there is a strictly positive probability that the project is not implemented. Therefore, for a project to be still implemented, the total number of supporting agents is exactly equal to the endogenous T, because otherwise an UP cascade would have been triggered. We illustrate the two scenarios with project implementation in Figure 1, which plots the difference between supporting agents and rejecting agents when n agents have arrived. Conditional on the project being implemented, if the total number of supporting agents is not $T^*(N) = 22$, then the state variable must have crossed the cascade threshold. The figure also includes a sample path that leads to an implementation failure because AoN threshold is not reached.

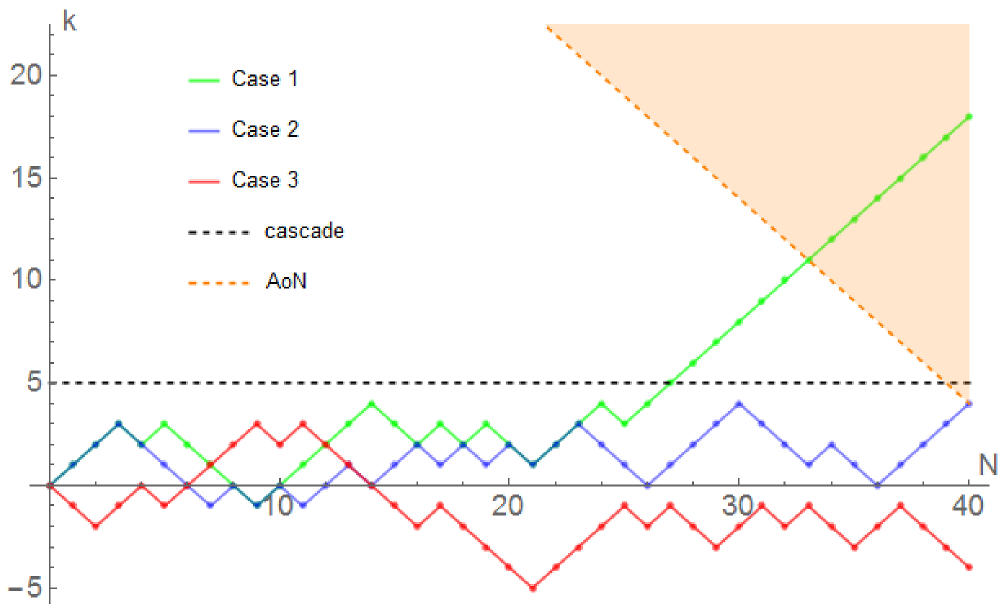


Figure 1: Evolution of support-reject differential

Simulated paths for $N = 40$, $q = 0.7$, $p^* = V_4 = 0.9673$, and AoN threshold $T^*(N) = 22$. Case 1 indicates a path that crosses the cascade trigger $\bar{k}(p) + 1 = 5$ at the 26th agent and all subsequent agents support regardless of their private signal; case 2 indicates a path with no cascade, but the project is still funded by the end of the fundraising; case 3 indicates a path where AoN threshold is not reached and the project is not funded. The orange shaded region above the AoN line indicates that the project is funded.

We next turn to the optimal pricing. Obviously, there is a tradeoff between setting a higher price to raise money in total given the number of supporters and setting a lower price to avoid DOWN cascades, as seen in studies such as Welch (1992). The optimal AoN target mitigate the concerns about DOWN cascades, but a higher AoN target itself is more difficult to reach. A higher price allows the proposal to extract more rent from each supporter, but at the same time reduces the number of supporters and probabilities of implementation.

In the proof of the Proposition we derive an explicit characterization of the proposer's expected profit as a function of price $p = V_k$, associated optimal threshold T^*, and number of potential agents N. Figure 2 illustrates how the entrepreneur's profit varies with price (and its corresponding optimal AoN threshold). The optimal price maximizes the profit.

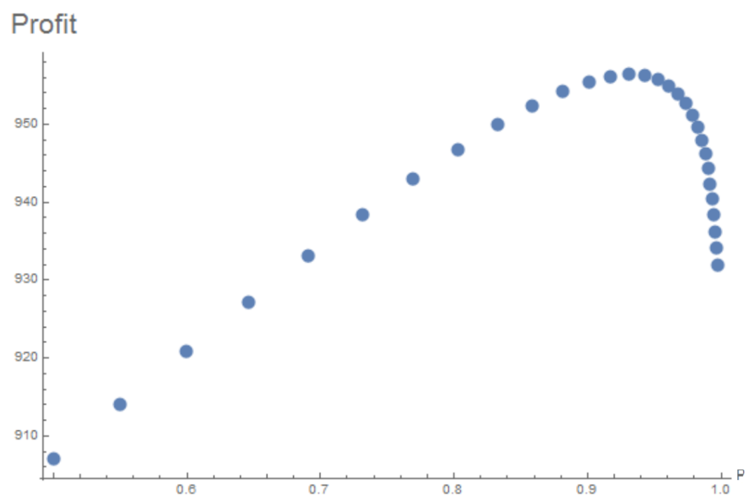


Figure 2: Proposal profit as a function of price with $N = 2000$, $\nu = 0$ and $q = 0.55$.

Unlike Lemma 2, Proposition 2 implies that the optimal price depends on the number of potential agents N. A financial technology (Internet-based platforms) that allows reaching a greater N thus has a fundamental impact.[16] In the proof, we demonstrate that a large size of agent base not only implies a higher price but also ensures a certain probability of implementing good projects.

Figure 3 shows the optimal pricing for different values of N, with the left panel plots the absolute price level and right panel plots the associated $\bar{k}$. In the limit, $\lim_{N\to\infty} p_N^* = 1$. With an endogenous AoN, the proposer can charge a higher price for a larger crowd, which can appear "overpriced" ex ante, i.e., $p > \mathbb{E}[V]$. Our findings on pricing are important

[16] In the standard cascades models, a DOWN cascade hurts the proposer significantly because subsequent agents all reject. The concern for DOWN cascades pushes down the optimal price, and can cause immediate start of an UP cascade, *independent of the number of agents* because the decisions of later agents have no impact on the first agent's payoffs (Welch, 1992).

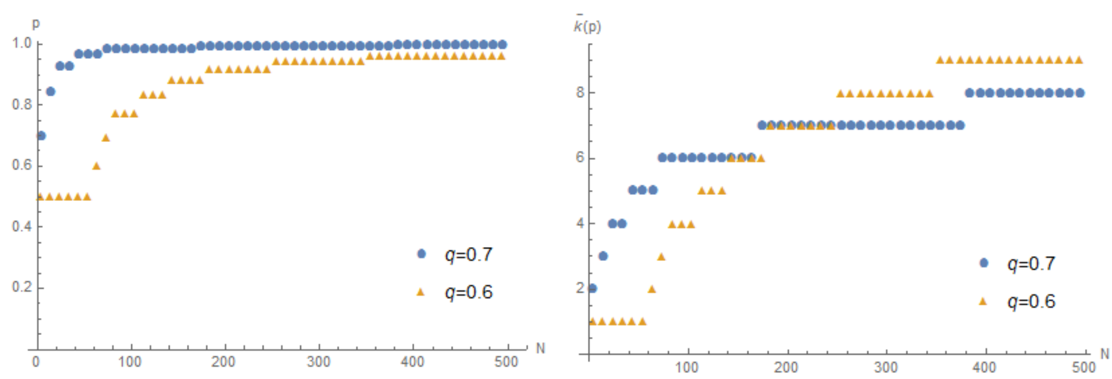
because the underpricing or overpricing of securities or products may affect the success or failure of a project proposal, and thus impact the real economy (Welch, 1992). We discuss these model implications next.

4 Implications of Thresholds and Large Crowds

Two key functionalities of modern financial markets and digital platforms are funding good projects and aggregating localized/decentralized information to inform investors and economic decisions. Meanwhile, one salient distinguishing feature of crowdfunding platforms from venture capital lies in the large crowds they access. For example, according to Kickstarter official statistics, as of November 2020, the crowdfunding platform has 18.87 million backers in total and the top 10 popular projects have 74,410 to 219,380 backers; the crowdfunding Center reports that fully funded projects have on average 300 backers. [17] We now examine the immediate implications of all-or-nothing thresholds for project implementation and information aggregation, as well as equilibrium outcomes as the crowd size gets large.

In particular, we show that threshold implementations improve proposal feasibility, project selection, and the accuracy (and thus utility) of aggregated information. The results primarily pertain to the general equilibrium with endogenous p and T, although results derived from the sub-game equilibrium solution equally apply for exogenous p and T. In the limit of large crowds, project implementation and information aggregation are fully efficient —

[17] See, for example, https://www.statista.com/statistics/288345/number-of-total-and-repeat-kickstarter-project-backers/ and https://www.statista.com/statistics/378054/most-backed-kickstarter-projects/; https://www.thecrowdfundingcenter.com/data/projects.



(a) Optimal price for $N = 0$ to 500.

(b) Corresponding $\bar{k}$ for $N = 0$ to 500.

Figure 3: Cascades and Optimal Prices as N Changes

results not obtainable in earlier models of dynamic observational learning and crowdfunding. Our findings demonstrate that threshold implementations have profound implications on financing projects and aggregating information and are crucial in market design.

4.1 Project Implementation

A financial marketplace serves to match capital with worthy projects. Given that individuals possess private signals and the observational learning setup, what is socially efficient is then ensuring good projects and good projects only are financed.

Proposal feasibility. Lemma 2 reveals a pricing upper bound in standard cascade models above which the proposal is infeasible: Good projects with production cost $\nu > q$ cannot be supported because even the break-even price triggers DOWN cascades. With threshold implementation, however, the proposer can charge $p > q$ and still implement the projects.

Proposition 3 (Proposal Feasibility)**.** *Without AoN thresholds, no project with $\nu > q$ can be implemented; with endogenous AoN thresholds, all projects have a positive ex ante probability to be implemented.*

The proposition follows directly from that charging $p \geq \nu$ does not trigger a DOWN cascade if T is set to be sufficiently high. As a result, crowdfunding and the like with endogenous AoN thresholds can enable financing of projects of higher production costs for which funding is otherwise infeasible. This is consistent with Mollick and Nanda (2015) which empirically documents that crowdfunding is more likely to finance projects with costly production that a group of experts would not finance in traditional settings.

Project selection. Without threshold implementation as in Welch (1992), UP cascades start from the very beginning and all projects are implemented, resulting in a poor project selection. With AoN thresholds, DOWN cascades do not occur before reaching the implementation threshold; neither do UP cascades start from the beginning. Good projects thus have a higher chance of reaching the target threshold due to the information aggregated publicly before an UP cascade starts. Project selection therefore improves.[18] We denote the

[18]Uni-directional cascade and threshold implementation also mean that offerings in our setting can fail whereas offerings never fail in Welch (1992). Our model thus helps explain why some offerings fail occasionally and/or are withdrawn, without invoking insider information as Welch (1992) does.

probabilities of missing a good project (Type I error) and financing a bad project (Type II error) by $\mathcal{P}^I = 1 - Pr(A_N \geq T|V = 1)$ and $\mathcal{P}^{II} = Pr(A_N \geq T|V = 0)$ respectively. While UP cascades do lead to some bad projects being financed, such Type II errors are not as frequent as in Welch (1992), in which all bad projects are financed with endogenous pricing and the probability of the cascade being correct is $\frac{1}{2}$.

AoN thresholds reduce underpricing, which in turn delays cascade and increases the probability of correct cascades (UP cascade when $V = 1$) given by

$$Pr(V = 1|p) = \frac{q^{\bar{k}(p)+1}}{q^{\bar{k}(p)+1} + (1-q)^{\bar{k}(p)+1}} \tag{9}$$

Because $\bar{k}(p)$ is weakly increasing in p (Proposition 1), one can show that the probability of a cascade being "correct" is larger than $\frac{1}{2}$, increasing in q, and weakly increasing in T.

Proposition 4 (Project Selection)**.** *Good projects are more likely to be implemented than bad projects. Moreover,* $\lim_{N\to\infty} \mathcal{P}_N^I = 0$ *and* $\lim_{N\to\infty} \mathcal{P}_N^{II} = 0$.

Whereas N does not matter in standard cascade models, threshold implementation links the timing and correctness of cascades to the size of the crowd! In Proposition 1 and Appendix A.4, we have already demonstrated that as $N \to \infty$, Type I error is eliminated even with exogenously given AoN thresholds. Appendix A.8 further elaborates on the two error probabilities, providing bounds on the errors and showing that with a large N, as is the case for Internet-based crowdfunding, the concern about implementing bad projects also goes away with endogenous thresholds. The wisdom of the crowd if fully harnessed to distinguish good projects from the bad ones. In particular, a larger crowd implies a higher endogenous optimal price, which in turn delays the arrival of UP cascades and reduces the probability of type II error. Project implementation becomes fully efficient.

Note that when the proposer has access to a large crowd, endogenous AoN achieve efficient implementation under informational constraints in the sense that good projects and good projects only are implemented. This does not imply that the scale of investment is the first best which would require everyone investing in a good project and not in a bad project. As for the allocation of surplus, the investors' share vanishes in the limit because the price approaches the true value of a good project, and the proposer eventually gets all the surplus from the project implementation.

4.2 Information Aggregation

Sequential investment processes such as crowdfunding allow the market to aggregate investors' private signals and (partially) reveal the aggregated information to the public. Such information can be important for resource allocation and business decision-making.

For example, For most projects on crowdfunding platforms, the entrepreneurs retain discretion over their own effort provision and the project's future commercialization—a form of real option whose exercise depends on the information aggregated in fundraising.[19] Studies such as (Chemla and Tinn, 2018) lay out the theoretical foundations for the information aggregated to be useful beyond the fundraising stage; Viotto da Cruz (2016) and Xu (2017) provide empirical evidence that entrepreneurs indeed benefit from the information aggregated from crowdfunding platforms when making real decisions. In particular, Xu (2017) documents in a survey of 262 unfunded Kickstarter entrepreneurs that after failing, 33% continued as planned.

Without threshold implementation, a support-gathering process produces little information because as soon as the public pool becomes slightly more informative than the signal of a single individual, individuals mimic the actions of predecessors and a cascade begins. But with AoN thresholds, the support observed (not necessarily received by the proposer because the project is not implemented when $A_N < T^*$) is informative about quality of the project. This is especially true when the support fails to reach AoN.

Proposition 5. *The crowdfunding process is informative of projects' quality and achieves full information aggregation in the large crowd limit. In other words,* $\mathbb{E}[V|\mathcal{H}_N]$ *is weakly increasing in* A_N *and* $\mathbb{E}[V|\mathcal{H}_N, A_N < T^*]$ *is strictly increasing in* A_N*, with* $\mathbb{E}[V|\mathcal{H}_N] \xrightarrow{prob.} V$ *as* $N \to \infty$*.*

Threshold implementations improve information aggregation for two reasons here. First, DOWN cascade is absent in equilibrium (except the last agent in special scenarios) with an endogenous AoN threshold, making rejections more informative. Second, the endogenous price p is sufficiently high that UP cascades do not arrive immediately, making supporting actions in history more informative. It is worth noting that $\mathbb{E}[V|\mathcal{H}_N]$ is also weakly increasing

[19] In our model, V can be interpreted as a transformation of the aggregate demand, which could be high ($V = 1$) or low ($V = 0$). Suppose that after the crowdfunding, an entrepreneur considers commercialization or abandoning the project (upon crowdfunding failure), and for simplicity the commercialization or continuation decision pays V (after normalization), but incurs an effort or reputation or monetary cost represented in reduced-form by I. Then the entrepreneur's expected payoff for the real option is $\max\{\mathbb{E}[V - I|\mathcal{H}_N], 0\}$.

in A_N with exogenous AoN thresholds, which is intuitive. The surprising part is what endogenous AoN thresholds imply in the large crowd limit.

Different from standard cascade models with DOWN cascade, conditional on failing to reach the AoN threshold, the proposer updates the belief more positively with more supporting agents. Our model further implies that the belief updates on V based on incremental support is smaller conditional on project implementation because it likely involves an UP cascade and information aggregation is more limited. This is consistent with Xu (2017), which finds that conditional on fundraising success, a 50% increase in pledged amount leads to a 9% increase in the probability of commercialization outside the crowdfunding platform — a small sensitivity of the update on project prospective to the level of support.

Note that the equilibrium characterization provided in Propositions 1 and 2 imply that when (p_N^*, T_N^*) are endogenous, all private signals become public and efficiently aggregated before an UP cascade starts. Therefore, the number of aggregated signals depends on the distribution of times at which an UP cascade starts. Because the proposer optimally increases the price with N, which delays the arrival of UP cascades, the number of agents N plays an important role for information aggregation.

In fact, we obtain perfect information aggregation in large markets, which is unachievable in settings with information cascades (Ali and Kartik, 2012) unless the action set maps to each agent's posterior belief one-to-one (Lee, 1993) — a condition we do not require.[20] In Appendix A.8, we provide further characterizations of errors and prices away from the large N limit, as well as the case with exogenous thresholds.

5 Discussion and Extensions

In this section, we characterize equilibrium outcomes when agents have options to wait, incur contribution and information acquisition costs, are heterogeneous in budget while AoN thresholds are in dollar amounts, or do not necessarily play the informer equilibrium. Our goal is three-fold: (i) to demonstrate that our findings about the impact of AoN on project implementation and information aggregation are robust, (ii) to derive new insights after enriching the model to be more realistic, and (iii) to demonstrate how the AoN feature leads

[20] Our convergence concept regarding information aggregation is consistent with the majority of previous theoretical studies such as Milgrom (1979) and Pesendorfer and Swinkels (1997).

to strategic considerations absent in most extant information cascade models, which are of game theoretical interests.

5.1 Option to Wait and Endogenous Ordering of Decisions

Information cascade models often fail to endogenize the ordering of agents' decision-making although in practice agents may choose to wait to observe more information.[21] We now verify that our main findings are robust to such options to wait.

For agent i who first arrives in period i, we denote her action in each period $t \geq i$ by $a_i^t \in \{-1, 0, 1\}$, where 0 means that agent i delays her decision in period t to the next period, and is a feasible action only when $i = t$ or $a_i^{t-1} = 0$, i.e., she has not supported or rejected the project yet. In any period t, after agent t's decision, all agents already waiting from earlier periods make decisions one by one (ordered by their first arrival time). For the ease of exposition, if agent i chooses not to wait at time t, $a_i^t \neq 0$, we write $a_i^l = a_i^t$, $\forall\ l > t$.

With the option to wait, for agent i at period $t \geq i$, the history can be summarized as

$$\mathcal{H}_i^t = \begin{cases} (a_1^1, a_2^2, a_1^2, a_3^3, a_1^3, a_2^3, \ldots, a_{t-2}^{t-1}, a_t^t) & \textit{if } i = t \\ (a_1^1, a_2^2, a_1^2, a_3^3, a_1^3, a_2^3, \ldots, a_{t-2}^{t-1}, a_t^t, a_1^t, a_2^t, \ldots a_i^t) & \text{if } i < t \end{cases} \tag{10}$$

and A_i^t can be defined as

$$A_i^t = \begin{cases} \sum_{1 \leq j \leq t-1} a_j^{t-1} \mathbb{1}_{a_j^{t-1}=1} + a_t^t \mathbb{1}_{a_t^t=1} & \textit{if } i = t \\ \sum_{i+1 \leq j \leq t-1} a_j^{t-1} \mathbb{1}_{a_j^{t-1}=1} + \sum_{1 \leq j \leq i} a_j^t \mathbb{1}_{a_j^t=1} + a_t^t \mathbb{1}_{a_t^t=1} & \text{if } i < t \end{cases} \tag{11}$$

The option to wait results in multiple equilibria due to the coordination problem on waiting decisions and off-equilibrium beliefs. Nevertheless, there exists an equilibrium similar to the one characterized in Proposition 1:

Proposition 6. *For any given (p, T), a subgame equilibrium exists in which those agents who would reject in Proposition 1 now delays their action as much as possible; those who support upon their first decision-making are the same supporting agents as in Proposition 1.*

[21]That said, potential contributors in crowdfunding often do not wait because of non-trivial attention costs. Moreover, in many cases the shares or products sold are often in limited supply, and waiting may cause an agent to miss out the opportunity.

To see this, if there is already an UP cascade, then no one wants to deviate to wait. Now suppose there is no cascade yet, then for agents with positive signals, supporting always dominates rejection and thus there is no need to wait. For agents with negative signals, waiting till the end weakly dominates rejection and they wait.

In terms of proposal feasibility, this equilibrium is qualitatively the same as the one in Proposition 2. If $\nu > q$, a proposal with $p > \nu$ would have a strictly positive success probability when the proposer commits to an AoN threshold. Our finding on project implementation with large crowds is also robust to option to wait, as the next proposition summarizes.

Proposition 7. *When $N \to \infty$, the optimal price $p_N^* \to 1$ even when agents can wait. Good projects and only good projects are implemented, with full public information aggregation.*

The intuition for the results is similar to that for Propositions 4 and 5. The absence of DOWN cascades helps us avoid missing good projects and the high price screens out bad projects whose valuation cannot be sufficiently high as information gradually gets aggregated. Observational learning remains equivalent as before because agents with different signals choose different actions. In equilibrium, before the arrival of an UP cascade, all agents infer supporting actions as good news and waiting as bad news, yielding exactly the same information aggregation as in the baseline model.[22]

5.2 Contribution and Information Acquisition Costs

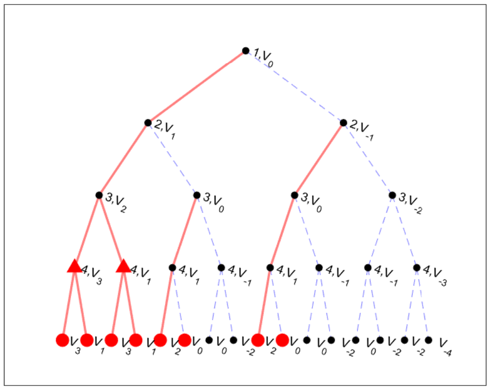
In practice, investing may incur an additional cost $\epsilon > 0$, which could be, for example, the opportunity cost from pre-committing the funds.[23] Notice that with Assumption 1, an agent supports even when the expected contribution payoff equals the contribution cost ϵ.

We first show that if the ϵ is sufficiently small, the equilibrium characterization in Proposition 1 still holds. That is, our results are not a knife-edge case driven by a specific assumption and are robust to small perturbations in the form of contributing costs.

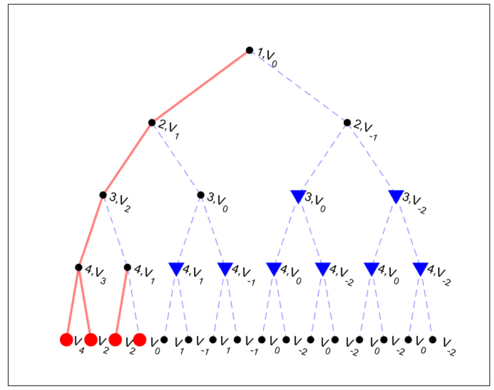
Proposition 8. *Given the agent base N and threshold T, for any price $p \in [V_k, V_{k+1})$, there exists a bound $\bar{\epsilon}(p,T) > 0$ such that for $\forall \epsilon \in (0, \bar{\epsilon}(p,T))$, the equilibrium entails the same outcomes as in the one characterized in Proposition 1 with the same threshold T and a modified price $p^* \equiv p + \epsilon$.*

[22] The option to wait may affect the optimal price p^* because agents with negative signal can still contribute if the posterior valuation after the information aggregation is good.

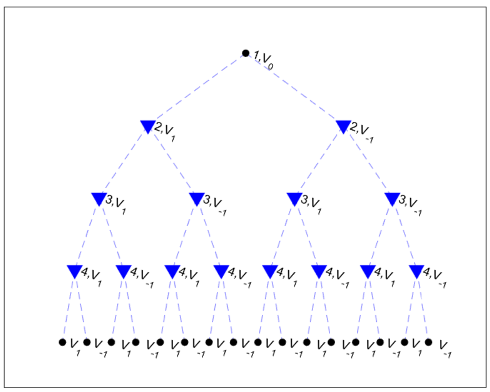
[23] We thank one anonymous referee for pointing this out.



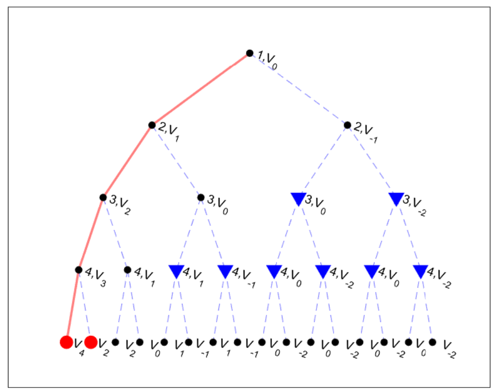
(a) The case with $\epsilon = 0$



(b) The case with $0 < \epsilon \leq V_2 - V_1$



(c) The case with $\epsilon > V_2 - V_1$



(d) How endogenous price helps when $\epsilon > V_2 - V_1$

Figure 4: Equilibrium dynamics as a function of contribution cost

If ϵ is sufficiently small, then it would not change any agent's equilibrium strategy and our main results are qualitatively unchanged. For larger ϵ, some agents may find the expected profit of investment lower than ϵ, which causes these agents to reject even if they observe positive private signals. This hinders information aggregation and in turn reduces the number of potential contributors, discourages other agents to contribute, further lowering the possibility of project implementation.

In general, a higher ϵ is associated with a lower chance of project implementation and less information aggregation. We use an example to illustrate how ϵ bounds the informativeness of the equilibria and affects information cascades. Consider the case with $N = 4, T = 2$, $\nu < p = V_1$, where $V_k = \frac{q^k}{q^k + (1-q)^k}$ is defined as in Proposition 1. The equilibrium is fully characterized by Proposition 1.

Figure 4 depicts the equilibrium dynamics with binomial trees. Nodes 1-4 index the agents, and the value V_k alongside each node indicates the public prior belief of $\Pr(V = 1)$. A left branch indicates a good private signal ($x_i = 1$), while a right branch indicates a bad one ($x_i = -1$). The big red terminal nodes indicate successful implementations and the corresponding values are the final posterior beliefs of $\Pr(V = 1|\mathcal{H}_N)$ after the branch sequence of signals. A red upward triangle indicate an UP cascade at that node, while a blue downward one indicates a DOWN cascade.

Figure 4a corresponds to $\epsilon = 0$. If we take the sixth terminal node in Figure 4a, the sequence of signals is given by: $\{1, -1, 1, -1\}$. Agents 1 and 3 choose to support the proposal, while Agents 2 and 4 reject it. The final posterior is V_0, which is the belief held by Agent 4 after receiving a bad signal $x_4 = -1$. However, for Agent 3, the expected value from supporting the project is V_1 because the project would not be implemented at terminals 7 and 8 (from the left). Note that, for Agent 1, when she chooses to support the proposal (if $x_1 = 1$), her expected payoff is a linear combination of V_2 and V_1, where V_2 is the equilibrium payoff when $x_2 = 1$ and V_1 the equilibrium payoff when $x_2 = -1$. Moreover, UP cascades occur after the signal sequence terminal node 1-4. Figure 4b displays the contribution nodes when the contribution costs $\epsilon \in (0, V_2 - V_1)$. Relative to Figure 4a, the proposal is less likely to be implemented (only the first four terminal nodes). There are also more DOWN cascades.

Finally, in Figure 4c, with sufficiently high contribution cost, the equilibrium becomes uninformative and DOWN cascade starts at the very beginning. Figure 4d highlights the importance of endogenous pricing and threshold design in improving financing and information aggregation. Compared to Figure 4c, if the proposer increases the threshold from $T = 2$ to $T = 3$, then the proposal can still be implemented after the signal sequences $\{1, 1, 1, 1\}$ and $\{1, 1, 1, -1\}$ which previously lead to implementation failures. Our baseline model's intuition goes through: a higher implementation threshold ensures that the project is not implemented with a sequence of negative signals which are more likely generated by low quality projects. This protection makes financing feasible and avoids some DOWN cascades to aggregate useful information.

Besides the absence of contribution cost, every agent acquires a private signal for free in the baseline model. But in reality, acquiring private signals may cost effort and agents may forego producing any information. A costly information production can have a similar impact as the contribution cost. To see this, denote the positive information production cost by ε and assume that an agent produces the information signal even when she is indifferent between costly learning or not (and that Assumption 1 still holds). When there is already an UP cascade, agents obviously support regardless of private signals and thus acquires no information. When there is no UP cascade yet, agents need to first decide whether to acquire information by comparing the expected profit with and without private signal (and potentially different contribution decisions).

Costly information acquisition differs from the contribution cost case in that agents need to make the information acquisition decision before their contribution decisions, and it maybe the case that one agent chooses to reject even after she pays the learning cost. Nevertheless, in Appendix A.13, we show that when ε is sufficiently small, then the equilibrium resembles the one characterized in Proposition 1. If ε is high, agents may choose not to produce information and becomes a free-rider. Similar to the contribution cost, this may leads to a lower expected contribution profit and less number of contributions, which in turn may further discourage information production by other agents. As the information cost ε increases, the equilibrium becomes less informative and the project is less likely to be implemented. The key takeaway is that contribution and information acquisition costs matter, but the economic mechanisms we highlight are not driven by knife-edge cases or the omission of these costs.

5.3 Investor Heterogeneity and Thresholds in Dollar Amounts

According to the Crowdfunding Center, fully-funded crowdfunding campaigns have an average of 300 backers and that successful campaigns, on the whole, rely on a large number of comparable, small contributions instead of a small number of huge individual contributions.[24] Therefore, our baseline assumptions of homogeneous contribution amount and threshold implementations based on the number of supporters reasonably balance tractability and reality. Nevertheless, many crowdfunded projects, unlike political proposals, requires a minimum dollar amount to be feasible. We now extend the model to illustrate the impact of investor heterogeneity in wealth and thresholds involving dollar amounts instead of the number of supporters. We demonstrate that our key insights remain robust. Moreover, we present a novel phenomenon of "prolonged learning" through partial support and briefly discuss numerical procedures for designing AoN thresholds in dollar amounts.

[24] *www.thecrowdfundingcenter.com/data/projects.*

As before, a project is presented to a sequence of agents $i = 1, \cdots, N$ who can either support or reject it. However, each agent i can be either rich ($\theta_i = H$) or poor ($\theta_i = L$), with dollar amount $H > L > 0$ and θ drawn i.i.d. with a prior such that $\Pr(\theta_i = H) = 1 - \Pr(\theta_i = L) = \lambda$. An H-type faces an investment choice set given by $\{H, L, 0\}$; an L-type agent has a choice set $\{L, 0\}$, choosing only between low support and rejection.[25]

The private signal x_i is realized according to the information structure as specified in the baseline model. We abuse the notation here to use T to denote the implementation threshold in terms of total dollar amount instead of the required number of supporters. We correspondingly denote the dollar amount of support collected until agent i using $A_i = \sum_{j=1}^{i} a_j$, for $1 \leq i \leq N$.

If the proposal is implemented eventually, the proposer charges a pre-specified "price" $0 < p < 1$ for each dollar supported, and each agent receives a return V per dollar invested in the proposal, which is either 0 or 1. We can alternatively view pL or pH as the actual amount the entrepreneur receives, then a given threshold T maps to pT as the threshold for the actual amount collected. Our convention in labeling $\{p, T\}$ is for easy comparison with the baseline model. Before characterizing the equilibrium, we illustrate the difficulty of such an analysis through the following numerical example.

Example 1. *Suppose* $N = 30, T = 10.1, H = 1, L = 0.3$. *Consider a history such that* $A_{i-1} = 9, k_{i-1} = \bar{k}(p) - 3$. *Suppose* $x_i = 1$ *and* $\theta_i = H$. *Should agent* i *fully support* ($a_i = H$)*, partially support* ($a_i = L$) *or reject* ($a_i = 0$)*?*

She gets nothing if she rejects. If she chooses full support H*, then* $A_i = A_{i-1} + 1 = 10$*, and* $k_i = k_{i-1} + 1 = \bar{k}(p) - 2$*. Now, even if Agent* $i + 1$ *gets a positive signal* $x_{i+1} = 1$*, she still chooses not to support, leading to DOWN cascade. Hence, Agent* i *still gets zero payoff. Finally, if she chooses partial support* L*, then* $A_i = A_{i-1} + 0.3 = 9.3$*, and* $k_i = k_{i-1} + 1 = \bar{k}(p) - 2$*. If the next three agents (i.e.,* $i + 1, i + 2, i + 3$*) all receives good signals and of type* L*, then we have* $a_{i+1} = a_{i+2} = a_{i+3} = L = 0.3, k_{i+3} = \bar{k}(p) + 1$ *and* $A_{i+3} = 10.2 > 10.1 = T$*.*

Considering all the options, partial support is evidently a dominant strategy because it generates a positive expected payoff. Therefore, the solution is much more subtle than the naive conjecture that one always contributes to the full extent if one supports. The

[25] We use discrete choices here to reflect that in crowdfunding, investors are typically given discrete choices on the amount they can invest. This specification also allows us to best convey intuition and insight: if we allow a continuum amount would be similar but the derivation is considerably more involved.

example reveals that it could be optimal for an agent to switch the contribution from H to L to enhance learning at the expense of slower fundraising. The question is under what conditions investors make such a switch.

Agents keep track of two statistics. One is the dollar gap between accumulated funding and the threshold, while the other one is the belief gap between current belief and the break-even belief k^*. Thus, when the dollar gap is small but the belief gap is still big, an H-type agent uses partial support to create "prolonged learning" because a partial support allows for more rounds of trials without triggering implementation or DOWN cascade. Such episodes of prolonged learning may occur multiple times before eventually the agent returns to full support or a DOWN cascade takes place, depending on whether the break-even belief $\bar{k}(p)$ is reached. The next proposition summarizes one equilibrium featuring potential prolonged learning with partial support L, even when full support to reach the funding target sooner is feasible.

Proposition 9. *There exists a pair (p^*, T^*) that maximizes the proposer's expected revenue. For any given (p, T), $a_i^* = 0$ for $\theta_i \in \{H, L\}$ whenever $x_i = -1$. When $x_i = 1$, type $\theta_i = L$ supports as long as she is not the gate-keeper ($A_{i-1} \geq T - L$ & $k_{i-1} < \bar{k}(p) - 1$); type $\theta_i = H$ supports before any cascade, but can switch from full support H to partial support L if the funding gap is small relative to the belief gap, i.e., $(\bar{k}(p) - k_{i-1})L < T - A_{i-1} < H + (\bar{k}(p) - k_{i-1} - 1)L$.*

Moreover, the entrepreneur can no longer extract the full surplus even as N goes infinity. To extract the full surplus, the proposer needs to set a price such that all agents are indifferent between support and rejection. This is problematic now because if the price is high enough (i.e., $p \to V_{\bar{k}}$, a necessary condition for full surplus extraction), then an agent will switch from high support to low support to prolong the campaign and information aggregation. Under such a strategy, there exist some signal sequences such that an UP cascade is more likely to happen, which generates a positive payoff to the entrepreneurs. Finally, note that the results hold for both endogenous and exogenous threshold and pricing. Because the minimum amount often depends on the nature of the project exogenous to the entrepreneur, it maps to an exogenous AoN target in dollar amounts in our model. Appendix A.15 provides a numerical procedure to search for a model solution with endogenous (p, T).

5.4 Free-Riders and Characterization of All Equilibria

Thus far, we have focused on informer equilibria. In this subsection, we characterize all other PBNEs—a daunting task most models of observational learning leave out—and show our key insights are robust. Maintaining the same tie-breaking rules (i.e., Assumption 1), we first show that all possible equilibria involve a group of "informers" and a group of "free-riders" whose actions before a cascade are ignored in equilibrium. Mathematically, agent i is a "free-rider" if $\mathbb{E}[V|\mathcal{H}_{i-1}] = \mathbb{E}[V|\mathcal{H}_i] < \bar{k}(p) + 1$ before any UP cascade. In other words, following sub-history $\mathcal{H}_{i-1}$, it is common knowledge that that subsequent agents would not update their beliefs based on agent i's action, even though an UP cascade has not started yet. Free-riders can be interpreted as irrational or stubborn to learn.

Although in both cascades and the case of free-riders an agent's action is uninformative, agents still can take informative actions after the free-rider's move, and information aggregation continues until a cascade starts or the game ends. Free-riding differs from an information cascade. We call an equilibrium with a positive number of free-riders a "free-rider equilibrium." Free-rider equilibria can be viewed as derivatives of the equilibrium characterized in Proposition 2 in the sense that on each equilibrium path, if one excludes all free-riders, then sub-game dynamics are exactly the same as the one described in Proposition 1.

In a free-rider equilibrium, who become free-riders is generally path-dependent (i.e., specific to realizations of the sequence of signals). Those agents essentially delegate their investment decision to the gate-keeper and this is common knowledge. In other words, they free-ride on information aggregation from subsequent investors. Similar to the informer equilibrium, a free-rider equilibrium differs from the equilibria in most information cascade models because coordination issues manifest themselves. Whether an agent becomes a free-rider depends on subsequent agents' beliefs and his beliefs on their beliefs, etc. Such phenomenon is absent in conventional models because the agent's expected payoff at the time of decision-making is independent of subsequent agents' actions.

To give an example, suppose $\nu < \frac{1}{2}$, $p = \frac{1}{2} = \frac{q^0}{q^0+(1-q)^0}$ and the target is $T = N$. Then there is a sub-game free-rider equilibrium in which all agents but the Nth one support regardless of their private signal, and the Nth agent supports if and only if $X_N = 1$. The next lemma provides the general characterization.

Lemma 3. *A PBNE is either an informer equilibrium or a free-rider equilibrium. If* $p \in$

$\{V_k, k = -1, 0, \dots N\}$, then all free-rider sub-game equlibria are weakly Pareto-dominated by the informer sub-game equilibrium described in Proposition 1, with free-rider sub-game equlibria involving at least two free-riders strictly Pareto-dominated.

The lemma states that given any history of actions, if agents decide to play a free-rider sub-game equilibrium that results in at least two free-riders, then agents are better off playing an informer sub-game equilibrium. The lemma also rules out mixed strategies because randomizing over free-riding and acting on signals are evidently dominated.

In the proof we argue that a free-rider would not reject in any equilibrium because she would not do so with a positive signal and the definition of free-riding implies that she also supports with a negative signal. The intuition behind Lemma 3 then is that every free-rider sub-game equilibrium resembles a subset of possible realization paths of an informer equilibrium with UP cascade early on, but "shifting" the agents after the UP cascade to the front instead. Investors prefer informer sub-game equilibrium because it induces more information aggregation and thus a higher chance of financing a good project.

A standard equilibrium selection is based on payoff dominance (Harsanyi and Selten, 1988). We thus focus on *Pareto-undominated* sub-game equilibria, which can be easily motivated in our context by communication among agents before they draw the signals. This weak refinement merely rules out nuisance equilibria such as the example given before the lemma where investors coordinate on Pareto inferior outcomes, but still allows the large class of free-rider equilibria for general $p \notin \{V_k, K = -1, 0, \dots N\}$. Whenever $p \in \{V_k, K = -1, 0, \dots N\}$, we only need to consider informer equilibria and free-rider equilibria with only one free-rider.

Note that nuisance equilibria can also be ruled out by considering agents' option to wait. Obviously, every agent observing signal $x_i = -1$ is better off waiting. So no matter what, free-rider equilibria cannot emerge because the proposer's payoff is dominated by that in the informer equilibrium when he sets $p \in \{V_k, K = -1, 0, \dots N\}$. This proves to be useful when analyzing the limiting behavior because the proposer can always resort to $p \in \{V_k, K = -1, 0, \dots N\}$ to bound his payoff above in the large crowd limit.

The next proposition shows that in the limit, Pareto-undominated free-rider equilibria deliver qualitatively the same results as informer equilibria do.

Proposition 10. *In any sequence of endogenous designs $\{p_N, T_N\}_{N=1}^{\infty}$ and Pareto-undominated* *sub-game equilibria, as $N \to \infty$, $p_N \to 1$, good projects and only good projects are implemented, and public information becomes arbitrarily informative, i.e.,* $\mathbb{E}[V|\mathcal{H}_N] \xrightarrow{prob.} V$.

The proposition implies that no matter which equilibrium is selected, in the limit the proposer charges a high enough price, which precludes DOWN cascades and ensures efficient project implementation and full information aggregation. In the proof, we actually show that for large N (even before reaching the limit), the implementation efficiency and information aggregation improve relative to that in standard information cascade settings without threshold implementations because in a free-rider equilibrium, the number of informers is unbounded as N goes up. Our earlier findings are therefore robust to considering free-rider equilibria. PBNEs feature efficient project implementation and full information aggregation in the limit of large crowds. Given that financing projects and aggregating information are arguably the most important functions of financial markets, the impact of threshold implementation, especially with large crowds, cannot be overstated.

6 Conclusion

We incorporate AoN thresholds into a classic model of information cascade, and find that agents' payoff interdependence results in uni-directional cascades in which agents rationally ignore private signals and imitate preceding agents only if the preceding agents decide to support. Information aggregation, proposal feasibility, and project selection all improve. In particular, when the number of agents grows large, equilibrium project implementation and information aggregation achieve the socially efficient levels, even under information cascades.

An important application of our model is that financial technologies such as Internet-based funding platforms can help entrepreneurs reach out to a larger agent base to better harness the wisdom of the crowd, as envisioned by the regulatory authorities. We highlight that specific features and designs such as endogenous AoN thresholds are crucial in capitalizing on these potential benefits, especially for sequential sales in the presence of informational frictions. For parsimony and generality, we have left out some details specific to individual applications. For example, third-party certification has significant impacts in equity crowdfunding (Knyazeva and Ivanov, 2017), and private values are equally important as product quality in reward-based crowdfunding. A project proposer may also price discriminate or control the information flow to potential investors. Specific applications taking into consideration these features as well as the joint information and mechanism design (using strategies beyond threshold implementation) definitely constitute useful future studies.

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Appendix A: Derivations and Proofs

For ease of exposition, we omit the equilibrium strategy terms $\{a_i^*\}_{i=1,2\dots,N}$ in conditional expectations $\mathbb{E}[V|\mathcal{H}_i]$ and state variable $k(\mathcal{H}_i)$.

A.1 Proof of Lemma 1

Proof. We prove the lemma by induction on the length of the history $l \in \{0, 1, \cdots, N\}$, where $\mathcal{H}_0 = \varnothing$. For $i = 0$, $\mathbb{E}[V|\mathcal{H}_0] = \frac{1}{2} = V_0$. Now, suppose the statement is true of all histories $\mathcal{H}_i (i \le l)$, that is, $\mathbb{E}[V|\mathcal{H}_i] = V_{k(\mathcal{H}_i)} = \frac{q^{k(\mathcal{H}_i)}}{q^{k(\mathcal{H}_i)}+(1-q)^{k(\mathcal{H}_i)}}$, for some $k(\mathcal{H}_i) \in \mathbb{Z}$.

Now, consider a history $\mathcal{H}_{l+1}$. Bayes' rule implies:

$$\begin{aligned}
\mathbb{E}[V|\mathcal{H}_{l+1}] &= \Pr(V = 1|\mathcal{H}_{l+1}) = \Pr(V = 1|\{a_{l+1}, \mathcal{H}_l\} \\
&= \frac{\Pr(\mathcal{H}_l) \Pr(V = 1|\mathcal{H}_l) \Pr(a_{l+1}|V = 1, \mathcal{H}_l)}{\sum_{j \in \{0,1\}} \Pr(\mathcal{H}_l) \Pr(V = j|\mathcal{H}_l) \Pr(a_{l+1}|V = j, \mathcal{H}_l)} \\
&= \frac{V_{k(\mathcal{H}_l)} \Pr(a_{l+1}|V = 1, \mathcal{H}_l)}{V_{k(\mathcal{H}_l)} \Pr(a_{l+1}|V = 1, \mathcal{H}_l) + (1 - V_{k(\mathcal{H}_l)}) \Pr(a_{l+1}|V = 0, \mathcal{H}_l)} \qquad (12)
\end{aligned}$$

Before we proceed, we formally define the "pooling" and "fully separating" strategies as below, and we shall discuss later why "partially separating" strategies do not appear in equilibrium.

Definition 4. *A strategy is said to be pooling after history* $\mathcal{H}_l$ *if* $\Pr(a_{l+1}\big|x_{l+1} = 1, \mathcal{H}_l) = \Pr(a_{l+1}\big|x_{l+1} = -1, \mathcal{H}_l) = 1$. *Similarly, a strategy is fully separating after history* $\mathcal{H}_l$ *if* $\Pr(a_{l+1} = 1\big|x_{l+1} = 1, \mathcal{H}_l) = 1$ *and* $\Pr(a_{l+1} = 1\big|x_{l+1} = -1, \mathcal{H}_l) = 0$. [26]

Iif agent $l + 1$ is playing a pooling strategy a_{l+1}, given the history $\mathcal{H}_l$ and the signal x_{l+1}, we have, $\forall j \in \{0, 1\}$

$$\Pr(a_{l+1}\big|V = j, \mathcal{H}_l) = \sum_{x_{l+1}} \Pr(x_{l+1}|V = j, \mathcal{H}_l) \Pr(a_{l+1}\big|x_{l+1}, V = j, \mathcal{H}_l) = 1.$$

which implies that $\mathbb{E}[V|\mathcal{H}_{l+1}] = \mathbb{E}[V|\mathcal{H}_l]$.

If, under equilibrium, agent $l + 1$ is playing a fully separating strategy, if $a_{l+1} = 1$, then

$$\Pr(a_{l+1} = 1\big|x_{l+1} = 1, V = 1, \mathcal{H}_l) \Pr(x_{l+1} = 1\big|V = 1, \mathcal{H}_l) = q,$$

and

$$\Pr(a_{l+1} = 1\big|x_{l+1} = -1, V = 1, \mathcal{H}_l) \Pr(x_{l+1} = -1\big|V = 1, \mathcal{H}_l) = 0,$$

which implies that $\Pr(a_{l+1} = 1\big|V = 1, \mathcal{H}_l) = q$.

[26] Note the indeterminacy when defining a pooling strategy, since a_{l+1} can depend on $\mathcal{H}_l$. We can similary define partially separating strategies.

Similarly,

$$\Pr(a_{l+1} = 1 \big| x_{l+1} = 1, V = 0, \mathcal{H}_l) \Pr(x_{l+1} = 1 \big| V = 0, \mathcal{H}_l) = 1 - q,$$

and

$$\Pr(a_{l+1} = 1 \big| x_{l+1} = -1, V = 0, \mathcal{H}_l) \Pr(x_{l+1} = -1 \big| V = 0, \mathcal{H}_l) = 0,$$

which implies that $\Pr(a_{l+1} = 1 \big| V = 0, \mathcal{H}_l) = 1 - q$. Hence, by equation (12), we get $\mathbb{E}[V|\mathcal{H}_{l+1}] = V_{k(\mathcal{H}_l)+1}$. By the same token, we can show that if $a_{l+1} = -1$, then $\mathbb{E}[V|\mathcal{H}_{l+1}] = V_{k(\mathcal{H}_l)-1}$.

Finally, we show that it is without loss of generality to ignore partially separating strategies because of the tie-breaking rule in Assumption 1. Denote $a = \Pr(V = 1|\mathcal{H}_l, x_{l+1} = 1) > \Pr(V = 1|\mathcal{H}_l, x_{l+1} = -1) = b$. There are five cases. (i) The agent with belief b is indifferent, then by Assumption 1, if $N - (l+1) \geq T - A_l - 1$, she will choose to support the project (i.e., $a_{l+1} = 1$), so it is a fully pooling strategy; otherwise, if $N - (l+1) < T - A_l - 1$, she will choose $a_{l+1} = -1$, leading to a fully separating strategy. (ii) The agent with belief a is indifferent, by Assumption 1, if $N - (l+1) \geq T - A_l - 1$, she will choose to support the project (i.e., $a_{l+1} = 1$), so it is a fully separating strategy; otherwise, if $N - (l+1) < T - A_l - 1$, she will choose $a_{l+1} = -1$, leading to a pooling strategy. (iii) The other three cases are trivial, including when the agent with belief b prefers supporting, when the agent with belief a strictly prefers rejection, and when the agent with belief a strictly prefers support while with belief b she prefers rejection. □ □

A.2 Proof of Lemma 2

Proof. For Agent 1, her posterior belief after observing $x_1 = 1$ is $\mathbb{E}[V|x_1 = 1] = q$. If $p > q$, then agent 1 chooses rejection regardless of her private signal and a DOWN cascade starts from the beginning for sure. We thus have the first part of the Lemma.

Similarly, $p = 1 - q = \mathbb{E}[V|x_1 = -1]$ induces an UP cascade starting from the beginning for sure, the entrepreneur or proposer has no incentive to charge $p < 1 - q$. Therefore, $p \in [1-q, q]$. For each $p \in (V_{k-1}, V_k]$, $p = V_k$ induces exactly the same number of supporting agents, so in the equilibrium proposer always charges $p^* = V_k$ for some $k \in \{-1, 0, 1, \ldots, N\}$. Consequently, only three prices are possible: $p_{-1} = 1 - q$, $p_0 = \frac{1}{2}$ and $p_1 = q$. Let $\Pi(p, N)$ be the expected profit when the price is p and there are $N \geq 2$ potential agents. Without AoN thresholds, $\Pi(p, N)$ is obviously increasing in N. Next, we examine the three possible optimal prices and show $p = 1 - q$ dominates.

- $p = 1 - q$: The total profit is $(1-q)N$ when $\nu = 0$;
- $p = \frac{1}{2}$: After the first two observations, $(x_1, x_2) = (-1, -1)$ induces a DOWN cascade yielding zero profit; $(1, -1)$ and $(1, 1)$ both induce an UP cascade at agent 1 due to the tie-breaking assumption, which leads to an expected payoff of $\frac{q+(1-q)}{2}\frac{1}{2}N$; $(-1, 1)$ does not change subsequent agents' belief. Therefore, $\Pi(p, N) = \frac{q+(1-q)}{2}\frac{1}{2}N + \frac{q(1-q)+(1-q)q}{2}\left(\frac{1}{2} + \Pi(p, N-2)\right) \leq$

$\frac{1}{4}N + (1-q)q\left(\frac{1}{2} + \Pi(p,N)\right)$. Thus $p = \frac{1}{2}$ is dominated by $p = 1-q$ if:

$$\Pi(p,N) \leq \frac{\frac{N}{4} + \frac{(1-q)q}{2}}{1-(1-q)q} \leq (1-q)N \ for\ N = 2, 3, \ldots, \tag{13}$$

which holds for $q \in \left(\frac{1}{2}, \frac{3}{4} + \frac{1}{4}(3^{\frac{1}{3}} - 3^{\frac{2}{3}})\right]$;

- $p = q$: After the first two observations, $(1,1)$ induces an UP cascade, $(-1,-1)$ and $(-1,1)$ both induce a DOWN cascade after agent 1, and $(1,-1)$ does not change the belief. The expected profit is $\Pi(p,N) = \frac{(1-q)^2+q^2}{2}qN + \frac{q(1-q)+(1-q)q}{2}(q + \Pi(p,N-2)) \leq \frac{(1-q)^2+q^2}{2}qN + (1-q)q(q + \Pi(p,N))$. Thus $p = q$ is dominated by $p = 1-q$ if:

$$\Pi(p,N) \leq \frac{\frac{(1-q)^2+q^2}{2}qN + q^2(1-q)}{1-(1-q)q} \leq (1-q)N \ for\ N = 2, 3, \ldots \tag{14}$$

  One can verify that the inequality holds for $q \in \left(\frac{1}{2}, \frac{3}{4} + \frac{1}{4}(3^{\frac{1}{3}} - 3^{\frac{2}{3}})\right]$.

□

A.3 Proof of Proposition 1

Proof. The evolution of a^* and k^* are given below, which is not included in the statement of the proposition for brevity. We will verify the expressions shortly.

From here on, we use subscript for a and A to avoid confusion, even though they are not strictly needed notation-wise. $a_i^*(x_i, \mathcal{H}_{i-1}) \equiv a^*(x_i, k(\mathcal{H}_{i-1}), A_{i-1}(\mathcal{H}_{i-1}))$ and posteriors $P(V=1|\mathcal{H}_i) = V_{k^*(\mathcal{H}_i)}$, where:

$$a_i^*(x_i, k, A_{i-1}) = \begin{cases} x_i & \text{if} \quad A_{i-1} < T-1 \ \& \ k \leq \bar{k}(p) \\ 1 & \text{if} \quad k > \bar{k}(p) \\ -1 & \text{if} \quad A_{i-1} \geq T-1 \ \& \ k < \bar{k}(p) - 1 \\ x_i & \text{if} \quad A_{i-1} \geq T-1 \ \& \ k \in \{\bar{k}(p), \bar{k}(p)-1\} \end{cases}, \tag{15}$$

$$k_i^*(\mathcal{H}_i) \equiv k^*(a_i, k_{i-1}, A_{i-1}) = \begin{cases} k_{i-1} + a_i & \text{if} \quad A_{i-1} < T-1 \ \& \ k_{i-1} \leq \bar{k}(p) \\ k_{i-1} & \text{if} \quad k_{i-1} > \bar{k}(p) \\ k_{i-1} & \text{if} \quad A_{i-1} \geq T-1 \ \& \ k_{i-1} < \bar{k}(p) - 1 \\ k_{i-1} + a_i & \text{if} \quad A_{i-1} \geq T-1 \ \& \ k_{i-1} \in \{\bar{k}(p), \bar{k}(p)-1\} \end{cases} \tag{16}$$

where $A_0 = 0$ and $k_i \equiv k(\mathcal{H}_i) \equiv k^*(a_i, k_{i-1}, A_{i-1})$, with $k_0 = 0$.

To prove that the characterization in Proposition 1 constitutes a PBNE, we first state and prove Lemma A.3 — the expected value of the adoption is bounded above by $V_{\bar{k}(p)+1}$. In fact, an UP cascades starts once a strong expectation is formed and it blocks further learning by subsequent agents. Anticipating this behavior from subsequent agents, early agents with negative signal do not support the proposal if an UP cascade has not started yet. It makes the support of the agents with positive signal informative for the subsequent agents. For the histories that either the AoN target or a cascade has been reached, the proof is trivial.

Suppose $\bar{k}(p) \geq -1$. Then, in every equilibrium, when it is still possible to reach the AoN target T, the following relation holds for every $2 \leq i \leq N$:

$$\mathbb{E}[V|\mathcal{H}_{i-1}] \leq V_{\bar{k}(p)+1} \tag{17}$$

In other words, there is an upper-bound on the expected value of the project as a function of p.

Proof. Suppose the contrary. Lemma 1(b) then implies that there exists an agent u with $\mathbb{E}[V|\mathcal{H}_u] = V_{\bar{k}(p)+2}$ and $\mathbb{E}[V|\mathcal{H}_{u-1}] = V_{\bar{k}(p)+1}$. Now given $\mathbb{E}[V|\mathcal{H}_{u-1}] = V_{\bar{k}(p)+1}$, however, u would accept the proposal regardless of her private signal because

$$\mathbb{E}[V|x_u, \mathcal{H}_{u-1}, A_N \geq T] \geq \mathbb{E}[V|x_u, \mathcal{H}_{u-1}] \geq \mathbb{E}[V|x_u = -1, \mathcal{H}_{u-1}] = V_{\bar{k}(p)} \geq p. \tag{18}$$

We remind the readers that all expectations are conditional on equilibrium strategies of other agents, which is not explicitly written for notational simplicity. The first inequality (from left) in (18) follows from the fact that the gate-keeper makes her decision based on an information set that fully nests $\mathcal{H}_{u-1}$. Therefore, her support positive updates agent u's belief.

Equation (18) shows that agent u's action is not informative for subsequent agents. Thus $\mathbb{E}[V|\mathcal{H}_u] = \mathbb{E}[V|\mathcal{H}_{u-1}] = V_{\bar{k}(p)+1}$, a contradiction. □

Now, we are ready to prove the equilibrium characterization. For notational ease, we replace $\bar{k}(p)$ by $\bar{k}$. We proceed by examining the optimal strategy for different histories:

$\mathbf{A_{i-1} < T - 1}$ and $\mathbf{k_{i-1} \leq \bar{k}}$

According to (3), the agent chooses to support if and only if $\mathbb{E}[V|x_i, \mathcal{H}_{i-1}, A_N \geq T] \geq p$. We examine two cases $x_i = 1$ and $x_i = -1$ separately:

- $x_i = 1$**:** If agent i chooses to reject, then she receives 0. If agent i supports, then consider a history $\mathcal{H}_N \succ \mathcal{H}_i = (\mathcal{H}_{i-1}, 1)$ in which the proposal is accepted. Denote the gate-keeper by g, namely g is the smallest integer that $A_g = T$. Note that g is a random variable depending on the history $\mathcal{H}$. Since the subsequent agents $\{i+1, i+2 \ldots\}$ perfectly infer $x_i = 1$ from the support of agent i, then agent g's information set fully nests that of agent i. We get:

$$\mathbb{E}[V|x_i, \mathcal{H}_{i-1}, A_N \geq T] = \mathbb{E}[V|x_i, \mathcal{H}_{i-1}, \mathbb{E}[V|x_g, \mathcal{H}_{g-1}] \geq p] = \mathbb{E}[V|\mathbb{E}[V|x_g, \mathcal{H}_{g-1}] \geq p] \geq p$$

In other words, it is optimal for agent i to support.

- $x_i = -1$**:** In this case, in the equilibrium, agent i's support would be misinterpreted by follow agents as a positive signal. In other words, $k(\mathcal{H}_i) = k(\mathcal{H}_{i-1}) + 1$, while the correct posterior should be $k(\mathcal{H}_i) = k(\mathcal{H}_{i-1}) - 1 = k(\mathcal{H}_{i-1}) + 1 - 2$. Moreover, if the proposal is accepted, Lemma A.3 implies $\mathbb{E}^g[V|x_g, \mathcal{H}_{g-1}] \in \{V_{\bar{k}}, V_{\bar{k}+1}\}$. Therefore, agent i, knowing that her signal is incorrectly inferred by her action, assigns an expected value bounded above by $V_{\bar{k}-1} < p$ conditional on her signal and project implementation. She thus optimally chooses $a_i = -1$, lest she gets negative expected payoff.

Therefore, when $k_{i-1} \leq \bar{k}$ and $A_{i-1} < T-1$, agent i follows her private signal and the subsequent agents update their beliefs accordingly, as specified in (15) and (16).

$\mathbf{A_{i-1} \geq T - 1}$ and $\mathbf{k_{i-1} \leq \bar{k}}$

In this case, agent i supports iff $\mathbb{E}[V|x_i, \mathcal{H}_{i-1}] = V_{k_{i-1}+x_i} \geq p$. The only case that the strategy is separating is when $k_{i-1} \in \{\bar{k}, \bar{k}-1\}$. Given this observation, it is easy to check that the strategies specified in (15) are optimal in this case.

$\mathbf{k_{i-1} > \bar{k}}$

Clearly $\mathbb{E}[V|x_i, \mathcal{H}_{i-1}] \geq V_{\bar{k}} \geq p$. In other words, she would gain at least $V_{\bar{k}} - p$ in expectation if the AoN target is reached. Therefore, it is not profitable to reject regardless of agent i's type, which proves the optimality in this case.

Finally, we show that the equilibrium strategy profile is the unique informer equilibrium. Note that an informer equilibrium is an equilibrium that all strategies are separating before reaching a cascade and the AoN threshold is possible to reach. Therefore, we only need to show that a cascade cannot occur when $A_{i-1} < T - 1$ and $k_{i-1} \leq \bar{k}$ (and the AoN threshold is possible to reach).[27]

Suppose the contrary that such a cascade exists in an equilibrium. Then, the agents should choose the same action after such a history. First, it is not possible that all agents support if this leads to project implementation, because the gate-keeper would certainly reject if she has a negative signal, and it contradicts the assumption that the gate-keeper is part of the cascade. Second, it is not possible that all reject because this would lead to the eventual rejection of the proposal. In this

[27]When $A_{i-1} \geq T-1$, the subsequent actions are irrelevant for agent i's payoff and her action only depends on the price and the expected value given x_i and $\mathcal{H}_{i-1}$. Therefore, the optimal strategy is the same across all equilibria.

case, all agents are indifferent between supporting and not supporting, and their rejections violate Assumption 1. The contradiction implies that the equilibrium is unique.

The characterization of UP cascades and DOWN cascades also follows directly. □

A.4 Large Crowds with Exogenous Price and AoN Targets

We examine the properties of the error probabilities as N becomes large.

Consider a fixed price $p > 0$ and an increasing sequence of AoN targets $\{T_N\}_{N=1}^{\infty}$, where $T_N \leq \lfloor \frac{N+\bar{k}(p)}{2} \rfloor$ for every N, and $\liminf_{N\to\infty} \frac{T_N}{N} \in (0, \frac{1}{2}]$. Let $\mathcal{P}_N^I = 1 - Pr(A_N \geq T_N|V = 1)$ and $\mathcal{P}_N^{II} = Pr(A_N \geq T_N|V = 0)$ denote the probabilities of missing a good project (Type I error) and financing a bad project (Type II error) respectively. Then, for any $\beta > \frac{q}{2q-1}$, there exist positive numbers M and $x < 1$, independent from the choice of p, such that if $T_N > \beta(\bar{k}(p) + 1)$:

$$\begin{gathered}\mathcal{P}_N^I < Mx^{T_N-\beta\bar{k}(p)} \quad \textit{and} \\ \left(1 - Mx^{T_N-\beta\bar{k}(p)}\right)\left(\frac{1-q}{q}\right)^{\bar{k}(p)+1} < \mathcal{P}_N^{II} < \left(\frac{1-q}{q}\right)^{\bar{k}(p)+1}.\end{gathered} \tag{19}$$

Proof. To derive the inequalities in (19), first we show that reaching the AoN target, i.e. $Pr(A_N \geq T_N|V = j)$, $j \in \{0, 1\}$, can be approximated with the probability of reaching an UP cascade before or at the same time of reaching the target. We denote this probability by η_N^j:

$$\eta_N^j = \bigcup_{n=\bar{k}(p)+1}^{2T_N-\bar{k}(p)-1} Pr(\sum_{i=1}^{n} x_i = \bar{k}(p) + 1|V = j) \qquad j \in \{0, 1\} \tag{20}$$

In particular, we show $\lim_{N\to\infty} |\mathcal{P}_N^I - (1 - \eta_N^1)| = \lim_{N\to\infty} |\mathcal{P}_N^{II} - \eta_N^0| = 0$. Then, we find the expressions for η_N^0 and η_N^1 by appealing to the reflection principle. Finally, we show that the sequence of $\{\eta_N^j\}_{N=1}^{\infty}$, $j \in \{0, 1\}$ are convergent and bounded by some algebraic functions.

First, let $\varphi_{\bar{k}(p)+1,i}^j$ be the probability of starting a cascade after agent i when $V = j$, for $j \in \{0, 1\}$. With an argument similar to the proof of Lemma A.5, one can show:

$$\varphi_{k+1,i}^1 = \frac{k+1}{i}\begin{pmatrix} i \\ \frac{i+k+1}{2} \end{pmatrix} q^{\frac{i+k+1}{2}}(1-q)^{\frac{i-k-1}{2}} \qquad \varphi_{k+1,i}^0 = \left(\frac{1-q}{q}\right)^{k+1}\varphi_{k+1,i}^1 \tag{21}$$

The following lemma shows $\varphi_{\bar{k}(p)+1,i}^j$ decreases in i for $i > \lceil (2\beta - 1)(\bar{k}(p) + 1) \rceil$. We use this lemma to show the error probabilities also converge to their limiting value with T_N.

There exists $x \in (0, 1)$ such that for every $p \geq 0$ and $i > \lceil (2\beta - 1)(\bar{k}(p) + 1) \rceil$, we have:

$$\frac{\varphi_{\bar{k}(p)+1,i}^j}{\varphi_{\bar{k}(p)+1,i-2}^j} < x \qquad j \in \{0, 1\} \tag{22}$$

Proof. Using the expressions for $\varphi^j_{\bar{k}(p)+1,i}$ in (21),

$$\begin{aligned}\frac{\varphi^j_{\bar{k}(p)+1,i}}{\varphi^j_{\bar{k}(p)+1,i-2}} &= 4q(1-q)\frac{(i-1)(i-2)}{(i+k+1)(i-k-1)} \\ &\leq 4q(1-q)\frac{i^2}{i^2-(k+1)^2} \end{aligned} \tag{23}$$

To ensure the RHS of equation (23) is strictly less than 1, we need that $i > \frac{\bar{k}(p)+1}{2q-1}$.

Moreover, recall that $\beta > \frac{q}{2q-1}$, and thus that $(2\beta-1) > \frac{1}{2q-1}$. Also note that the RHS of equation (23) is decreasing in i^2. Thus, if we take $i \geq \lceil (2\beta-1)(\bar{k}(p)+1)\rceil$, we have equation (23) holds with

$$x = 4q(1-q)\frac{(2\beta-1)^2}{(2\beta-1)^2-1}.$$

It is easy to verify that $x \in (0,1)$. □

Corollary .2. *For a fixed p and conditional on either $V=1$ or $V=0$, the probability of reaching the AoN target without an UP cascade converges to zero as T_N and N become arbitrarily large.*

Proof. According to (21) and with a similar argument to the proof of Lemma A.5(b), the probability of reaching the AoN target without an UP cascade is $(q^j(1-q)^{1-j})^{-1}\varphi^j_{\bar{k}(p)+1,2T_N-\bar{k}(p)+1}$. Note that to find the probability, we used the fact that the target in this case should be reached exactly at agent $s_{T_N} = 2T_N - \bar{k}(p)$. Now, by appealing to Lemma A.4 , we have:

$$\begin{aligned}\lim_{T_N\to\infty} \varphi^j_{\bar{k}(p)+1,2T_N-\bar{k}(p)+1}(q^j(1-q)^{1-j})^{-1} &\leq \lim_{T_N\to\infty} x\varphi^j_{\bar{k}(p)+1,2T_N-\bar{k}(p)-1}(q^j(1-q)^{1-j})^{-1} \\ &\leq \lim_{T_N\to\infty} x^{\lfloor T_N-\frac{\bar{k}(p)-1}{2}-\frac{i}{2}\rfloor}\varphi^j_{\bar{k}(p)+1,i}(q^j(1-q)^{1-j})^{-1} \\ &= 0.\end{aligned}$$

□

Corollary .2 shows that as N and T_N go to infinity, the AoN target is reached by an UP cascade with probability one. In other words, we proved our first claim that the following asymptotic results hold:

$$\lim_{N\to\infty} |\mathcal{P}^I_N - (1-\eta^1_N)| = \lim_{N\to\infty} |\mathcal{P}^{II}_N - \eta^0_N| = 0$$

Together with Lemma A.8, we are ready to prove the inequalities in (19). For the first inequality, note that:

$$\mathcal{P}^I_N < 1 - \sum_{i=\bar{k}(p)+1}^{s_{T_N}} \varphi^1_{\bar{k}(p)+1,i}$$

where $s_{T_N} = 2T_N - (\bar{k}(p)+1)$. It is easy to check that if an UP cascade starts by the time that the target is reached, then the UP cascade should start at agent s_{T_N} at the latest. Lemma A.4 and equation (34) imply the following provided $s_{T_N} > (2\beta-1)(\bar{k}(p)+1)$:

$$1 - \sum_{i=\bar{k}(p)+1}^{s_{T_N}} \varphi^1_{\bar{k}(p)+1,i} = \sum_{j=1}^{\infty} \varphi^1_{\bar{k}(p)+1,2j+s_{T_N}} < \sum_{j=1}^{\infty} x^{0.5(s_{T_N}-(2\beta-1)(\bar{k}(p)+1))+j} \varphi^1_{\bar{k}(p)+1,\lceil(2\beta-1)(\bar{k}(p)+1)\rceil+\varepsilon_{\bar{k}(p)}}$$
$$< \frac{x^{0.5(s_{T_N}-(2\beta-1)(\bar{k}(p)+1))}}{1-x} \varphi^1_{\bar{k}(p)+1,\lceil(2\beta-1)(\bar{k}(p)+1)\rceil+\varepsilon_{\bar{k}(p)}} = \frac{x^{T_N-\beta(\bar{k}(p)+1))}}{1-x} \varphi^1_{\bar{k}(p)+1,\lceil(2\beta-1)(\bar{k}(p)+1)\rceil+\varepsilon_{\bar{k}(p)}}$$

where $\varepsilon_{\bar{k}(p)}$ is chosen either zero or one such that makes $\lceil\beta(\bar{k}(p)+1)\rceil + \varepsilon_{\bar{k}(p)} + \bar{k}(p) + 1$ an even number. Now, suppose $M \equiv \sup_{k\geq 0} \frac{x^{-\beta}}{1-x}\varphi^1_{k+1,\lceil(2\beta-1)(k+1)\rceil+\varepsilon_k}$. Note that (34) ensures the existence of M and imply $M \leq 1$. Hence, we have

$$\mathcal{P}^I_N < 1 - \sum_{i=\bar{k}(p)+1}^{s_{T_N}} \varphi^1_{\bar{k}(p)+1,i} < Mx^{T_N-\beta\bar{k}(p)} \tag{24}$$

which is the first inequality in (19). The second inequality can be similarly shown by using the second equation in (33). □

Lemma A.4 shows that when the price is fixed, the probability of reaching the AoN target increases, as the probability of having an UP cascade increases. While fewer good projects are missed as N increases (type I error decreases), more bad projects are also implemented (type II error increases).

The proof for Lemma A.4 also shows that in the limit, all good projects are implemented.

Corollary .3. *For fixed p and a sequence $\{T(N)\}_{N=1}^{\infty}$, if $\liminf_{N\to\infty} \frac{T(N)}{N} \in (0,1)$, then a good project with $V=1$ is implemented with an UP cascade with probability 1, as $N\to\infty$.*

A large agent base in a sense improves the implementation of good projects. The intuition is that when $V=1$ and T_N is sufficiently large, the law of large numbers implies that with a high probability, the number of agents with favorable signals among the first T_N agents exceeds the ones with negative signals by a large amount, which in turn guarantees implementation through an UP cascade.[28] Therefore, all good projects are implemented as N becomes arbitrarily large. However, if $V=0$ and few agents with favorable signals happen to be concentrated at the beginning of the queue, then an UP cascade may start too early. In this case, the agents ignore the negative private signals obscured by the UP cascade and support a bad project.

[28]It follows from the assumption that $T_N \leq \lfloor\frac{N+\bar{k}(p)}{2}\rfloor$.

A.5 Proof of Proposition 2

Proof. We prove the proposition through multiple steps and lemmas. In Step 1, we show a profit maximizer proposer optimally chooses a price in $\{V_{-1}, V_0, \ldots, V_N\}$. Therefore, the optimal pair (p^*, T^*) belongs to set $\{(p,T)|p \in \{V_{-1}, \ldots, V_N\}, T \in \{1, \ldots, N\}\}$ with finite number of elements, which ensures the existence of a solution to the proposer's optimization problem. In step 2-6, we show $T^* = \lfloor \frac{N+k^*}{2} \rfloor$, by showing T^* is a dominant choice compared to all $T > T^*$ and $T < T^*$, respectively in Steps 2 and 3 , when $k^* > 0$. For $k^* = 0$, we proves the optimality of T^* for $q(1-q) > \frac{1}{6}$ and $q(1-q) \leq \frac{1}{6}$, respectively in Steps 4 and 5. In step 6, We finish the proof by showing T^* is optimal when $k^* = -1$.

Step 1: Existence of the Solution

First, It is also straightforward to see that $p \geq \nu$, because any equilibrium price $p < \nu$ would generate a negative return for the proposer and is strictly dominated by $p = \nu$. We next show $p^* \in \{V_{-1}, V_0, \ldots, V_N\}$. Note that all $p < V_{-1}$ are suboptimal since an UP cascade starts from the very first agent and all agents would support if $p \leq V_{-1}$. Moreover, clearly the posterior of the agents never exceeds V_N, therefore all $p > V_N$ are suboptimal as well. Thus we focus on the case $p \in [V_{-1}, V_N]$.

For any $p \in (V_{k-1}, V_k]$, $k \in \{0, 1, \ldots, N\}$, in the sub-game the agents follow the same equilibrium strategy profile specified in (15), as it only depends on $\bar{k}(p)$ and T. It implies any choice of $p \in (V_{k-1}, V_k)$ induce the same $\bar{k}(p)$ and is dominated by $p = V_k$, for $k \in \{0, 1 \ldots N\}$. Consequently, $p^* \in \{V_{-1}, V_0, \ldots, V_N\}$. Notice that the set of T^* is also finite as $T \in \{1, 2, \ldots, N\}$. Then the proposer only needs to choose from a finite set of pairs (p, T), which guarantees the existence of solution.

Some Helpful Definitions and Results

For the rest of the proof, suppose the optimal pair is (p^*, T^*), where $p^* = V_{k^*}$. Moreover, we say a sequence of signals $x \in X$ is *"T-supported"* for some $1 \leq T \leq N$, if the proposal is accepted for pair (V_{k^*}, T). Following the discussion in section A.4 , we define $s_{T_N} = 2T_N - (\bar{k}(p) + 1)$. It is easy to check that if an UP cascade starts by the time that the target is reached, then the UP cascade should start at agent s_{T_N} at the latest.

The following two lemmas are useful for our analysis.

Suppose sequence $x \in X$ is $(T-1)$-supported and not T-supported, for some $T \leq T^*$. Then, there are at least $T-1$ positive signals in x. Furthermore, if h_{T-1} is the agent that has the $(T-1)$th positive signal in the queue, then $s_{T-1} \leq N-2$ and $(h_{s_{T-1}}, x_{h_{T-1}+1}, x_{h_{T-1}+2}) = (1, -1, -1)$.

Proof. First, we show that if a sequence $x' \in X$ induces an UP cascade, then more than T^* agents support the proposal. To see this, suppose an UP cascade starts after agent $r < N$, which implies

$\sum_{i=1}^{r} x_i = k^*+1$. Therefore, since all the subsequent agents support the proposal, the total number of supporters is

$$\frac{r+k^*+1}{2} + N - r > \frac{N+k^*+1}{2} > T^*$$

Given this result, since x is not T-supported, the AoN target $T-1$ cannot be reached by an UP cascade, and the existence of at least $T-1$ positive signals is necessary.

To see $h_{T-1} \leq N-2$, note that when the AoN target is $T-1$ and x is $T-1$-supported, the support of h_{T-1} requires that the following condition holds:

$$T - 1 - (h_{T-1} - (T-1)) \geq k^* \Rightarrow h_{T-1} \leq 2(T^*-1) - k^* \leq N-2$$

The only claim is left to show is that $(x_{h_{T-1}}, x_{h_{T-1}+1}, x_{h_{T-1}+2}) = (1,-1,-1)$. It is resulted from the assumption that x is not T-supported. Therefore, it implies that both $\sum_{i=1}^{h_{T-1}+1} x_i$ and $\sum_{i=1}^{h_{T-1}+2} x_i$ should be strictly less than k^*. The result is straightforward from the observation that $\sum_{i=1}^{h_{T-1}} x_i = k^*$. □

Suppose sequence $x \in X$ is $(T-1)$-supported, for some $T \leq T^*$. Then, there exists an injective function $x'(x)$ that maps each sequence x to a distinct sequence $x'(x)$ such that $x'(x)$ is T-supported. The number of supporting agents in $x'(x)$ for T is weakly higher than the one in x for $T-1$.

Proof. The proof of Lemma A.5 shows that if a sequence is $(T-1)$-supported and induces an UP cascade, then it is also T-supported and also induces an UP cascade. Consider $x'(x) = x$, the number of supporting agents for both cases then are the same. If a sequence x is both $(T-1)$-supported and T-supported and there is no UP cascade, then consider again $x'(x) = x$, and the number of supporting agents for both cases are the same. Suppose $\tilde{X}_{T-1}$ is the set of all sequence of signals that they are $(T-1)$-supported, but not T-supported. Similarly, suppose $\tilde{X}_T$ is the set of sequence of signals that are T-supported, but not $(T-1)$-supported. From Lemma A.5, every sequence of signals $x \in \tilde{X}_{T-1}$ can be rewritten as $x = (\underbrace{\dots}_{h_{T-1}-1}, 1, -1, -1, \dots)$. There exists a corresponding sequence $x' = (\underbrace{\dots}_{h_{T-1}-1}, -1, 1, 1, \dots) \in \tilde{X}_T$, in which only the three middle signals are reversed. Based on results in Proposition 1, there are exactly $T-1$ supporters in x (since a DOWN cascade starts at agent $h_{T-1}+2$), while there are at least T supporters in $x'(x)$. By construction, each x has a distinct image $x'(x)$, so $x'(x)$ is an injective function. □

Over the next couple of steps, we show next the optimal AoN Threshold is $T^* = \lfloor \frac{N+k^*}{2} \rfloor$.

Step 2: The Proof of $\pi(p^*, T) < \pi(p^*, T^*)$, for $T > T^*$ and $k^* > 0$

We simply show that if a sequence of signals is T-supported for some $T > T^*$, then it is T^*-supported as well. Finally, we show that there exists at least one sequence that is T^*-supported but not T-supported for any $T > T^*$.

To prove the first claim, suppose $x \in X$ is a T-supported sequence for some $T > T^*$. Denote s_j the agent that makes the j'th support. There are two possibilities. If s_{T^*} is part of an UP cascade, namely she supports regardless of her private signals, then the formation of the UP cascade is not affected by reducing the AoN target to T^*. Therefore, x is also T^*-supported.

If s_{T^*} is not part of an UP cascade, then all the first T^* supporters have a positive private signal. If the AoN target is reduced to T^*, it does not affect the decision of all agents $i \leq s_{T^*-1}$. Agent s_{T^*}, as the gate-keeper, supports only if the total number of positive signals exceed the negative ones by at least k^*, i.e. $\sum_{i=1}^{s_{T^*}} x_i \geq k^*$. It is the case, because:

$$\sum_{i=1}^{s_{T^*}} x_i = 2T^* - s_{T^*} \geq 2T^* - N + (T - T^*) \geq 2T^* - N + 1 \geq k^*$$

The first inequality comes from the fact that $s_{T^*} + (T - T^*) \leq s_T \leq N$. Therefore, x is T^*-supported too. Moreover, it is easy to check that the following sequence is T^*-supported and not T-supported for any $T > T^*$: If $N + k^*$ is even, consider a sequence that its first $\frac{N-k^*}{2}$ elements are -1 and the last $\frac{N+k^*}{2}$ elements are 1. If $N + k^*$ is odd, then consider a sequence that its first $\frac{N-k^*-1}{2}$ elements and the last element are -1 and the remaining elements are 1.

Step 3: The Proof of $\pi(p^*, T) < \pi(p^*, T^*)$, for $T < T^*$ and $k^* > 0$

Notice that given Proposition 1 and $k^* > 1$, if $T < k^*$ then the project will not be implemented for sure. Thus we only consider the case when $k^* \leq T$. We first show the probability of reaching the AoN target ($P(A_N \geq T|T)$) strictly increases with T for $k^* \leq T \leq T^*$. Then, we show the expected number of supporters conditional on reaching the AoN target ($\mathbb{E}[A_N|A_N \geq T, T]$) is also increasing with T. These two results are enough to conclude that the proposer's expected profit is increasing in T for $k^* \leq T \leq T^*$.

First, we show $P(A_N \geq T|p^*, T) \geq P(A_N \geq T-1|p^*, T-1)$, for $T \leq T^*$. We aim to show $Pr(\tilde{X}_T) \geq Pr(\tilde{X}_{T-1})$. But we need a useful lemma here. Let $\varphi_{k+1,i}$ denote the probability that an UP cascade starts after agent i. Since each agent privately observes either $x_i = 1$ or $x_i = -1$ and her decision perfectly reveals her private signal before an UP cascade starts, the arrival of an UP cascade is equivalent to the first passage time of a one-dimension biased random walk. Then using results on hitting times from (Van der Hofstad and Keane, 2008), we can compute $\varphi_{k+1,i}$:

Suppose $\bar{k}(p) = k$ (recall $\bar{k}(p)$ is defined in Proposition 1), then we have

(a) The probability that an UP cascade starts after agent i is

$$\varphi_{k+1,i} = \frac{k+1}{i} \begin{pmatrix} i \\ \frac{i+k+1}{2} \end{pmatrix} [q(1-q)]^{\frac{i-k-1}{2}} \frac{(1-q)^{k+1} + q^{k+1}}{2}, \tag{25}$$

where

$$\left(\begin{array}{c} i \\ \frac{i+j}{2} \end{array}\right) = \begin{cases} \frac{i!}{\frac{i+j}{2}!\frac{i-j}{2}!} & \text{if } i \geq j \text{ and } j+i \text{ even;} \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

(b) For a given pair of price and threshold (p, T), the probability of reaching the threshold at agent i without a prior UP cascade is $\frac{q^k+(1-q)^k}{q^{k+1}+(1-q)^{k+1}}\varphi_{k+1,i+1}$.

Proof. First, we restate a standard result of the hitting time theorem (Van der Hofstad and Keane, 2008) [**Hitting Time Theorem**] Fix $n \geq 1$. Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of independently and identically distributed random variables Y_i taking values in $\{\cdots, -2, -1, 0, 1\}$. Define $S_n = \sum_{i=0}^{n} Y_i$, where $Y_0 = 0$. Define the stopping (hitting) time $\tau_k = \inf\{m \geq 0 : S_m = k\}$. Then,

$$\Pr(\tau_k = n) = \frac{k}{n}\Pr(S_n = k). \tag{27}$$

Now, we are ready to prove lemma A.5.

Part (a). Note that an UP cascade starts after agent i if and only if $\tau_{k+1} = i$, once we replace Y_i in lemma A.5 with X_i in our setting. Moreover, we can directly derive $\Pr(S_i = k+1|V \in \{0,1\})$ by combinitorial calcualtion as below:

$$\Pr(\tau_{k+1} = i|V = 1) = \frac{k+1}{i}\Pr(S_i = k+1|V = 1) = \frac{k+1}{i}\binom{i}{\frac{i+k+1}{2}}q^{\frac{i+k+1}{2}}(1-q)^{\frac{i-k-1}{2}},$$

and

$$\Pr(\tau_{k+1} = i|V = 0) = \frac{k+1}{i}\Pr(S_i = k+1|V = 0) = \frac{k+1}{i}\binom{i}{\frac{i+k+1}{2}}(1-q)^{\frac{i+k+1}{2}}q^{\frac{i-k-1}{2}}.$$

Moreover, note that

$$\varphi_{k+1,i} = \Pr(\tau_{k+1} = i) = \sum_{j\in\{0,1\}}\Pr(V = j)\Pr(\tau_{k+1} = i|V = j).$$

The proof concludes by simple algebra.

Part (b). Denote $A = \{\mathcal{H}_i : \text{Reaching the threshold at } i \text{ without prior UP cascade}\}$. Obviously, $k(\mathcal{H}_i) = \bar{k}(p)$. Thus, if $x_{i+1} = 1$, then we have an UP cascade starting at $i+1$, or $\tau_{k+1} = i+1$. Hence, we have:

$$\Pr(A|V = j) = \frac{\Pr(\tau_{k+1} = i+1|V = j)}{\Pr(x_{i+1} = 1|V = j)}.$$

for $j \in \{0,1\}$. This further implies

$$\begin{aligned}\Pr(A) &= \sum_{j\in\{0,1\}} \Pr(V=j)\Pr(A|V=j) \\ &= \sum_{j\in\{0,1\}} \Pr(V=j)\frac{\Pr(\tau_{k+1}=i+1|V=j)}{\Pr(x_{i+1}=1|V=j)}\end{aligned}$$

By plugging the expressions for both $\Pr(\tau_{k+1} = i+1|V=j)$ (see the proof of Part (a) above) and $\Pr(x_{i+1} = 1|V=j)$, the proof concludes. □

Back to the main proof for Step 3. Following the proof of Lemma A.5 , for every sequence of signals $x = (\underbrace{\dots}_{h_{T-1}-1}, 1, -1, -1, \dots) \in \tilde{X}_{T-1}$, there exists a corresponding sequence $x'(x) = (\underbrace{\dots}_{h_{T-1}-1}, -1, 1, 1, \dots) \in \tilde{X}_T$, in which only the three middle signals are reversed. $x'(x)$ is an injective function, therefore,

$$Pr(\tilde{X}_T) = \frac{1}{2}\left[Pr(\tilde{X}_T|V=1) + Pr(\tilde{X}_T|V=0)\right] \geq \frac{1}{2}\left[\frac{q}{1-q}Pr(\tilde{X}_{T-1}|V=1) + \frac{1-q}{q}Pr(\tilde{X}_{T-1}|V=0)\right]$$
$$\Rightarrow Pr(\tilde{X}_T) - Pr(\tilde{X}_{T-1}) \geq \frac{2q-1}{2}\left[\frac{1}{1-q}Pr(\tilde{X}_{T-1}|V=1) - \frac{1}{q}Pr(\tilde{X}_{T-1}|V=0)\right]$$

The first inequality comes from the fact that $x'(x)$ is an injection but not necessarily a bijection. Consequently, we only need to show $\frac{Pr(\tilde{X}_{T-1}|V=1)}{Pr(\tilde{X}_{T-1}|V=0)} \geq \frac{1-q}{q}$. To see this, note that

$$\mathbb{E}\left[V|x \in \tilde{X}_{T-1}\right] = \mathbb{E}\left[V|\sum_{i=1}^{h_{T-1}+2} x_i\right] = V_{k^*-2} \Rightarrow \frac{Pr(\tilde{X}_{T-1}|V=1)}{Pr(\tilde{X}_{T-1}|V=0)} = \frac{V_{k^*-2}}{1-V_{k^*-2}} \geq \frac{V_{-1}}{1-V_{-1}} = \frac{1-q}{q}$$

We thus have $Pr(\tilde{X}_T) \geq Pr(\tilde{X}_{T-1})$ for $T \leq T^*$ and $k^* > 0$.

We next discuss the number of supporters conditional on reaching the AoN target. Based on the proof of Lemma A.5 , we know that for each $x \in X$ that is $(T-1)$-supported, there exists a distinct sequence $x_T \in X$ that is T-supported and has at least the same number of supporting agents. Moreover, for $x \in \tilde{X}_{T-1}$, the corresponding x_T has strictly higher number of supporters.

As a result, the probability of reaching the AoN target is greater for T than $T-1$, and the expected number of supporters conditional on reaching the target is strictly greater for T than $T-1$. Therefore, for any $\nu < V_N$, $\pi(p^*, T) > \pi(p^*, T-1)$ for $k^* \leq T \leq T^*$. By combining the results of Steps 2 and 3, we conclude that the optimal AoN target $T^\star = \lfloor \frac{N+k^\star}{2} \rfloor$ if $k^* > 0$.

Step 4: Optimality of T^* when $k^* = 0$, $q(1-q) > \frac{1}{6}$

For $k^* = 0$ ($p^* = \frac{1}{2}$), the proof of step 2 also applies here, so $\pi(\frac{1}{2}, T) < \pi(\frac{1}{2}, T^*)$, for $T > T^*$. Therefore, we only need to show $\pi(\frac{1}{2}, T) < \pi(\frac{1}{2}, T^*)$ for $T < T^*$. To be more specific, we will show that any strategy $(p=\frac{1}{2}, T-1)$, $2 \leq T \leq T^*$, is dominated by $(\frac{1}{2}, T)$. Given $p=\frac{1}{2}$, for any sequence x that has agent $2T$ as part of an UP cascade, it is both $(T-1)$-supported and T-supported, and the number of supporters are the same for $T-1$ and T. So the proposer is indifferent between $(\frac{1}{2}, T-1)$ and $(\frac{1}{2}, T)$ when agent $2T$ is part of an UP cascade.

Define $\mathbb{Q}_m = \{x | \sum_{i=1}^{j} x_i \leq 0, \forall j \leq m, \sum_{i=1}^{m} x_i = 0\}$. For any $x \in \mathbb{Q}_m$, agent m is not part of an UP cascade. Then we can characterize $\tilde{X}_{T-1}$ and $\tilde{X}_T$:

$$\tilde{X}_{T-1} = \{x | x \in \mathbb{Q}_{2T-2}, x_{2T-1} = x_{2T} = -1\}$$

$$\tilde{X}_T = \mathbb{Q}_{2T} / \{x | x \in \mathbb{Q}_{2T-2}, x_{2T-1} = -1, x_{2T} = 1\}$$

By Lemma A.5, we find the probability of $\mathbb{Q}_m$:

$$Pr(\mathbb{Q}_m) = \frac{1}{2} Pr(\mathbb{Q}_m | V=1) + \frac{1}{2} Pr(\mathbb{Q}_m | V=0) = \frac{1}{2} \frac{Pr(\mathbb{H}_{m+1}^U | V=1)}{q} + \frac{1}{2} \frac{Pr(\mathbb{H}_{m+1}^U | V=0)}{1-q},$$

where $\mathbb{H}_m^U$ is the set of histories that an UP cascade starts after agent m. Let $\pi(p, T, \mathbb{Z})$ be the expected revenue on event set $\mathbb{Z}$, given strategy (p, T). Then:

$$\begin{aligned}
\frac{\pi(\frac{1}{2}, T, \tilde{X}_T)}{\pi(\frac{1}{2}, T-1, \tilde{X}_{T-1})} &\geq \frac{Pr(\tilde{X}_T)}{Pr(\tilde{X}_{T-1})} \frac{T}{T-1} = \frac{Pr(\mathbb{Q}_{2T}) - Pr(\mathbb{Q}_{2T-2}) q(1-q)}{Pr(\mathbb{Q}_{2T-2}) \frac{(1-q)^2+q^2}{2}} \frac{T}{T-1} \\
&= \frac{\frac{1}{2} \frac{Pr(\mathbb{H}_{2T+1}^U | V=1)}{q} + \frac{1}{2} \frac{Pr(\mathbb{H}_{2T+1}^U | V=0)}{1-q} - \frac{1}{2} \frac{Pr(\mathbb{H}_{2T-1}^U | V=1) q(1-q)}{q} - \frac{1}{2} \frac{Pr(\mathbb{H}_{2T-1}^U | V=0) q(1-q)}{1-q}}{\left[\frac{1}{2} \frac{Pr(\mathbb{H}_{2T-1}^U | V=1) q^2}{1-q} + \frac{1}{2} \frac{Pr(\mathbb{H}_{2T-1}^U | V=0)(1-q)^2}{q}\right] \frac{T-1}{T}} \\
&= \frac{6T}{T+1} \frac{q(1-q)}{(1-q)^2+q^2} \geq \frac{4q(1-q)}{1-2q(1-q)}
\end{aligned}$$

where, the first inequality comes from the fact that any $x \in \tilde{X}_{T-1}$ has exactly $T-1$ supporting agents while any $x \in \tilde{X}_T$ has at least T supporters, and the last inequality applies the fact that $T \geq 2$. When $q(1-q) > \frac{1}{6}$, $\pi\left(\frac{1}{2}, T, \tilde{X}_T\right) > \pi\left(\frac{1}{2}, T-1, \tilde{X}_{T-1}\right)$, thus $\left(\frac{1}{2}, T-1\right)$ is dominated by $\left(\frac{1}{2}, T\right)$.

Step 5: Optimality of T^* when $k^* = 0$, $q(1-q) \leq \frac{1}{6}$

Similar to step 4, we only need to show for any $2 \leq T \leq T^*$,$(\frac{1}{2}, T-1)$ is a dominated strategy. To be specific, we will show that $(\frac{1}{2}, T-1)$ is dominated by (q, T). To achieve this, we decompose all possible implementation histories under strategy $(\frac{1}{2}, T-1)$ into several sets, and shows that for each set, there is a corresponding distinct set of implementation histories under strategy (q, T) associated with more profit.

When $(1-q)q \leq \frac{1}{6}$, we have $q \geq \frac{1}{2} + \frac{\sqrt{3}}{6} > \frac{3}{4}$. For $p^* = \frac{1}{2}$ and AoN threshold $T-1$, given the definition of T^*, the project would be implemented either when there is already an UP cascade by agent $2T-2$ or when there is no UP cascade by agent $2T-2$ and the $2T-2$th agent is the $T-1$th supporting agent. It suffices to show that in each scenario, the alternative strategy fares better for the proposer.

1. When there is already an UP cascade by agent $2T-2$, let $\mathbb{H}_i^U$ be the set of histories that result in an UP cascade starting after agent $i \leq 2T-2$. Given any $\mathcal{H}_i \in \mathbb{H}_i^U$, $\mathbb{E}[V|k(\mathcal{H}_i)] = q$, and denote the number of supporters by $A_N(\mathbb{H}_i^U)$. If $x_{i+1} = 1$, then there would be an UP cascade starting after agent $i+1$ for the strategy $(p^* = q, T)$, and the number of supporting agents is also $A_N(\mathbb{H}_i^U)$. Let $\pi(p, T|\mathbb{H}_i^U)$ be the expected payoffs for the proposer conditional on strategy (p, T) and event $\mathbb{H}_i^U$. $(p^* = q, T)$ dominates $(\frac{1}{2}, T-1)$ conditional on $\mathbb{H}_i^U$ because

$$\begin{aligned}
\pi(q, T|\mathbb{H}_i^U) &\geq (q-\nu)A_N(\mathbb{H}_i^U)[Pr(V=1|\mathbb{H}_i^U)q + Pr(V=0|\mathbb{H}_i^U)(1-q)] \\
&= A_N(\mathbb{H}_i^U)(q-\nu)[(1-q)^2+q^2] > A_N(\mathbb{H}_i^U) \times \left(\frac{3}{4}-\nu\right) \times (1-2(1-q)q) \\
&\geq A_N(\mathbb{H}_i^U) \times \left(\frac{3}{4}-\nu\right) \times \frac{2}{3} \geq \left(\frac{1}{2}-\nu\right) A_N(\mathbb{H}_i^U) = \pi\left(\frac{1}{2}, T-1|\mathbb{H}_i^U\right)
\end{aligned}$$

2. When there is no UP cascade by agent $2T-2$ but the $(2T-2)$th agent is the $(T-1)$th supporting agent, $x \in \mathbb{Q}_{2T-2}$. Consider the following two sets of histories for strategy (q, T):

   (a) $\mathbb{Q}_{2T-1}^A = \{x | x \in \mathbb{Q}_{2T-2}, x_{2T-1} = 1\}$:

   Obviously the threshold T is met. Since given any $\mathcal{H}_{2T-2} \in \mathbb{Q}_{2T-2}$, there are equal number of positive and negative signals by agent $2T-2$, we have

   $$Pr(\mathbb{Q}_{2T-1}^A) = Pr(\mathbb{Q}_{2T-2})[Pr(V=1|\mathbb{Q}_{2T-2})q + Pr(V=0|\mathbb{Q}_{2T-2})(1-q)] = \frac{1}{2}Pr(\mathbb{Q}_{2T-2})$$

   We then discuss the expected number of supporting agents:

   For event $\mathbb{Q}_{2T-2}$ under strategy $(\frac{1}{2}, T-1)$, when $x_{2T-1} = 1$, the UP cascade starts after agent $2T-1$ and the number of supporting agents is $N-T+1$, the maximum conditional on $\mathbb{Q}_{2T-2}$. The associated conditional probability is $Pr(x_{2T-1} = 1|\mathbb{Q}_{2T-2}) = \frac{1}{2}$. For event $\mathbb{Q}_{2T-1}^A$ under strategy (q, T), if $x_{2T} = 1$, the UP cascade starts after agent $2T$ and the number of supporting agents is also $N-T+1$. The associated conditional probability is $Pr(x_{2T} = 1|\mathbb{Q}_{2T-1}^A) = q^2 + (1-q)^2 \geq \frac{2}{3} > \frac{1}{2}$.

   For event $\mathbb{Q}_{2T-2}$ under strategy $(\frac{1}{2}, T-1)$, $Pr\left(x_{2T-1} = -1|\mathbb{Q}_{2T-2}\right) = \frac{1}{2}$. On the other hand, for event $\mathbb{Q}_{2T-1}^A$, under strategy (q, T), $Pr\left(x_{2T} = -1|\mathbb{Q}_{2T-1}^A\right) = 2q(1-q) < \frac{1}{2}$. For each possible subsequence $z = \{x_{2T}, x_{2T+1}, \ldots, x_N\}$ of a sequence $x \in \mathbb{Q}_{2T-2}$, and $x_{2T-1} = -1$, let $A_N(\frac{1}{2}, T-1|\mathbb{Q}_{2T-2}, x_{2T-1} = -1, z)$ be the associated number of supporting agents under strategy $(\frac{1}{2}, T-1)$. The corresponding subsequence $z' = \{x_{2T}, x_{2T+1}, \ldots, x_{N-1}\}$ satisfies $A_N(q, T|\mathbb{Q}_{2T-1}^A, x_{2T} = -1, z') > A_N(\frac{1}{2}, T-1|\mathbb{Q}_{2T-2}, x_{2T-1} = -1, z)$. This inequility holds because in each scenario, right before the corresponding subsequence $z(z')$ starts, the the posterior is $\bar{k}(p)-1$, the project would be implemented for sure, and there are T supporters in $\mathbb{Q}^A_{2T-1}$ before z' but only $T-1$ supporting decisions in $\mathbb{Q}_{2T-2}$ before z. So $\mathbb{E}[A_N(q,T)|\mathbb{Q}^A_{2T-1}, x_{2T}=-1] \geq \mathbb{E}[A_N(\frac{1}{2}, T-1)|\mathbb{Q}_{2T-2}, x_{2T-1}=-1]$.

Then:

$$\begin{aligned}
\mathbb{E}\left[A_N(q,T)|\mathbb{Q}^A_{2T-1}\right] &= \sum_{-1,1} Pr(x_{2T}=i|\mathbb{Q}^A_{2T-1})\mathbb{E}\left[A_N(q,T)|\mathbb{Q}^A_{2T-1}, x_{2T}=i\right] \\
&= (q^2+(1-q)^2)(N-T+1) + 2q(1-q)\mathbb{E}\left[A_N(q,T)|\mathbb{Q}^A_{2T-1}, x_{2T}=-1\right] \\
&> \frac{1}{2}(N-T+1) + \frac{1}{2}[A_N(q,T)|\mathbb{Q}^A_{2T-1}, x_{2T}=-1] \\
&> \frac{1}{2}(N-T+1) + \frac{1}{2}[A_N(\frac{1}{2},T-1)|\mathbb{Q}_{2T-2}, x_{2T-1}=-1] \\
&= \sum_{-1,1} Pr(x_{2T-1}=i|\mathbb{Q}_{2T-2})\mathbb{E}\left[A_N(\frac{1}{2},T-1)|\mathbb{Q}_{2T-2}, x_{2T-1}=i\right] \\
&= \mathbb{E}\left[A_N(\frac{1}{2},T-1)|\mathbb{Q}_{2T-2}\right],
\end{aligned}$$

where the first inequality comes from the fact that $N-T-1$ is the maximum number of possible supporters conditional on $\mathbb{Q}_{2T-2}$.

(b) Consider the set $\mathbb{Q}^B_{2T-1} = \{x | \sum_{i=1}^{j} x_i \leq 1, \forall 1 \leq j \leq 2T-1, \sum_{i=1}^{2T-3} x_i = 1, x_{2T-2} = -1, x_{2T-1} = 1\}$:

$\mathbb{Q}^B_{2T-1}$ is the event that there is no UP cascade (with respect to $p^* = q$) by agent $2T-2$, and $k(\mathcal{H}_{2T-3}) = 1$, $x_{2T-2} = -1$ and $x_{2T-1} = 1$. Obviously the threshold T is met. Notice that for strategy (q,T), histories in this set are distinct from those we have discussed (in case 1 we only cover UP cascades for strategy (q,T)). For any sequence $x \in \mathbb{Q}^B_{2T-1}$, there is a mapping $x^A(x) = \{x_{2T-2}, x_1, x_2, \ldots, x_{2T-3}, x_{2T-1}, \ldots\}$. $x^A(x)$ is a bijection that establishes a one-to-one mapping between finite sets $\mathbb{Q}^B_{2T-1}$ and $\mathbb{Q}^A_{2T-1}$. Following the discussion in part (a), we have $Pr(\mathbb{Q}^B_{2T-1}) = Pr(\mathbb{Q}^A_{2T-1}) = \frac{1}{2}Pr(\mathbb{Q}_{2T-2})$, and the expected number of supporters conditional on event $\mathbb{Q}^B_{2T-1}$ and strategy (q,T) is higher than that of event $\mathbb{Q}_{2T-2}$ and strategy $(\frac{1}{2}, T-1)$.

Since $Pr(\mathbb{Q}^B_{2T-1}) + Pr(\mathbb{Q}^A_{2T-1}) = Pr(\mathbb{Q}_{2T-2})$, and in either case there are more supporting agents that generating higher profit $q-\nu > \frac{1}{2}-\nu$, So $(p^*=q, T)$ dominates $(\frac{1}{2}, T-1)$ when there is no cascade and $(1-q)q \leq \frac{1}{6}$.

Step 6: Optimality of T^* when $k^* = -1$

Note that for $k^* = -1$ $(p^* = 1-q)$, an UP cascade is reached from the first agent. Therefore, regardless of the choice of T, all the agents support the proposal. Therefore, $T = T^*$ is an optimal choice.

In conclusion, step 2-6 show that $T = T(p^*)$ is the proposer's weakly dominant strategy, and it is a strictly dominant strategy whenever different T choices may lead to different equilibrium expected proceeds.

Endogenous Price and AoN Target

We now prove the last part of the proposition. Suppose A_N^k is the number of supporters when there are N agents, $p = V_k$ and $T = \lfloor \frac{N+k}{2} \rfloor$. Note that A_N^k is a random variable depending on the sequence of signals. Furthermore, the proposer's profit is $\pi(V_k, \lfloor \frac{N+k}{2} \rfloor, N) = (V_k - \nu)\mathbb{E}[A_N^k \mathbb{1}_{\{A_N^k \geq \lfloor \frac{N+k}{2} \rfloor\}}]$, where $\mathbb{E}[A_N^k]$ is the expected number of supporters when $p = V_k$. For the rest of the proof, let $S_N^k \equiv A_N^k \mathbb{1}_{\{A_N^k \geq \lfloor \frac{N+k}{2} \rfloor\}}$.

Now, we want to provide an upper bound for $\mathbb{E}[S_N^k - S_N^{k+1}]$, the expected number of supporters lost due to the price increase from V_k to V_{k+1}. According to Lemma A.4, Lemma A.8 and Corollary .2 , if $\frac{N}{2} > (\beta - \frac{1}{2})k$ and N and k are sufficiently large, then the probability of not reaching the target with an UP cascade for $p = V_{k+1}$, while reaching the target with $p = V_k$ is at most $1 - Mx^{\frac{N}{2} - (\beta - \frac{1}{2})(k+1)}$ when $V = 1$ and $(\frac{1-q}{q})^{k+1}(1 - Mx^{\frac{N}{2} - (\beta - \frac{1}{2})(k+1)})$ when $V = 0$. Therefore:

$$\mathbb{E}[S_N^k - S_N^{k+1}] \leq (1-\delta)\mathbb{E}\left[S_N^k - S_N^{k+1} | S_N^{k+1} \geq \left\lfloor \frac{N+k+1}{2} \right\rfloor\right] + \delta N, \tag{28}$$

where $\delta = Mx^{\frac{N}{2} - (\beta - \frac{1}{2})k}$. Note that in (28), we use the fact that $\frac{1}{2}(1 + (\frac{1-q}{q})^{k+1}) < 1$. In fact, we provide the bound conditional on $V = 1$. The proof for $V = 0$ is similar.

In Corollary .2 , we showed that the probability of reaching the target without reaching an UP cascade goes to zero for arbitrarily large values of k and N. Furthermore, it is clear that if an UP cascade is reached for $p = V_{k+1}$, it should follow an UP cascade for $p = V_k$. Now, in Lemma A.5 , we find an upper bound for the expected number of supporters lost due to the price increase.

Suppose C_i is the probability that the UP cascade when $V = 1$ for $p = V_{k+1}$ starts $2i + 1$ periods after that for $p = V_k$, then

(a) $C_i \leq \frac{1}{2i+1} q^{i+1}(1-q)^i \binom{2i+1}{i+1}$; **(b)** $\sum_{i=0}^{\infty} iC_i < \frac{q}{1-4q(1-q)}$.

Proof. **(a)** Following the lemma A.5, suppose an UP cascade starts at $u \leq \lfloor \frac{N+k}{2} \rfloor$ when $p = V_k$. The probability that an UP cascade starts at $u + 2i + 1$ for $p = V_{k+1}$ conditional on u (and $u + 2i + 1 \leq \lfloor \frac{N+k+1}{2} \rfloor$) is the probability of reaching an UP cascade when $k = 0$. Moreover, $\frac{1}{2i+1} q^{i+1}(1-q)^i \binom{2i+1}{i+1}$ is the probability of such cascade conditional on $V = 1$, which is higher than the one conditional on $V = 0$.

(b) The inequality can be easily derived by noting that $\binom{2i+1}{i+1} < 2^{2i+1}$, $\frac{i}{2i+1} < \frac{1}{2}$ and using the inequality obtained in part (a). □

Note that $\sum_{i=0}^{\infty} iC_i$ is an upper bound on the expected number of supporters lost, conditional on eventually reaching an UP cascade. Therefore, we can modify (28) as follows:

$$\mathbb{E}[S_N^k - S_N^{k+1}] < B(1-\delta) + \delta N, \tag{29}$$

where $B \equiv \frac{q}{1-4q(1-q)}$. We next use this result to find an optimality condition for k_N^*. Note that for the optimal price $p_N^* = V_{k_N^*}$, we should have:

$$\begin{aligned}
V_{k_N^*}\mathbb{E}\left[S_N^{k_N^*}\right] - V_{k_N^*+1}\mathbb{E}\left[S_N^{k_N^*+1}\right] \geq 0 &\Rightarrow V_{k_N^*}(\mathbb{E}\left[S_N^{k_N^*} - S_N^{k_N^*+1}\right]) > (V_{k_N^*+1} - V_{k_N^*})\mathbb{E}\left[S_N^{k_N^*+1}\right] \\
&\Rightarrow V_{k_N^*}(B(1-\delta^*) + \delta^* N) > \frac{N + k_N^* + 1}{2}(1-\delta^*)(V_{k_N^*+1} - V_{k_N^*}) \\
&\Rightarrow \frac{V_{k_N^*}}{V_{k_N^*+1} - V_{k_N^*}} > \frac{(1-\delta^*)\frac{N+k_N^*+1}{2}}{B(1-\delta^*) + \delta^* N} \Rightarrow \frac{2q}{2q-1}(\frac{q}{1-q})^{k_N^*} > \frac{(1-\delta^*)\frac{N+k_N^*+1}{2}}{B(1-\delta^*) + \delta^* N},
\end{aligned}$$

where $\delta^* \equiv M x^{\frac{N}{2} - (\beta - \frac{1}{2})k_N^*}$. Note that if $\frac{k_N^*}{\ln N} \to 0$, then $\delta^* N$ converges to zero as N goes to infinity. Therefore, for large enough values of N, we should have:

$$\frac{2q}{2q-1}(\frac{q}{1-q})^{k_N^*} > \frac{1}{2B}N \Rightarrow \ln\frac{2q}{2q-1} + k_N^* \ln\frac{q}{1-q} > \ln\frac{1}{2B} + \ln N \Rightarrow \liminf_{N\to\infty} \frac{k_N^*}{\ln N} \geq \frac{1}{\ln\frac{q}{1-q}},$$

which is a contradiction with $\frac{k_N^*}{\ln N} \to 0$. There thus exists $a > 0$ such that $\liminf_{N\to\infty} \frac{k_N^*}{\ln N} \geq a > 0$. It implies that there exist a, b such that $k_N^* > a \ln N + b$ for every N. We then have,

$$\begin{aligned}
1 - p_N^* = \frac{(1-q)^{k_N^*}}{(1-q)^{k_N^*} + q^{k_N^*}} &< (\frac{1-q}{q})^{k_N^*} < (\frac{1-q}{q})^{a\ln N + b} = N^{a\ln\frac{1-q}{q} + b\frac{\ln\frac{1-q}{q}}{\ln N}} \\
\Rightarrow N^{a\ln\frac{q}{1-q}}(1-p_N^*) &< N^{\frac{b\ln\frac{1-q}{q}}{\ln N}} = e^{b\ln\frac{1-q}{q}} \Rightarrow N^{\gamma_1}(1-p_N^*) < \gamma_2,
\end{aligned}$$

where $\gamma_1 = a\ln\frac{q}{1-q} > 0$ and $\gamma_2 = e^{b\ln\frac{1-q}{q}} > 0$. This completes the proof. □

As an extended discussion, the following corollary tells us that a large size of agent base not only implies a higher price but also ensures a certain probability of implementing good projects.

Corollary .4. *For a given ν and an arbitrary $y \in (0,1)$, as long as $N \geq (2\beta-1)\bar{k}(\nu) + 1 + \frac{2[\ln(1-y) - \ln M]}{\ln x}$, $\exists\ p \geq \nu$ and an AoN target T such that $\Pr(A_N \geq T_N | p, T, V = 1) \geq y$.*

Proof. The result follows from the first inequality provided in Lemma A.4 for $p = \nu$. □

To ensure a good project is implemented with a probability higher than y, it is sufficient to have an agent base of $(2\beta-1)\bar{k}(\nu) + 1 + \frac{2[\ln(1-y) - \ln M]}{\ln x}$ or higher.

Profit and Pricing with Large Crowds

For any N_1 and its corresponding optimal price V_{k_1}, there exists a finite $\overline{N}$ such that for $\forall N \geq \overline{N}$, the corresponding optimal price $p > V_{k_1}$.

Proof. First, notice that given Lemma A.5, when the price is $p = V_{-1} = 1 - q$, the proposer's expected profit is $(1 - q - \nu)N$. Given a price $p = V_k$, $k \in \{0, 1, \ldots, N\}$, the proposer's expected profit is

$$\pi(V_k, N) = \begin{cases} (V_k - \nu)\left[\sum_i^N \varphi_{k+1,i}(N - \frac{i-k-1}{2}) + \frac{(1-q)^k q + (1-q)q^k}{(1-q)^{k+1} + q^{k+1}} \varphi_{k+1,N} \frac{N+k-1}{2}\right] & \text{if } k + N \text{ odd;} \\ (V_k - \nu)\left[\sum_i^{N-1} \varphi_{k+1,i}(N - \frac{i-k-1}{2}) + \frac{(1-q)^{k+1} + q^{k+1}}{(1-q)^{k+2} + q^{k+2}} \varphi_{k+1,N+1} \frac{N+k}{2}\right] & \text{if } k + N \text{ even.} \end{cases} \tag{30}$$

We then proceed to prove the following Lemma: Let $\bar{k}(\nu) \in \{0, 1, 2, \ldots\}$ be the smallest integer satisfying $V_{\bar{k}(\nu)} \geq \nu$. For each $k \in \{\bar{k}(\nu), \bar{k}(\nu) + 1, \bar{k}(\nu) + 2, \ldots\}$, there exists a finite positive integer $\underline{N}(k)$ such that for $\forall\ N \geq \underline{N}(k)$, $\pi(V_k, T^*(V_k), N) > \pi(V_{k-1}, T^*(V_{k-1}), N)$, where $T^*(V_k)$ is defined as $\lfloor \frac{N+k}{2} \rfloor$.

Proof. To show the existence of $\underline{N}(k)$, we first prove the existence of $\underline{N}(0)$, then proceed to the $k \geq 1$ case. From the standard Gambler's Ruin problem we know that as $N \to \infty$, for a given V_k, the conditional probability that an UP cascade occurs at a finite time is 1 if $V = 1$, and $\frac{(1-q)^{k+1}}{q^{k+1}}$ if $V = 0$ (Feller, 1968, Page 347 Eq. 2.8).

Note that $\pi(V_{-1}, N) = (1 - q - \nu)N$. Furthermore, for $p = V_0 = \frac{1}{2}$, an UP cascade starts if $\sum_{j=1}^i x_j = 1$ for some $1 \leq j < 2T$, where T is the AoN target. Because $(1 - q)q < \frac{1}{4}$, we have:

$$\begin{aligned} \lim_{N\to\infty} \frac{\pi(V_0, T_N^*, N)}{N} &= (V_0 - \nu)\left(Pr(V = 1) + Pr(V = 0)\frac{1-q}{q}\right) \\ &= \left(\frac{1}{2} - \nu\right)\left(\frac{1}{2} + \frac{1-q}{2q}\right) > \left(\frac{1}{2} - \nu\right) 2(1 - q) \\ &> 1 - q - \nu = V_{-1} - \nu, \end{aligned}$$

where $T_N^* = \lfloor \frac{N}{2} \rfloor$. Since $\varphi_{1,i}$ is strictly positive, there exists a strictly positive integer $N_1(0)$ such that:

$$(V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} > 1 - q - \nu$$

Let $D = (V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} - (1 - q - \nu) > 0$, $Q = (V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} \frac{i-1}{2}$, and $\underline{N}(0)$ be the smallest integer that is larger than $\max\{N_1(0), \frac{Q}{D}\}$. Then for any $N \geq \underline{N}(0)$:

$$\begin{aligned} \pi(V_0, T_N^*, N) &\geq (V_0 - \nu) \sum_{i=1}^{\underline{N}(0)} \varphi_{1,i}(N - \frac{i-1}{2}) \geq N(V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} - Q \\ &\geq \frac{Q}{D} D + (1 - q - \nu)N - Q = (1 - q - \nu)N. \end{aligned}$$

Now, we prove the existence of $\underline{N}(k)$ for $k > 0$. For each $k \geq 1$, and the time i arrival rate $\varphi_{k+1,i+1}$, there exists a corresponding $\varphi_{k,i}$ for price V_{k-1}. For each i, we have:

$$\frac{(V_k-\nu)\varphi_{k+1,i+1}}{(V_{k-1}-\nu)\varphi_{k,i}} \geq \frac{V_k\varphi_{k+1,i+1}}{V_{k-1}\varphi_{k,i}} = \frac{V_k \frac{k+1}{i+1} \frac{(i+1)!}{\frac{i+k+2}{2}!\frac{i-k}{2}!}[(1-q)q]^{\frac{i-k}{2}} \frac{(1-q)^{k+1}+q^{k+1}}{2}}{V_{k-1}\frac{k}{i}\frac{i!}{\frac{i+k}{2}!\frac{i-k}{2}!}[(1-q)q]^{\frac{i-k}{2}}\frac{(1-q)^k+q^k}{2}}$$
$$= q\frac{k+1}{k}\frac{i}{\frac{i+k}{2}+1}\left(1+\frac{[q(1-q)]^{k-1}(q-(1-q))^2}{((1-q)^k+q^k)^2}\right).$$

Since $\lim_{i\to\infty} q\frac{i}{\frac{i+k}{2}+1} = 2q > 1$, for each k, the ratio $\frac{V_k\varphi_{k+1,i+1}}{V_{k-1}\varphi_{k,i}}$ is monotonically increasing in i for large enough values of i, and consequently, there exists an integer N_1 that $\frac{V_k\varphi_{k+1,i+1}}{V_{k-1}\varphi_{k,i}} \geq 1$ whenever $i \geq N_1$. We then have

$$\begin{aligned}
&\lim_{N\to\infty}(V_k-\nu)\sum_{i=1}^{N-1}\varphi_{k+1,i+1} &&= (V_k-\nu)\left(\frac{1}{2}+\frac{\frac{(1-q)^{k+1}}{q^{k+1}}}{2}\right)\\
=\ &\frac{V_k-\nu}{V_k}\frac{1}{2}\frac{q^k}{q^k+(1-q)^k}\frac{(1-q)^{k+1}+q^{k+1}}{q^{k+1}} &&= \frac{V_k-\nu}{V_k}\frac{1}{2q}\frac{(1-q)^{k+1}+q^{k+1}}{(1-q)^k+q^k}\\
>\ &\frac{V_k-\nu}{V_k}\frac{1}{2q}\frac{(1-q)^k+q^k}{(1-q)^{k-1}+q^{k-1}} &&= \frac{V_k-\nu}{V_k}V_{k-1}\left(\frac{1}{2}+\frac{\frac{(1-q)^k}{q^k}}{2}\right)\\
\geq\ &(V_{k-1}-\nu)\left(\frac{1}{2}+\frac{\frac{(1-q)^k}{q^k}}{2}\right) &&= \lim_{N\to\infty}(V_{k-1}-\nu)\sum_{i=1}^{N}\varphi_{k,i},
\end{aligned}$$

where we have used Cauchy-Schwarz inequality to derive $(q^{k+1}+(1-q)^{k+1})(q^{k-1}+(1-q)^{k-1}) > (q^k+(1-q)^k)^2$.

Given that $\lim_{N\to\infty}\sum_{i=1}^{N}\varphi_{k+1,i+1}$ converges to a finite number, there exists an integer $N_2 \geq N_1$ such that:

$$D \equiv (V_k-\nu)\sum_{i=1}^{N_2-1}\varphi_{k+1,i+1} - (V_{k-1}-\nu)\sup_{N\geq N_2}\left\{\sum_{i=1}^{N}\varphi_{k,i} + \frac{q^{k-1}+(1-q)^{k-1}}{(1-q)^k+q^k}\varphi_{k,N+1}\right\} > 0,$$

where $\frac{q^{k-1}+(1-q)^{k-1}}{(1-q)^k+q^k}\varphi_{k,N+1}$ is the probability that there is no UP cascade and agent N is the Tth supporting agent given price V_{k-1}. Let $Q \equiv (V_k-\nu)\sum_{i=1}^{N_2-1}\varphi_{k+1,i+1}\frac{i-k}{2}$. Then for each k, let $\underline{N}(k)$ be the smallest integer that is larger than $\max\{N_2, \frac{Q}{D}\}$. Then for any $N \geq \underline{N}(k)$:

$$\pi\left(V_k, \left\lfloor\frac{N+k}{2}\right\rfloor, N\right) - \pi\left(V_{k-1}, \left\lfloor\frac{N+k-1}{2}\right\rfloor, N\right) > ND - Q > \underline{N}(k)D - Q \geq 0.$$

The proposition follows. □

Given Lemma A.5, for $\forall N_1$ and corresponding optimal price V_{k_1}, one can construct $\overline{N} \equiv$

$\max\{\underline{N}(\bar{k}(\nu)), \underline{N}(\bar{k}(\nu)+1), \underline{N}(\bar{k}(\nu)+2)\dots\underline{N}(k_1)\}$, such that for $\forall N \geq \overline{N}$, we have

$$\begin{aligned}\pi(V_{k_1+1}, T^*(V_{k_1+1}), N) > \pi(V_{k_1}, T^*(V_{k_1}), N) > \pi(V_{k_1-1}, T^*(V_{k_1-1}), N)\\ > \pi(V_{k_1-2}, T^*(V_{k_1-2}), N) > \cdots > \pi(V_{\bar{k}(\nu)}, T^*(V_{\bar{k}(\nu)}), N).\end{aligned} \tag{31}$$

That is to say, for $\forall N \geq \overline{N}$, the optimal price $V^* \geq V_{k_1+1} > V_{k_1}$. Moreover, this result implies that $\lim_{N\to\infty} k^*(N) = \infty$, suggesting that $\lim_{N\to\infty} p_N^* = 1$. □

A.6 Proof of Corollary 1

Proof. Denote the $(T^*-1)^{th}$ supporting agent by i, then we show that there is no DOWN cascade unless the following holds simultaneously:

1. $N + k^*$ is odd;
2. There is no UP cascade;
3. $N - 3 \leq i \leq N - 1$;
4. $x_j = -1, \forall \quad i < j \leq N - 1$.

First of all, if there is an UP cascade, then there would be no DOWN cascade. Second, if there are less than $T^* - 1$ supporting agents, then from Proposition 1 there would be no DOWN cascade. Third, if there is no UP cascade by the T^*th supporting agent i, then by the construction of $T^* = \lfloor \frac{N+\bar{k}(p^*)}{2} \rfloor$, $i = 2T^* - \bar{k}(p^*) \geq N - 1$, and there would be no DOWN cascade.

The only remaining case is when there is no UP cascade by the $T^* - 1$th supporting agent i, since there is no UP cascade yet then by the construction of T^*, $i \geq 2(T^* - 1) - \bar{k}(p^*) \geq N - 3$. To be specific:

1. If $N + \bar{k}(p^*)$ is even, then $i \geq N - 2$. When $i = N$, then then from Proposition 1 there would be no DOWN cascade. When $i \in \{N-2, N-1\}$, $k(\mathcal{H}_i) = \bar{k}(p^*) + N - 2 - i$, and there would be no DOWN cascade (a DOWN cascade starts after $k = \bar{k}(p^*) - 2$).
2. If $N + \bar{k}(p^*)$ is odd, then $i \geq N - 3$. When $i = N$, then then from Proposition 1 there would be no DOWN cascade. When $i \in \{N-3, N-2, N-1\}$, $k(\mathcal{H}_i) = \bar{k}(p^*) + N - 3 - i$, and there would a DOWN cascade after agent $N - 1$ if all agents $i < j \leq N - 1$ observes negative signals.

□

A.7 Proof of Proposition 3

Proof. According to Lemma 2, the proposer is never able to cover the cost when no threshold is set (equivalently, when $T = 1$). For the second part, it is enough to consider the case $p = V_N$

and $T = N$. With a positive probability, all agents receive a positive signal, in which case all projects with a cost not exceeding V_N are financed. Since the agents' posterior cannot exceed V_N after any history of actions, projects with $\nu > V_N$ cannot be financed through the threshold implementation. $\square$

A.8 Proof of Proposition 4

Proof. To prove the first part, i.e. $Pr(A_N \geq T|V = 1) \geq Pr(A_N \geq T|V = 0)$, we only need to show

$$\frac{Pr(V = 1|A_N \geq T)}{Pr(V = 0|A_N \geq T)} \geq 1 \qquad \forall p \in \mathbb{R}_{++}, T \in \{1, \ldots, N\} \tag{32}$$

since good projects and bad projects are equally likely. For $p \leq V_{-1}$, An UP cascade starts from the first agent, regardless of the sequence of the private signals and the AoN target is certainly reached. Hence, Inequality (32) holds for $p \leq V_{-1} = 1 - q$. For $p > V_{-1}$ and a sequence $x \in X$ for which the target is reached, let $s_T(x)$ be the T'th supporter. Therefore, the law of iterative expectations implies:

$$\frac{Pr(V = 1|A_N(x) \geq T)}{Pr(V = 1|A_N(x) \geq T) + Pr(V = 0|A_N(x) \geq T)} = \mathbb{E}[V|A_N(x) \geq T] = \mathbb{E}[V|\mathcal{H}_{s_T(x)}] \geq V_{\bar{k}(p)} \geq \frac{1}{2}$$

Inequality (32) clearly follows and the proof for the first part is complete. The remaining claims can be easily verified using (9).

For the second part of the statement, we prove it in several steps: **Step 1:** $\lim_{N\to\infty} \frac{1}{N}\pi(p_N^*, T_N^*, N) = \frac{1}{2}(1 - \nu)$

We first prove the following lemma concerning probability limits. For a fixed price $p \geq \frac{1}{2}$, if N and T_N go to infinity, then we have:

$$\lim_{N\to\infty} \mathcal{P}_N^I = 0 \qquad\qquad \lim_{N\to\infty} \mathcal{P}_N^{II} = \left(\frac{1-q}{q}\right)^{\bar{k}(p)+1} \tag{33}$$

Proof. Note that when $V = 1$, we can see that, according to the law of large numbers, the probability of not starting an UP cascade by the $T_N - 1$'th supporter goes to zero, because:

$$\lim_{T_N\to\infty} P\left(\sum_{i=1}^{s_{T_N-1}} x_i \leq \bar{k}(p)\right) = \lim_{T_N\to\infty} P\left(\frac{1}{s_{T_N-1}}\sum_{i=1}^{s_{T_N-1}} x_i \leq \frac{\bar{k}(p)}{s_{T_N-1}}\right)$$
$$\leq \lim_{T_N\to\infty} P\left(|\frac{1}{s_{T_N-1}}\sum_{i=1}^{s_{T_N-1}} x_i - \mathbb{E}[x_i|V=1]| \geq \mathbb{E}[x_i|V=1] - \frac{\bar{k}(p)}{s_{T_N-1}}\right) = 0$$

where $s_{T_N-1} = 2(T_N - 1) - (\bar{k}(p) + 1)$ is the agent after whom an UP cascade starts if she is the $T_N - 1$'th supporter and no UP cascade starts before her. Moreover, in the last inequality, we used that fact that $\mathbb{E}[x_i|V=1]=2q-1>0$, hence $\lim_{T_N\to\infty}\mathbb{E}[x_i|V=1]-\frac{\bar{k}(p)}{s_{T_N-1}}=2q-1>0$. This result shows that when $V=1$ the AoN target is reached with probability one as the target is set arbitrarily large, i.e. $\lim_{N\to\infty}\mathcal{P}_N^I=0$.

As a result, we should have:

$$\sum_{i=0}^{\infty}\varphi^1_{\bar{k}(p)+1,\bar{k}(p)+2i+1}=1\Rightarrow\sum_{i=0}^{\infty}\varphi^0_{\bar{k}(p)+1,\bar{k}(p)+2i+1}=\left(\frac{1-q}{q}\right)^{\bar{k}(p)+1} \tag{34}$$

Equation (34) shows the probability of reaching an UP cascade when $V=0$ and the target is set arbitrarily large. Since earlier we showed that as N and T_N go to infinity, the AoN target is reached by an UP cascade with probability one, then the expression in (34) gives the probability of type-two error for arbitrary large values of N, which proves the second equation in Lemma A.8.

$\square$

Given the expressions provide in Lemma A.8, we have:

$$\lim_{N\to\infty}\frac{1}{N}\pi(V_k,T_N^k,N)=\frac{1}{2}(V_k-\nu)(1+(\frac{1-q}{q})^{k+1})\Rightarrow\lim_{k\to\infty}\lim_{N\to\infty}\frac{1}{N}\pi(V_k,T_N^k,N)=\frac{1}{2}(1-\nu)$$

Suppose $T_N^k=\lfloor\frac{N+k}{2}\rfloor$. Clearly, $\pi(p_N^*,T_N^*,N)\geq\pi(V_k,T_N^k,N)$, for every k. Therefore

$$\lim_{k\to\infty,N\to\infty}\frac{1}{N}\pi(p_N^*,T_N^*,N)\geq\frac{1}{2}(1-\nu) \tag{35}$$

Note that in expectation, the investors at least break even. To be more concrete, we know $\mathbb{E}[V-p_N^*|A_N\geq T_N^*]\geq 0$ for the optimal choice of (p_N^*,T_N^*). Given the total surplus of efficient investment is $\frac{1}{2}(1-\nu)$. Then

$$\lim_{k\to\infty,N\to\infty}\frac{1}{N}\pi(p_N^*,T_N^*,N)\leq\frac{1}{2}(1-\nu) \tag{36}$$

which completes the proof.

Step 2: $\lim_{N\to\infty}\mathcal{P}_N^{II}=0$

Note that in expectation, the investors at least break even. To be more concrete, we know $\mathbb{E}[V-p_N^*|A_N\geq T_N^*]\geq 0$ for the optimal choice of (p_N^*,T_N^*). It implies for any N, the following relation for the probability of errors should hold:

$$\frac{1}{2}(1-\mathcal{P}_N^I)(1-p_N^*)-\frac{1}{2}\mathcal{P}_N^{II}p_N^*\geq 0 \tag{37}$$

We know $p_N^*\to 1$ as N goes to infinity. Therefore, (37) implies $\mathcal{P}_N^{II}\to 0$.

Step 3: $\lim_{N\to\infty}\mathcal{P}_N^I=0$

Now, we show $\lim_{N\to\infty}\mathcal{P}_N^I=0$. To see this, first note that the following inequality holds for every N

$$\frac{1}{N}\pi(p_N^*, T_N^*, N) < \frac{1}{2}(1 - \mathcal{P}_N^I + \mathcal{P}_N^{II})(p_N^* - \nu) \tag{38}$$

Step 1 shows that the left hand side in (38) goes to $\frac{1}{2}(1-\nu)$ as N becomes arbitrarily large. It implies:

$$\lim_{N\to\infty} 1 - \mathcal{P}_N^I + \mathcal{P}_N^{II} \geq 1 \Rightarrow \lim_{N\to\infty} \mathcal{P}_N^I \leq \lim_{N\to\infty} \mathcal{P}_N^{II} = 0 \Rightarrow \lim_{N\to\infty} \mathcal{P}_N^I = 0$$

□

A.9 Proof of Proposition 5

Proof. First, we prove $\mathbb{E}[V|A_N]$ is weakly increasing in $A_N < \lfloor\frac{N+\bar{k}(p)}{2}\rfloor$.

1. $A_N < T^* - 1$**:** According to (15), all agents act based on their private signals when $A_{i-1} < T - 1$ and $k_{i-1} \leq \bar{k}(p)$. Moreover, we know that if an UP cascade starts after some agent i in the queue, then the total support is at least $\frac{i+\bar{k}(p)+1}{2} + N - i > \lfloor\frac{N+\bar{k}(p)}{2}\rfloor \geq T$. Therefore, certainly no UP cascade starts in this case. These observations imply that there are exactly A_N positive signals in x, or in other words $\sum_{i=1}^{N} x_i = 2A_N - N$. Therefore, $\mathbb{E}[V|A_N] = V_{2A_N-N}$ for $A_N < T - 1$. It is clear that the function is strictly increasing in A_N.

2. $A_N = T^* - 1$**:** In this case, with an argument similar to the previous case, we can show that there are at least $T - 1$ positive signals. However, it is possible that a DOWN cascade starts after the $T-1$'th supporter, which implies the expected value stays at a value not exceeding $V_{\bar{k}(p)-2}$ for the agents in the cascade, including the last agent. Since $\bar{k}(p) - 2 \geq 2(T-1) - N$, the expected value in this case is strictly bounded below by $V_{\bar{k}(p)-2}$, which implies that the increase from $A_N = T^* - 2$ to $A_N = T^* - 1$ strictly improves the publicly perceived valuation.

By comparing the posteriors in the cases mentioned above, we can get that $\mathbb{E}[V|\mathcal{H}_N]$ is weakly increasing in A_N. Also the Case 1 indicates that $\mathbb{E}[V|\mathcal{H}_N, A_N < T^*]$ is strictly increasing in A_N. In fact, with exogenous AoN, a potential UP cascade in Case 1 simply means A_N weakly improves the perceived valuation. The statement that $\mathbb{E}[V|\mathcal{H}_N]$ is weakly increasing in A_N holds generally.

For the large crowd limit, note that Proposition 4 shows that

$$\lim_{N\to\infty} \Pr(A_N \geq T_N^\star | V = 0) = \lim_{N\to\infty} \Pr(A_N < T_N^\star | V = 1) = 0.$$

Bayes rule then implies that

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}[V|A_N \geq T_N^\star] &= \lim_{N\to\infty} \Pr(V = 1|A_N \geq T_N^\star) = 1, \\ \lim_{N\to\infty} \mathbb{E}[V|A_N < T_N^\star] &= \lim_{N\to\infty} \Pr(V = 1|A_N < T_N^\star) = 0. \end{aligned} \tag{39}$$

Now, we prove that $\lim_{N\to\infty} \Pr(|\mathbb{E}[V|\mathcal{H}_N]-V| > \varepsilon) = 0$, for $\varepsilon > 0$. Suppose the contrary. Then, $\exists B_N \in \sigma(\mathcal{H}_N)$ such that $\Pr(B_N) > \delta$ and $|\mathbb{E}[V|\mathcal{H}_N(\omega)] - V| > \varepsilon, \forall \omega \in B_N$, for some $\delta > 0$ and an arbitrary N. This implies that $\mathbb{E}[V|\mathcal{H}_N] \in (\varepsilon, 1-\varepsilon), \forall \mathcal{H}_N \in B_N$. This contradicts equation (39). To see it, denote $C = \{A_N \geq T_N^\star\}$. Note that $B_N \cap C \in \sigma(\mathcal{H}_N)$ and $B_N^c \cap C \in \sigma(\mathcal{H}_N)$, and thus $\mathbb{E}[V|B_N \cap C)]$ and $\mathbb{E}[V|B_N^c \cap C)]$ are well-defined. Moreover, we can assume that $\Pr(B_N \cap C) \geq \frac{\delta}{2}$; otherwise, $\Pr(B_N \cap C^c) \geq \frac{\delta}{2}$ and we can then apply the same reasoning to the set C^c:

$$
\begin{aligned}
\mathbb{E}[V|A_N \geq T_N^\star] &= \frac{\Pr(B_N \cap C)}{\Pr(B_N \cap C) + \Pr(B_N^c \cap C)} \mathbb{E}[V|B_N \cap C] \\
&+ \frac{\Pr(B_N^c \cap C)}{\Pr(B_N \cap C) + \Pr(B_N^c \cap C)} \mathbb{E}[V|B_N^c \cap C] \\
&\leq \frac{\delta}{2}(1-\varepsilon) + \left(1 - \frac{\delta}{2}\right) \times 1 = 1 - \frac{\delta\varepsilon}{2} < 1
\end{aligned}
$$

This contradicts equation (39) by taking the limit on both sides. Thus, we have $\mathbb{E}[V|\mathcal{H}_N] \xrightarrow{p} V$, which completes the proof.

$\square$

A.10 Proof of Proposition 6

Proof. For any given pair of (p, T), there exists an equilibrium with strategy profile a_i^{j*} and posteriors $P(V = 1|\mathcal{H}_{i-1}^i) = V_{k^*(\mathcal{H}_{i-1}^i)}$, where:

$$
a_i^{i*} = \begin{cases} \mathbb{1}_{\{x_i=1\}} & \textit{if } A_{i-1}^{i-1} < T-1,\ k_{i-1} \leq \bar{k}(p),\ i < N \\ 1 & \text{if } k_{i-1} > \bar{k}(p) \\ 0 & \text{if } A_{i-1}^{i-1} \geq T-1,\ k_{i-1} < \bar{k}(p) - 1 \\ \mathbb{1}_{\{x_t=1\}} & \text{if } A_{i-1}^{i-1} \geq T-1,\ k_{i-1} \in \{\bar{k}(p), \bar{k}(p)-1\} \end{cases} \tag{40}
$$

and for $j > i$,

$$
a_i^{j*} = \begin{cases} 2\mathbb{1}_{\{k_N \geq \bar{k}(p)\}} - 1 & \text{if } j = N \ \ \& \ \ a_i^{N-1} = 0 \\ a_i^{i*} & \textit{otherwise} \end{cases} \tag{41}
$$

$$
k_i^*(\mathcal{H}_{i-1}^i) = \begin{cases} k_{i-1} + (2a_i^i - 1) & \text{if } A_{i-1}^{i-1} < T-1,\ k_{i-1} \leq \bar{k}(p),\ i < N \\ k_{i-1} & \text{if } \ k_{i-1} > \bar{k}(p) \\ k_{i-1} & \text{if } A_{i-1}^{i-1} \geq T-1, \quad k_{i-1} < \bar{k}(p) - 1 \\ k_{i-1} + (2a_i^i - 1) & \text{if } A_{i-1}^{i-1} \geq T-1, \quad k_{i-1} \in \{\bar{k}(p), \bar{k}(p)-1\} \end{cases} \tag{42}
$$

To see these, suppose agent i observes $x_i = 1$, she has no incentive to deviate. If she chooses rejection or waiting, then all follow agents misinterpret her action and update their beliefs as if $x_i = -1$. This results in failures for some project that should be financed if i correctly reveals her information.

If agent i observes $x_i = -1$, as we discussed in the baseline model, if there is an UP cascade she chooses to invest. When there is no UP cascade yet, she has no incentive to invest, and waiting is a weakly dominating strategy since she can always reject latter. Thus her first action of waiting still reveals her information. $\square$

A.11 Proof of Proposition 7

Proof. When agents have options to wait, investors with negative signals invest if they are part of an UP cascade and the project would be implemented. From the proof for Proposition A.5, we have

$$\lim_{N\to\infty}(V_k - \nu)\sum_{i=1}^{N-1}\varphi_{k+1,i+1} \geq \lim_{N\to\infty}(V_{k-1} - \nu)\sum_{i=1}^{N}\varphi_{k,i}.$$

From the proof for Proposition A.5, p goes to 1 when N goes to infinity. Because $Pr(x_i = 1|V = 1) = q > 1 - q$, Feller (1968), page 347 equation 2.8 shows that the probability that an UP cascade takes place from some finite agent is 1 when $V = 1$, and $\frac{(1-q)^{\bar{k}(p)+1}}{q^{\bar{k}(p)+1}}$ when $V = 0$. So all good project would be implemented almost surely when N goes to infinity, and bad projects would be abandoned almost surely as p goes to 1 when N goes to infinity. $\square$

A.12 Proof of Proposition 8

Proof. We denote the equilibrium characterized in Proposition 1 with the same threshold T and price $p^* \equiv p + \epsilon$ as $\mathbb{G}$. Define the expected profit of agent i with posterior $V_{k(\mathcal{H}_{i-1})+x_i}$ (conditional on her private signal as well) in equilibrium G as $\chi(p, i, T, \mathcal{H}_{i-1}, x_i, a_i)$. Define $\bar{\epsilon}(p, T)$ as

$$\bar{\epsilon}(p, T) = \min\{\min\{\chi(p, i, T, \mathcal{H}_{i-1}, x_i, a_i = 1))\}_{\{\chi>0\}}, V_{k+1} - p\}. \tag{43}$$

Now we verify that $\mathbb{G}$ still holds. If there is already an UP cascade as in $\mathbb{G}$, given other agents' strategies, the project would be implement for sure if agent i supports, then by construction agent i's expected profit (conditional on her private signal) would be at least

$$V_{k+1} - p \geq \epsilon. \tag{44}$$

So agent i will support regardless of her private signal.

Consider the case when there is no UP cascade yet and agent i observes a negative signal. Same to the discussion in the proof of Proposition 8, agent i's expected profit would be negative if she chooses support, and she finds it optimal to reject the project.

If there is no UP cascade yet and agent i's private signal is positive, then by the construction of $\bar{\epsilon}(p, T)$, agent i would choose support. $\square$

A.13 Discussion of Information Acquisition Costs

Given the agent base N and threshold T, for any $p \in (V_{k-1}, V_k]$, there exists a bound $\overline{\varepsilon}(p, T) > 0$ such that for $\forall \varepsilon \in (0, \overline{\varepsilon}(p, T))$, the equilibrium is as characterized in Proposition 1 with the same threshold T and price p.

Note that if $p = V_k$, the project is implemented if and only if there is an UP cascade, which is the case if and only if under the same sequence of signals, there would be an UP cascade in the equilibrium characterized in Proposition 1 with the same threshold T and price p. In both cases, agents learn if and only if there is no UP cascade yet and it is still possible to reach the threshold.

Proof. We denote the equilibrium characterized in Proposition 1 with agent base N, threshold T, and price p as $\mathbb{G}_1$. Define the expected investment profit of agent i with posterior $V_{k(\mathcal{H}_{i-1})}$ in equilibrium G conditional on a positive and negative signal as $\chi_1(p, i, T, \mathcal{H}_{i-1})$ and $\chi_0(p, i, T, \mathcal{H}_{i-1})$, respectively. Let $\mathbb{K}$ be the set of agent i and associated history $\mathcal{H}_{i-1}$ such that:

$$\mathbb{K} \equiv \{i, \mathcal{H}_{i-1} | \chi_1(p, i, T, \mathcal{H}_{i-1}) > 0, \chi_0(p, i, T, \mathcal{H}_{i-1}) < 0\}. \tag{45}$$

Define $\overline{\varepsilon}(p, T)$ as

$$\overline{\varepsilon}(p, T) = \min \left\{ V_{k(\mathcal{H}_{i-1})} \chi_1(p, i, T, \mathcal{H}_{i-1}), -(1 - V_{k(\mathcal{H}_{i-1})}) \chi_0(p, i, T, \mathcal{H}_{i-1}) \right\}_{\mathbb{K}}. \tag{46}$$

Now we verify that when $p \in (V_{k-1}, V_k)$, then $\mathbb{G}$ still holds.

1. If there is an UP cascade as in $\mathbb{G}$, then agent i finds it optimal to contribute regardless of her potential private signal. Agent i chooses not to produce information and contribute.

2. If there is no UP cascade yet, and in the equilibrium $\mathbb{G}$ agent i contribute if she observes a good signal, then $\chi_1(p, i, T, \mathcal{H}_{i-1}) > 0$. If agent i chooses not to acquire the private signal and contribute anyway, then her expected profit is

$$V_{k(\mathcal{H}_{i-1})} \chi_1 - \varepsilon > V_{k(\mathcal{H}_{i-1})} \chi_1(p, i, T, \mathcal{H}_{i-1}) + (1 - V_{k(\mathcal{H}_{i-1})}) \chi_0(p, i, T, \mathcal{H}_{i-1}). \tag{47}$$

   Because by construction $V_{k(\mathcal{H}_{i-1})} \chi_1 - \varepsilon > 0$, agent i finds it optimal to acquire the information and contribute if and only if she observes a positive signal.

3. If there is no UP cascade yet, and in the equilibrium G agent i finds it impossible to reach the threshold and chooses not to contribute, then $\chi_1(p, i, T, \mathcal{H}_{i-1}) = 0$. Agent chooses not to acquire information and not to contribute.

Finally we verify that case for $p = V_k$. The only difference is that if there is no UP cascade yet, and it is impossible to reach an UP cascade in the future, then $\chi_1(p, i, T, \mathcal{H}_{i-1}) = 0$, and agent chooses not to acquire information and not to contribute. $\square$

A.14 Proof for Lemma 3

Proof. Any equilibrium involves a sub-game equilibrium following the proposer's decision on p and T. We only need to show that any sub-game equilibrium is either the informative one characterized in Proposition 1 or one involves a group of free-riders whose actions before a cascade are ignored.

For any agent observing a high signal, it is her dominating strategy to contribute when there is a positive probability to reach the AoN threshold (and her action would be irrelevant if the project would not be implemented for sure). For an agent observing a low signal, if she knows that in the equilibrium the subsequent agents update their beliefs based on her action, then she always rejects as discussed in the proof for Proposition 1. However, if she knows her supporting the proposal does not positively update the subsequent agents' posteriors, then for any rational off-equilibrium belief (that is to say, if subsequent agent observes a rejection instead of a support action, they do not positively update their beliefs), supporting becomes a dominant action, since it allows her to free-ride on the gate-keeper's decision. Hence, she would support the proposal.

Having proven that a PBNE is either an informer or free-rider equilibrium, we next show that if $p \in \{V_k, K = -1, 0, \dots N\}$, then in any free-rider sub-game equilibrium, the project would be implemented only if there is an UP cascade. Suppose agent i is a free-rider (given the history $\mathcal{H}_{i-1}$) in a free-rider sub-game equilibrium, then $a_i = 1$. Otherwise, suppose the contrary that there exists a free-rider sub-game equilibrium in which agent i always rejects the project, then the project would only be implemented when the public posterior after the Tth supporting agent is at least p. However, if agent i's private signal is $x_i = 1$, then her conditional posterior is strictly higher than p, suggesting that she has incentive to deviate.

If agent i always supports the project, then she must have no incentive to deviate when her private signal is $x_i = -1$. Given $x_i = -1$, if the public posterior after the Tth supporting agent is $V_{\bar{k}(p)+1}$, then agent i's conditional posterior is $V_{\bar{k}(p)} = p$. If the public posterior after the Tth supporting agent is $V_{\bar{k}(p)}$, then agent i's conditional posterior is $V_{\bar{k}(p)-1} < p$. For agent i, the following inequality must hold for her individual rationality of investing:

$$\varphi(V_{\bar{k}(p)} - p) + Q(V_{\bar{k}(p)-1} - p) = Q(V_{\bar{k}(p)-1} - p) \geq 0, \tag{48}$$

where φ is the probability that the public posterior after the Tth supporting agent is $V_{\bar{k}(p)+1}$ and Q is the probability that the public posterior after the Tth supporting agent is $V_{\bar{k}(p)} = p$, conditional on the history $\mathcal{H}_{i-1}$ and agent i's private observation $x_i = -1$. In other words, conditional on $x_i = -1$, agent i breaks even when the public posterior after the Tth supporting agent is $V_{\bar{k}(p)+1}$

and loses money when the public posterior after the Tth supporting agent is $V_{\bar{k}(p)}$. For the agent to be rationally free riding by always supporting, it must be $Q = 0$, that is, the project is implemented with an UP cascade.

We next show that for every investor, the informer sub-game equilibrium weakly Pareto dominates all free-rider sub-game equilibria. In a free-rider sub-game equilibrium, for each realization path $x \in X$ that results in the project implementation, let h_T be the Tth supporting agent. Consider the a corresponding $x'(x)$: if there exists a sequence of free-riders $\{j_1, j_2, \dots\}$, then move the sequence of signals $\{x_{j_1}, x_{j_2}, \dots\}$ to the right of signal x_{h_T}. For each free-rider sub-game equilibrium, $x'(x)$ is an injective function that maps each sequence x to a distinct sequence $x'(x)$ such that $x'(x)$ is a realization path that results in an Up cascade (and thus project implementation) in the informer sub-game equilibrium. Now consider agent i in the free-rider sub-game equilibrium, if she observes $x_i = -1$, she breaks even if she is a free-rider, and also receives 0 when she is either an informer or in an UP cascade. Now suppose agent i observes $x_i = 1$, for each realization path x that results in the project implementation, she always chooses the same action and receives the same payoff on the corresponding realization path $x'(x)$ in the informer sub-game equilibrium. Since x and $x'(x)$ are equally likely to happen, agent i is weakly better off in the informer sub-game equilibrium.

Finally, suppose a free-rider sub-game equilibrium involves at least two free-riders. Then with positive probability that there exists a sequence x with at least two free-riders all observing positive private signals and the project is implemented. Similar to the discussion above, on the realization path x, right after the $(T-1)$th supporting agent the public posterior is $V_{\bar{k}(p)}$. Also notice that the $(T-1)$th supporting agent must be the $(N-1)$th agent, for otherwise $Q > 0$. In the free-rider sub-game equilibrium, consider the following realization path $\hat{x}(x)$: the first $N-1$ signals are the same as the ones in sequence x, while $x_N = -1$. Given the sequence $\hat{x}(x)$, the project would not be implemented in the free-rider sub-game equilibrium, but would be implemented in the informer sub-game equilibrium. Moreover, given at least two more positive signals, all investing agents receive positive expected profit on the sequence $\hat{x}(x)$. Therefore, this free-rider equilibrium is strictly dominated by the informer equilibrium. The proposition ensues. $\square$

A.15 Proof of Proposition 9 and Discussion on Numerical Procedure

First, we take (p, T), whether exogenous or endogenously determined, as given, to derive the subgame equilibrium. We then prove the existence of the optimal design and provide a numerical procedure for equilibrium seraching, when the design of (p, T) is endogenous.

Proof. We first define the concept of *no early under-support* and use it as an equilibrium refinement to ensure tractibility.

Definition 5. *NEU: An equilibrium satisfies the no-early under-support (NEU) property if an agent i always choose high support (i.e., $a_i = H$) whenever $T - A_{i-1} \geq H + (\bar{k}(p) - k_{i-1} - 1)L$ & $k_{i-1} \leq \bar{k}(p) - 1$, excluding the following two cases that $k_{i-1} = \bar{k}(p) - 1$ and that $k_{i-1} = \bar{k}(p) - 2$ & $T - A_{i-1} = H + L$. Equivalently, $T - A_{i-1} > H + L$.*

Intuitively speaking, the definition says that an agent should fully support whenever it is still possible to achieve threshold implementation.[29] If agent i chooses full support H, the belief regarding project quality goes up from k_{i-1} to $k_{i-1}+1$. Now, there is still enough room for learning since the funding gap $T - A_{i-1} - H$ is big enough if we face a sequence of $\bar{k}(p) - k_{i-1} - 1$ good signals so as to push the belief to break even without triggering threshold implementation. Moreover, NEU can be achieved by specifications on the agents' off-equilibrium actions. If an agent does not act accordingly and switch to partial support L too early, all subsequent agents coordinate on punishment by unanimously rejecting the project. One can embed the coordinated punishment as part of the equilibrium to justify NEU.

The next lemma states an incomplete subgame equilibrium characterization: For any given pair of (p, T), there exists an equilibrium with strategies $a_i^*(x_i, \theta_i, \mathcal{H}_{i-1}) \equiv a^*(x_i, \theta_i, k(\mathcal{H}_{i-1}), A_{i-1}(\mathcal{H}_{i-1}))$ and posteriors $P(V = 1|\mathcal{H}_i) = V_{k^*(\mathcal{H}_i)}$.

i) For all on equilibrium history (satisfying NEU),

$$a_i^*(x_i, H, k_{i-1}, A_{i-1}) = \begin{cases} \mathbb{1}_{\{x_i=1\}} * H & \text{if} \quad A_{i-1} < T - H - L \\ \mathbb{1}_{\{x_i=1\}} * L & \text{if} \quad A_{i-1} > T - H - (\bar{k}(p) - k_{i-1} - 1)L \ \& \ k_{i-1} \leq \bar{k}(p) - 2 \\ H & \text{if} \quad k_{i-1} > \bar{k}(p) \\ 0 & \text{if} \quad A_{i-1} \geq T - L \ \& \ k_{i-1} < \bar{k}(p) - 1 \\ \mathbb{1}_{\{x_i=1\}} * H & \text{if} \quad A_{i-1} \geq T - L \ \& \ k_{i-1} \in \{\bar{k}(p), \bar{k}(p) - 1\} \end{cases} \tag{49}$$

$$a_i^*(x_i, L, k_{i-1}, A_{i-1}) = \begin{cases} \mathbb{1}_{\{x_i=1\}} * L & \text{if} \quad A_{i-1} < T - L \ \& \ k_{i-1} \leq \bar{k}(p) \\ L & \text{if} \quad k_{i-1} > \bar{k}(p) \\ 0 & \text{if} \quad A_{i-1} \geq T - L \ \& \ k_{i-1} < \bar{k}(p) - 1 \\ \mathbb{1}_{\{x_i=1\}} * L & \text{if} \quad A_{i-1} \geq T - L \ \& \ k_{i-1} \in \{\bar{k}(p), \bar{k}(p) - 1\} \end{cases} \tag{50}$$

[29]In Definition 5, we exclude two special cases and more discussions about this is provided after lemma A.15.

$$k_i^*(\mathcal{H}_i) = \begin{cases} k_{i-1} + \mathbb{1}_{\{a_i \neq 0\}} - \mathbb{1}_{\{a_i = 0\}} & \text{if} \quad A_{i-1} < T - L \ \ \& \ \ k_{i-1} \leq \bar{k}(p) - 1 \\ k_{i-1} & \text{if} \quad k_{i-1} > \bar{k}(p) \\ k_{i-1} & \text{if} \quad A_{i-1} \geq T - L \ \ \& \ \ k_{i-1} < \bar{k}(p) - 1 \\ k_{i-1} + \mathbb{1}_{\{a_i \neq 0\}} - \mathbb{1}_{\{a_i = 0\}} & \text{if} \quad A_{i-1} \geq T - L \ \ \& \ \ k_{i-1} \in \{\bar{k}(p), \bar{k}(p) - 1\} \end{cases} \tag{51}$$

where $k_0 = 0$ and $A_0 = 0$.

ii) For all off-equilibrium path histories whenever NEU is violated,

$a_j = 0$ and $k_j^* = k_i^*$, for all $j \in \{i+1, \cdots, N\}$.

Proof. The proof needs to check the incentives for two types of agents for both the on-equilibrium path histories and off-equilibrium path histories.

- *On-equilibrium path histories.*
  - Type L's incentives.

    Note that for all on equilibrium histories, the strategy and belief for type $\{\theta_i = L\}$ are essentially the same as the exogenous case in the benchmark model.
  - Type H's incentives.

    First, note that line 3-5 for the strategy specification in Eq. (49) are almost identical to the exogenous case in the benchmark model and thus omitted.

    Second, in light of NEU, there are two types of on equilibrium path histories. i) *histories with big funding gap (relative to the belief gap to break even)*. Formally, this is defined as that in Definition 5, that is, $T - A_{i-1} \geq H + (\bar{k}(p) - k_{i-1} - 1)L$. Now we check that agent i of type $\{\theta_i = H\}$ has an incentive to choose high support H. If she rejects, the payoff is trivially zero. If she chooses low support, all subsequent agents switches to rejection and the project will never be implemented. If she chooses high support H, the payoff is strictly positive since there is enough room to push the belief to break even and the sequence of $\bar{k}(p) - k_{i-1} - 1$ successive positive signals suffices for this purpose.

    ii) *histories with big belief gap (relative to funding gap to break even).* In other words, $T - A_{i-1} < H + (\bar{k}(p) - k_{i-1} - 1)L$. The payoff is zero if agent i rejects or choose high support H. For the latter case, the belief cannot be pushed above to break even without triggering threshold implementation. This is not possible under the most favorable sequence of signals where all subsequent signals are positive. In contrast, the funding gap may be still big enough to allow for sufficient learning under low support so that more rounds of trials can be utilized.

- *Off-equilibrium path histories.* We only discuss the off-equilibrium path histories when property 1 is violated, that is, when the agents switch to low support L whereas they are supposed to use full support H. This only happens when agent i is rich and observes a good signal $x_i = 1$. The arguments here only help us pick up an equilibrium satisfying property 1, but it does not negate the existence of other informative equilibrium.

  Recall that, $A_{i-1} < T - H - L$. Now, after observing $a_i = L$ (agent i should have used H), $A_i < T - H$. Let us consider what will agent $i+1$ do in the scenario. The best belief is given by $k_i = \bar{k}(p) - 1$. Now, given all subsequent agents chooses to rejects, agent $i+1$ cannot not implement the project even if the signal $x_{i+1} = 1$, since $A_{i+1} \leq A_i + H < T$. This implies that $a_{i+1} = 0$ is incentive-compatible for agent $i+1$. The same reasoning applies to other agents.

The proof for lemma A.15 concludes. □

Before we proceed, we note that both the rich and the poor reject when $x_i = -1$, and that the equilibrium strategy for $\theta_i = L$ has already been specified. Hence, the following analysis only concern the four cases above when $\theta_i = H$ and $x_i = 1$. Recall that $V_k = \frac{q^k}{q^k+(1-q)^k}$ and we use the following notation

$$f(x) = q \times x + (1 - q) \times (1 - x).$$

$$a \wedge b = \min\{a, b\}, \text{ and } a \vee b = \max\{a, b\}$$

For ease of reference, we define

$$
\begin{aligned}
p_{11}^* &= \frac{HV_{\bar{k}} - f(V_{\bar{k}})V_{\bar{k}+1}L}{H - f(V_{\bar{k}})L} \\
p_{12}^* &= \frac{HV_{\bar{k}} - f(V_{\bar{k}})V_{\bar{k}+1}L - (1-f(V_{\bar{k}}))f(V_{\bar{k}-1})V_{\bar{k}}L}{H - L[f(V_{\bar{k}}) + (1-f(V_{\bar{k}}))f(V_{\bar{k}-1})]} \\
p_{21}^* &= \frac{V_1}{V_2}, \quad p_{22}^* = \frac{V_3}{V_4}, \quad p_{23}^* = \frac{W_1}{W_2}, \quad p_{24}^* = \frac{W_3}{W_4} \quad \text{and} \quad p_{34}^* = \frac{Z_1}{Z_2} \\
V_1 &= Hf(V_{\bar{k}})V_{\bar{k}+1} + H(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})V_{\bar{k}} - Lf(V_{\bar{k}})V_{\bar{k}+1} \\
&- L(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})f(V_{\bar{k}})V_{\bar{k}+1} \\
V_2 &= Hf(V_{\bar{k}}) + H(1-f(V_{\bar{k}}))f(V_{\bar{k}-1}) - Lf(V_{\bar{k}}) - L(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})f(V_{\bar{k}}) \\
V_3 &= Hf(V_{\bar{k}})V_{\bar{k}+1} + H(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})V_{\bar{k}} - Lf(V_{\bar{k}})V_{\bar{k}+1} \\
&- L(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})\Big(\lambda V_{\bar{k}} + (1-\lambda)f(V_{\bar{k}})V_{\bar{k}+1}\Big) \\
V_4 &= Hf(V_{\bar{k}}) + H(1-f(V_{\bar{k}}))f(V_{\bar{k}-1}) - Lf(V_{\bar{k}}) \\
W_1 &= V_{\bar{k}+1}f(V_{\bar{k}})\Big(H - L - L(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})\Big) \\
&+ V_{\bar{k}}(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})\Big(H - Lf(V_{\bar{k}-1})(1-f(V_{\bar{k}})) - (1-f(V_{\bar{k}-1}))f(V_{\bar{k}-2})\Big) \\
W_2 &= f(V_{\bar{k}})\Big(H - L - L(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})\Big) \\
&+ (1-f(V_{\bar{k}}))f(V_{\bar{k}-1})\Big(H - Lf(V_{\bar{k}-1})(1-f(V_{\bar{k}})) - (1-f(V_{\bar{k}-1}))f(V_{\bar{k}-2})\Big) \\
W_3 &= V_{\bar{k}+1}f(V_{\bar{k}})\Big(H - L - L(1-f(V_{\bar{k}}))(1-\lambda)\Big) + V_{\bar{k}}(1-f(V_{\bar{k}}))f(V_{\bar{k}-1}) \\
&\times \Big(H - L\big[\lambda + (1-\lambda)(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})\big] - L(1-f(V_{\bar{k}-1}))f(V_{\bar{k}-2})\Big) \\
W_4 &= f(V_{\bar{k}})\Big(H - L - L(1-f(V_{\bar{k}}))(1-\lambda)\Big) + (1-f(V_{\bar{k}}))f(V_{\bar{k}-1}) \\
&\times \Big(H - L\big[\lambda + (1-\lambda)(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})\big] - L(1-f(V_{\bar{k}-1}))f(V_{\bar{k}-2})\Big) \\
Z_1 &= V_{\bar{k}}\Big(H - [1-f(V_{\bar{k}})]f(V_{\bar{k}-1})L - [1-f(V_{\bar{k}-1})]f(V_{\bar{k}-2})L\Big) - LV_{\bar{k}+1}f(V_{\bar{k}}) \\
Z_2 &= H - [1-f(V_{\bar{k}})]f(V_{\bar{k}-1})L - [1-f(V_{\bar{k}-1})]f(V_{\bar{k}-2})L - Lf(V_{\bar{k}})
\end{aligned}
$$

We impose the following restriction to simplify analysis.

Assumption 2. *(Small income gap):* $L < H < 2L$.[30]

Lemma A.15 is not complete since there exist four cases unspecified, including: 1) $k_{i-1} = \bar{k}(p) - 1$ & $A_{i-1} = T - H$; 2) $k_{i-1} = \bar{k}(p) - 1, A_{i-1} \in [T - H - L, T - H)$; 3) $k_{i-1} = \bar{k}(p) - 2$, $A_{i-1} = T - H - L$; and 4) $k_{i-1} = \bar{k}(p) - 1$ & $A_{i-1} > T - H$. Note that the first three cases corresponds to the two cases excluded in Definition 5, while the fourth case corresponds to the missing case in line 2 in (49). These four cases have an important common feature, that is, we need

[30]The assumption here is imposed to ease presentation. Note that the coordinated punishment in the proof works as long as $A_{i-1} < T - H - L$ for any arbitrary $H > L > 0$. However, when $H \geq 2L$, there will be more rents related to the incentive compatibility conditions.

to establish incentive compatibility when agent i is rich (i.e., $\theta_i = H$). An important ramification of this feature is that, the rich type agent is endogenously endowed with bargaining power since if the rent is too low, she can always switch to low support to accelerate learning, which can generate strictly positive payoff under a sufficiently optimistic signal sequence. On the one hand, this problem arises because coordinated punishment device is no longer applicable. On the other hand, when the belief is more optimistic and/or the funding gap is bigger, a rich agent enjoys more bargaining power and thus a higher rent generated by incentive compatibility conditions.

It is straightforward to see that whenever $x_i = -1$, $a_i^* = 0$ for $\theta_i \in \{H, L\}$. When $x_i = 1$, type $\theta_i = L$ behaves according to Proposition A.15, but type $\theta_i = H$, under the four cases mentioned above behaves as follows.

- Case 1: $k_{i-1} = \bar{k}(p) - 1$ & $A_{i-1} = T - H$ (and $x_i = 1$ & $\theta_i = H$).

$$a_i^*(x_i, \theta_i, k_{i-1}, A_{i-1}) = \begin{cases} H, & N = i \\ H * \mathbb{1}(p \le p_{11}^*) + L * \mathbb{1}(p > p_{11}^*), & N = i+1 \\ H * \mathbb{1}(p \le p_{12}^*) + L * \mathbb{1}(p > p_{12}^*), & N \ge i+2 \end{cases} \tag{52}$$

- Case 2: $k_{i-1} = \bar{k}(p) - 1, A_{i-1} \in [T - H - L, T - H)$ (and $x_i = 1$ & $\theta_i = H$).

  1. $k_{i-1} = \bar{k}(p) - 1, A_{i-1} \in [T - 2L, T - H)$

$$a_i^*(x_i, \theta_i, k_{i-1}, A_{i-1}) \in \{0, L, H\}$$

  2. $k_{i-1} = \bar{k}(p) - 1, A_{i-1} \in [T - H - L, T - 2L)$

$$a_i^*(x_i, \theta_i, k_{i-1}, A_{i-1}) = \begin{cases} \in \{0, L, H\}, & N = i \\ H, & N \in \{i+1, i+2\} \\ H * \mathbb{1}\big(p \le p_{11}^* \wedge p_{22}^*\big) & +L * \mathbb{1}\big(p \in (p_{11}^* \wedge p_{22}^*, p_{11}^*]\big) \\ +H * \mathbb{1}\big(p \in (p_{11}^*, p_{11}^* \vee p_{21}^*]\big) & +L * \mathbb{1}\big(p > p_{11}^* \vee p_{21}^*\big), \\ & N = i+3 \\ H * \mathbb{1}\big(p \le p_{12}^* \wedge p_{24}^*\big) & +L * \mathbb{1}\big(p \in (p_{12}^* \wedge p_{24}^*, p_{12}^*]\big) \\ +H * \mathbb{1}\big(p \in (p_{12}^*, p_{12}^* \vee p_{23}^*]\big) & +L * \mathbb{1}\big(p > p_{12}^* \vee p_{23}^*\big), \\ & N \ge i+4 \end{cases} \tag{53}$$

- Case 3: $k_{i-1} = \bar{k}(p) - 2$, $A_{i-1} = T - H - L$ (and $x_i = 1$ & $\theta_i = H$).

$$a_i^*(x_i, \theta_i, k_{i-1}, A_{i-1}) = \begin{cases} \in \{0, L, H\}, & N = i \\ H, & N = i+1 \\ H * \mathbb{1}(p \leq p_{11}^*) + L * \mathbb{1}(p > p_{11}^*), & N = i+2 \\ H * \mathbb{1}(p \leq p_{34}^*) + L * \mathbb{1}(p > p_{34}^*), & N \geq i+3 \end{cases} \tag{54}$$

- Case 4: $k_{i-1} = \bar{k}(p) - 1$ & $A_{i-1} > T - H$ (and $x_i = 1$ & $\theta_i = H$).

$$a_i^*(x_i, \theta_i, k_{i-1}, A_{i-1}) = \begin{cases} H & A_{i-1} \geq T - L \\ H & A_{i-1} \in (T-H, T-L) \text{ \& } N = i \\ H * \mathbb{1}(p \leq p_{11}^*) & +L * \mathbb{1}(p > p_{11}^*) \\ & A_{i-1} \in (T-H, T-L) \text{ \& } N = i+1 \\ H * \mathbb{1}(p \leq p_{12}^*) & +L * \mathbb{1}(p > p_{12}^*) \\ & A_{i-1} \in (T-H, T-L) \text{ \& } N \geq i+2 \end{cases} \tag{55}$$

We prove it case by case.

Case 1: $k_{i-1} = \bar{k}(p) - 1$ & $A_{i-1} = T - H$ (and $x_i = 1$).

Case 1.1: $N = i$. Now, agent i is the last one, and she will choose full support H, and gets a payoff $H(V_{\bar{k}} - p)$.

Case 1.2: $N = i + 1$. Now, agent i is the second to last agent. By choosing high support H, she gets $H(V_{\bar{k}} - p)$. If agent i chooses low support L, with probability $f(V_{\bar{k}})$, $x_{i+1} = 1$ and it leads to UP cascade; with complementary probability $1 - f(V_{\bar{k}})$, $x_{i+1} = x_N = -1$ and the project will not be implemented since it does not break even. Thus,

$$\Big(f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}})) \times 0\Big) \times L$$

or equivalently, agent i chooses the high support H if and only if

$$p \leq \frac{HV_{\bar{k}} - f(V_{\bar{k}})V_{\bar{k}+1}L}{H - f(V_{\bar{k}})L} \equiv p_{11}^* \tag{56}$$

Case 1.3: $N \geq i + 2$.

- By choosing rejection, the payoff is 0.
- By choosing H, the payoff is given by $H(V_{\bar{k}} - p)$.

- By choosing L, $A_i = T - H + L$ and $k_i = \bar{k}$. The threshold implementation is not triggered, and it depends on the signals x_{i+1} and x_{i+2}.
  - $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$, since $V_{\bar{k}}$ is the new prior of the project being good at period $i+1$. Now, $k_{i+1} = \bar{k}+1$, and UP cascade happens and $A_{i+1} \geq T - H + 2L \geq T$, and thus the project is implemented. Correspondingly, the payoff is $(V_{\bar{k}+1} - p)$.
  - $x_{i+1} = -1$ occurs with complementary probability $1 - f(V_{\bar{k}})$. Thus, $k_{i+1} = \bar{k} - 1$, $A_{i+1} = T - H + L$.
    - $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, independent of being rich or poor, agent $i+2$ will support, and thus $k_{i+2} = \bar{k}$ and $A_{i+2} \geq T - H + 2L \geq T$. Correspondingly, the return is $V(\bar{k}) - p$ per dollar invested.
    - $x_{i+2} = -1$, which implies that $k_{i+2} = \bar{k} - 2$ and recall that $A_{i+1} = T - H + L$. These two conditions imply DOWN cascade and the return is 0.

To summarize, partial support L yields a payoff

$$\big(f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}})f(V_{\bar{k}-1})(V_{\bar{k}} - p)\big) \times L$$

Hence, agent i of the rich type $\theta_i = H$ will switch to L whenever

$$\Big(f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}})\{f(V_{\bar{k}-1})(V_{\bar{k}} - p) + (1 - f(V_{\bar{k}-1})) * 0\}\Big) \times L \geq H(V_{\bar{k}} - p) \quad (57)$$

Equation (57) highlights the endogenous bargaining power by the rich agent. It is now impossible for the proposer to subtract all surplus as that in the benchmark model since the rich type agent can switch to low support L when the coordinated punishment is not available, which may generate UP cascade and the agent can receive a positive rent. For instance, if the price $p \to V_{\bar{k}}$, then agent i will always switch to low support L, which guarantees a rent no less than $f(V_{\bar{k}})(V_{\bar{k}+1} - p) \times L$. This, in turn, implies that the full surplus extraction is not plausible.

We can simplify equation (57) to find out that agent i of type $\theta_i = H$ will choose H only when

$$p \leq \frac{HV_{\bar{k}} - f(V_{\bar{k}})V_{\bar{k}+1}L - (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})V_{\bar{k}}L}{H - L[f(V_{\bar{k}}) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})]} \equiv p^*_{12} \quad (58)$$

It is easy to check that $p^*_{12} < V_{\bar{k}}$, which negates the possibility of full surplus extraction.

Now, (52) summarizes the discussion above, which concludes the analysis for case 1. ■

Case 2: $k_{i-1} = \bar{k}(p) - 1$ **and** $A_{i-1} \in [T - H - L, T - H)$ We divide it into two cases.

Case 2.1: $k_{i-1} = \bar{k}(p) - 1$**,** $T - 2L \leq A_{i-1} < T - H$ **(and** $x_i = 1$**).**

This case is very straightforward, and it is optimal for agent i of type $\theta_i = H$ to use full support H. Given that $H \in (L, 2L)$, Agent i's type is not important anymore, since either $2L$, $H + L$ or $2H$ lead to the same future threshold implementation. Hence, it is always optimal to use H as long as the return is non-negative.

Case 2.1.1: $N = i$**.** It is impossible to get enough funds to achieve threshold implementation, and agent i can choose any action from $\{0, L, H\}$.

Case 2.1.2: $N = i + 1$**.**

- By choosing rejection, the payoff is 0. Thus, $U_i(0) = 0$.
- By choosing H, then $A_i \in [T - 2L + H, T)$ and $k_i = \bar{k}$. Since the threshold is not reached, we need to check the signal x_{i+1}.
  1. x_{i+1} occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+1} = \bar{k} + 1$ and $A_{i+1} = A_i + a_{i+1} \geq T$. The return per dollar invested is $V_{\bar{k}+1} - p$.
  2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$, which implies that $k_{i+1} = \bar{k} - 1$ and $A_{i+1} = A_i \in [T - 2L + H, T)$. The project is not implemented and the return is 0.

  To summarize,

$$U_i(H) = H \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) * 0 \right\} = H f(V_{\bar{k}})(V_{\bar{k}+1} - p)$$

- By choosing L, then $A_i \in [T - L, T - H + L)$ and $k_i = \bar{k}$. Since the threshold is not reached, we need to check the signal x_{i+1}.
  1. x_{i+1} occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+1} = \bar{k} + 1$ and $A_{i+1} = A_i + a_{i+1} \geq T$. The return per dollar invested is $V_{\bar{k}+1} - p$.
  2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$, which implies that $k_{i+1} = \bar{k} - 1$ and $A_{i+1} = A_i \in [T - 2L + H, T)$. The project is not implemented and the return is 0.

  To summarize,

$$U_i(H) = L \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) * 0 \right\} = L f(V_{\bar{k}})(V_{\bar{k}+1} - p)$$

  Note that, we always have

$$U_i(H) \geq U_i(L) \geq U_i(0).$$

  Henceforth, it is optimal for a rich agent i to use full support H.

Case 2.1.3: $N \geq i+2$.

- By choosing rejection, the payoff is 0, that is, $U_i(0) = 0$.

- By choosing H, then $A_i \in [T-2L+H, T)$ and $k_i = \bar{k}$. Since the threshold is not reached, it depends on future signals, including x_{i+1} and x_{i+2}.

  1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$, which implies that $k_{i+1} = \bar{k}+1$ and $A_{i+1} = A_i + a_{i+1} \geq T$. Thus, the return per dollar invested is $(V_{\bar{k}+1} - p)$.
  2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$, which implies that $k_{i+1} = \bar{k}-1$ and $A_{i+1} = A_i \in [T-2L+H, T)$. For this case, project implementation depends further on x_{i+2}.
     - $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} + 1 = \bar{k}$. Agent $i+2$ chooses to support and $A_{i+2} = A_{i+1} + a_{i+2} \geq T$. The project is implemented and the return is $(V_{\bar{k}} - p)$.
     - $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} - 1 = \bar{k} - 2$ and $A_{i+2} = A_{i+1} \in [T-L, T)$. Along this history, it is impossible to break even and DOWN cascade arises. Obviously, the return is 0.

  To summarize, by choosing full support H, it generates a payoff

$$U_i(H) = H \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + [1 - f(V_{\bar{k}})]\, f(V_{\bar{k}-1})(V_{\bar{k}} - p) \right\}$$

- By choosing L, then $A_i \in [T-L, T-H+L)$ and $k_i = k_{i-1} + 1 = \bar{k}$. The project is not implemented and it depends on future signals. Before we proceed, we divide the analysis into two cases.

  1. x_{i+1} occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+1} = k_i + 1 = \bar{k}+1$, and there will be an UP cascade. Note that $A_{i+1} \geq A_i + L = T$, and thus the project is implemented with a return of $(V_{\bar{k}+1} - p)$.
  2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Now, $k_{i+1} = \bar{k} - 1$ and $A_{i+1} = A_i \in [T-L, T-H+L)$. Hence, the threshold implementation is undetermined.
     - $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} + 1 = \bar{k}$ and $a_{i+2} \geq L$. Thus, $A_{i+2} = A_{i+1} + a_{i+2} \geq T$ and the project is implemented with a return of $(V_{\bar{k}} - p)$.
     - $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} - 1 = \bar{k} - 2$, and $A_{i+2} = A_i \geq T - L$. Thus, there will be DOWN cascade since the threshold implementation will be triggered with beliefs bounded by $\bar{k} - 1$, which generates a strict loss. Henceforth, the return is 0.

To summarize, by choosing low support L, the expected payoff for agent i of type $\theta_i = H$ is given by:

$$U_i(L) = L \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + [1 - f(V_{\bar{k}})]\, f(V_{\bar{k}-1})(V_{\bar{k}} - p) \right\}$$

Note that we always have

$$U_i(H) \geq U_i(L) \geq U_i(0).$$

Hence, it is optimal to use H when agent i is rich. ■

Case 2.2: $k_{i-1} = \bar{k}(p) - 1$**,** $T - H - L \leq A_{i-1} < T - 2L$ **(and** $x_i = 1$**).**

Case 2.2.1: $N = i$**.**

It is impossible to get enough funds to achieve threshold implementation, and agent i can choose any action from $\{0, L, H\}$.

Case 2.2.2: $N = i + 1$**.**

- By choosing rejection, the payoff is 0, that is, $U_i(0) = 0$.
- By choosing H, then $A_i \in [T - L, T - 2L + H)$ and $k_i = \bar{k}$. Since the threshold is not reached, it depends on the signal in the last period, x_{i+1}.
    1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, the belief $k_{i+1} = \bar{k} + 1$, and thus $a_{i+1} \geq L$. The proposal is implemented since $A_{i+1} \geq A_i + L \geq T$. The return per dollar invested is $(V_{\bar{k}+1} - p)$.
    2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Now, the belief $k_{i+1} = \bar{k} - 1$, and thus $a_{i+1} = 0$. The proposal is abandoned and thus the return is 0.

  The expected payoff of choosing H is given by

  $$U_i(H) = H f(V_{\bar{k}})(V_{\bar{k}+1} - p)$$

- By choosing L, then $A_i \in [T - H, T - L)$ and $k_i = \bar{k}$. Since the threshold is not reached, it depends on the signal in the last period, x_{i+1}.
    1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, the belief $k_{i+1} = \bar{k} + 1$. In order to reach the proposal threshold, we need $a_i > T$, which only happens when $\theta_{i+1} = H$. The return per dollar invested is $(V_{\bar{k}+1} - p)$ when $\theta_{i+1} = H$. In other words, the proposal is abandoned when $\theta_{i+1} = L$, leading to a return of 0.
    2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Now, the belief $k_{i+1} = \bar{k} - 1$, and thus $a_{i+1} = 0$. The proposal is abandoned and thus the return is 0.

The expected payoff of choosing H is given by

$$U_i(L) = \lambda L f(V_{\bar{k}})(V_{\bar{k}+1} - p)$$

Henceforth, $a_i^* = H$ when $N = i + 1$.

Case 2.2.3: $N = i + 2$**.**

- By choosing rejection, the payoff is 0, that is, $U_i(0) = 0$.
- By choosing H, then $A_i \in [T - L, T - 2L + H)$ and $k_i = \bar{k}$. Since the threshold is not reached, it depends on future signals, including x_{i+1} and x_{i+2}.

  1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$, which implies that $k_{i+1} = \bar{k} + 1$ and $A_{i+1} = A_i + a_{i+1} \geq T$. Thus, the return per dollar invested is $(V_{\bar{k}+1} - p)$.
  2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$, which implies that $k_{i+1} = \bar{k} - 1$ and $A_{i+1} = A_i \in [T - L, T - 2L + H)$. For this case, project implementation depends further on x_{i+2}.

     – $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} + 1 = \bar{k}$. Agent $i + 2$ chooses to support and $A_{i+2} = A_{i+1} + a_{i+2} \geq T$. The project is implemented and the return is $(V_{\bar{k}} - p)$.

     – $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} - 1 = \bar{k} - 2$ and $A_{i+2} = A_{i+1} \in [T - L, T)$. Along this history, it is impossible to break even and DOWN cascade arises. Obviously, the return is 0.

  To summarize, by choosing full support H, it generates a payoff

$$U_i(H) = H \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + [1 - f(V_{\bar{k}})]\, f(V_{\bar{k}-1})(V_{\bar{k}} - p) \right\}$$

- By choosing L, then $A_i \in [T - H, T - L)$ and $k_i = k_{i-1} + 1 = \bar{k}$. The project is not implemented and it depends on future signals.

  1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+1} = k_i + 1 = \bar{k} + 1$, and there will be an UP cascade. If $\theta_{i+1} = H$, then the threshold implementation is triggered instantly at period $i + 1$. Otherwise, if $\theta_{i+1} = L$, then $A_{i+1} = A_i + L \in [T - H + L, T)$ and thus the project will be implemented at period $i + 2$. In either case, the return per dollar invested is $(V_{\bar{k}+1} - p)$.
  2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Now, $k_{i+1} = \bar{k} - 1$ and $A_{i+1} = A_i \in [T - H, T - L)$. Hence, the threshold implementation is undetermined.

– $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} + 1 = \bar{k}$. Since agent $i+2$ is the last one, the threshold can be reached only when $\theta_{i+2} = H$ (which happens with probability λ). The return under this case is $(V_{\bar{k}} - p)$. When $\theta_{i+2} = L$, it is impossible to reach the implementation threshold, and thus the proposal is abandoned. In this case, the return is 0.

– $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} - 1 = \bar{k} - 2$, and $A_{i+2} = A_i \in [T - H, T - L)$. The threshold is not reached, and thus the proposal is abandoned. The return is 0.

To summarize, the payoff of choosing L is given by

$$U_i(L) = L \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})\lambda(V_{\bar{k}} - p) \right\}$$

Note that $U_i(H) > U_i(L) > U_i(0)$. Henceforth, $a_i^* = H$ when $N = i + 2$.

Case 2.2.4; $N = i + 3$.

Note that when agent i chooses rejection or high support (i.e., $a_i \in \{0, H\}$), threshold implementation will be determined within the next two periods. Hence, the analysis coincides with that in case 2.2.3.

- By choosing rejection, the payoff is 0, that is, $U_i(0) = 0$.
- By choosing H,

$$U_i(H) = H \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + [1 - f(V_{\bar{k}})]\, f(V_{\bar{k}-1})(V_{\bar{k}} - p) \right\}$$

- By choosing L, then $A_i \in [T - H, T - L)$ and $k_i = k_{i-1} + 1 = \bar{k}$. The project is not implemented and it depends on future signals.

1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+1} = k_i + 1 = \bar{k} + 1$, and there will be an UP cascade. If $\theta_{i+1} = H$, then the threshold implementation is triggered instantly at period $i+1$. Otherwise, if $\theta_{i+1} = L$, then $A_{i+1} = A_i + L \in [T - H + L, T)$ and thus the project will be implemented at period $i + 2$. In either case, the return per dollar invested is $(V_{\bar{k}+1} - p)$.
2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Now, $k_{i+1} = \bar{k} - 1$ and $A_{i+1} = A_i \in [T - H, T - L)$. Hence, the threshold implementation is undetermined.

   – $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} + 1 = \bar{k}$.

(a) $\theta_{i+2} = H$ occurs with probability λ. Now, agent $i+2$ faces the same decision problem as agent i in case 1.2.[31] Specifically, if $p > p^*_{11}$, agent $i+2$ prefers low support L, and the return under this case is given by

$$f(V_{\bar{k}})(V_{\bar{k}+1} - p)$$

Otherwise, when $p \leq p^*_{11}$, agent $i+2$ chooses high support H. Correspondingly, the return will be $(V_{\bar{k}} - p)$.

(b) $\theta_{i+2} = L$ occurs with probability $1 - \lambda$. Agent $i+2$ will choose low support (i.e., $a_{i+2} = L$) and thus $A_{i+2} = A_{i+1} + L \in [T - H + L, T)$ and $k_{i+2} = \bar{k}$. This coincides with the case above when a rich agent $i+2$ chooses L, and thus return is given by $f(V_{\bar{k}})(V_{\bar{k}+1} - p)$.[32]

– $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} - 1 = \bar{k} - 2$, and $A_{i+2} = A_i \in [T - H, T - L)$. The threshold is not reached yet. However, the proposal is abandoned *de facto*, because even after a good signal $x_{i+3} = 1$, the belief cannot break even (i.e., $k_{i+3} = \bar{k} - 1$). Henceforth, the return is 0.

To summarize, the payoff of choosing low support L is given by

1. When $p > p^*_{11}$,

$$U_i(L|p > p^*_{11}) = L \times \left\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})f(V_{\bar{k}})(V_{\bar{k}+1} - p) \right\}$$

This implies that agent $i+2$ will choose H only when

$$p \leq p^*_{21} := \frac{V_1}{V_2}$$

where

$$V_1 = Hf(V_{\bar{k}})V_{\bar{k}+1} + H(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})V_{\bar{k}} - Lf(V_{\bar{k}})V_{\bar{k}+1} - L(1-f(V_{\bar{k}}))f(V_{\bar{k}-1})f(V_{\bar{k}})V_{\bar{k}+1}$$

and

$$V_2 = Hf(V_{\bar{k}}) + H(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) - Lf(V_{\bar{k}}) - L(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})f(V_{\bar{k}})$$

[31] In case 1.2, $k_{i-1} = \bar{k} - 1$ and $A_{i-1} = T - H$. However, the underlying decision problem coincides since H will trigger implementation, and L will not in both cases.

[32] The threshold is not reached and thus depends on the signal x_{i+3}. Specifically, $x_{i+3} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+3} = \bar{k} + 1$ and $a_{i+3} \geq L$. This implies that $A_{i+3} = A_{i+2} + a_{i+3} > T$. The project ends with an UP cascade and the return per dollar invested is $(V_{\bar{k}+1} - p)$. In contrast, when $x_{i+3} = -1$, then $k_{i+3} = \bar{k} - 1$. Hence, the proposal is abandoned and the return is 0.

2. When $p \leq p_{11}^*$,

$$U_i(L|p \leq p_{11}^*) = L \times \Big\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) \\ \times \Big(\lambda(V_{\bar{k}} - p) + (1 - \lambda)f(V_{\bar{k}})(V_{\bar{k}+1} - p) \Big) \Big\}$$

This implies that agent $i+2$ will choose H only when

$$p \leq p_{22}^* := \frac{V_3}{V_4}$$

where

$$V_3 = Hf(V_{\bar{k}})V_{\bar{k}+1} + H(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})V_{\bar{k}} - Lf(V_{\bar{k}})V_{\bar{k}+1} \\ - L(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})\Big(\lambda V_{\bar{k}} + (1 - \lambda)f(V_{\bar{k}})V_{\bar{k}+1} \Big)$$

and

$$V_4 = Hf(V_{\bar{k}}) + H(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) - Lf(V_{\bar{k}}) \\ - L(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})(\lambda + (1 - \lambda)f(V_{\bar{k}}))$$

Case 2.2.5: $N \geq i+4$.
For reasons stated in case 2.2.4, we have

- By choosing rejection, the payoff is 0, that is, $U_i(0) = 0$.
- By choosing H,

$$U_i(H) = H \times \Big\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + [1 - f(V_{\bar{k}})]\, f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big\}$$

- By choosing L, then $A_i \in [T - H, T - L)$ and $k_i = k_{i-1} + 1 = \bar{k}$. The project is not implemented and it depends on future signals.

  1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+1} = k_i + 1 = \bar{k} + 1$, and there will be an UP cascade. If $\theta_{i+1} = H$, then the threshold implementation is triggered instantly at period $i+1$. Otherwise, if $\theta_{i+1} = L$, then $A_{i+1} = A_i + L \in [T - H + L, T)$ and thus the project will be implemented at period $i+2$. In either case, the return per dollar invested is $(V_{\bar{k}+1} - p)$.
  2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Now, $k_{i+1} = \bar{k} - 1$ and $A_{i+1} = A_i \in [T - H, T - L)$. Hence, the threshold implementation is undetermined.

- $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} + 1 = \bar{k}$.
  (a) $\theta_{i+2} = H$ occurs with probability λ. In this case, agent $i+2$ faces the same decision problem as agent i in case 1 as long as there exists two more agents (i.e., $N \geq i+4$).[33] In other words, if $p > p^*_{12}$, agent $i+2$ prefers low support L, and the return under this case is given by

  $$f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})(V_{\bar{k}} - p)$$

  Otherwise, when $p \leq p^*_{12}$, agent $i+2$ chooses high support H. Correspondingly, the return will be $(V_{\bar{k}} - p)$.

  (b) $\theta_{i+2} = L$ occurs with probability $1 - \lambda$. Agent $i+2$ will choose low support (i.e., $a_{i+2} = L$) and thus $A_{i+2} = A_{i+1} + L \in [T - H + L, T)$ and $k_{i+2} = \bar{k}$. The threshold is not reached and thus depends on future signals.
    * $x_{i+3} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, $k_{i+3} = \bar{k} + 1$ and $a_{i+3} \geq L$. This implies that $A_{i+3} = A_{i+2} + a_{i+3} > T$. The project ends with an UP cascade and the return per dollar invested is $(V_{\bar{k}+1} - p)$.
    * $x_{i+3} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. $k_{i+3} = \bar{k} - 1$, $a_{i+3} = 0$ and $A_{i+3} \in [T - H + L, T)$.
      i. $x_{i+4} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+4} = \bar{k}$, $a_{i+4} \geq L$ and $A_{i+4} > T$. The project is implemented with a return of $V_{\bar{k}} - p$.
      ii. $x_{i+4} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+4} = \bar{k} - 2$, $a_{i+4} = 0$ and $A_{i+4} = A_{i+3} > T - L$. From now on, the DOWN cascade arises since the project can only be implemented with a belief bounded above by $\bar{k} - 1$ (with a strict loss). The return is 0.

- $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+2} = k_{i+1} - 1 = \bar{k} - 2$, and $A_{i+2} = A_i \in [T - H, T - L)$. The threshold is not reached yet and it depends on future signal x_{i+3} and x_{i+4}. Actually, only two successive good signals can implement the project. Specifically,
  (a) $x_{i+3} = 1$ occurs with probability $f(V_{\bar{k}-2})$. Now, $k_{i+3} = k_{i+2} + 1 = \bar{k} - 1$, and $a_{i+3} = L$. This is because for a rich agent $i+3$ (i.e., $\theta_{i+3} = H$), choosing H triggers threshold implementation with a belief below the break-even point. Thus, we have $A_{i+3} \in [T - H + L, T)$.
    * $x_{i+4} = 1$ occurs with probability with $f(V_{\bar{k}-1})$. This implies that $k_{i+4} = \bar{k}$ and $a_{i+4} \geq L$. Hence, the project is implemented since $A_{i+4} = A_{i+3} + a_{i+4} \geq T - H + 2L > T$. The return is $(V_{\bar{k}} - p)$.

[33] In case 1, $k_{i-1} = \bar{k} - 1$ and $A_{i-1} = T - H$. However, the underlying decision problem coincides since H will trigger implementation, and L will not in both cases.

* $x_{i+4} = -1$ occurs with probability with $1 - f(V_{\bar{k}-1})$. This implies that $k_{i+4} = \bar{k} - 2$ and DOWN cascade arises since the belief cannot be pushed as high as the break-even point. The return is 0.

(b) $x_{i+3} = -1$ occurs with probability $1 - f(V_{\bar{k}-2})$. Now, $k_{i+3} = k_{i+2} - 1$, and $A_{i+3} = A_{i+2} \in [T - H, T - L)$. Note that we have a DOWN cascade along this history, since the future belief is bounded above by $\bar{k} - 1$ when the threshold T is reached. Hence, the return is 0.

To summarize, given $x_i = 1$ and $\theta_i = H$,

1. When $p > p^*_{12}$, by choosing low support L, the expected payoff is given by:

$$U_i(L|p \in (p^*_{12}, V_{\bar{k}}]) = L \times \Big\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + [1 - f(V_{\bar{k}})] \times$$
$$\Big(f(V_{\bar{k}-1}) \Big[f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big] + (1 - f(V_{\bar{k}-1}))f(V_{\bar{k}-2})f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big) \Big\}$$

   Equivalently, agent i of type $\theta_i = H$ will choose H only when

$$p \leq p^*_{23} := \frac{W_1}{W_2}$$

   where

$$\begin{aligned} W_1 = {} & V_{\bar{k}+1} f(V_{\bar{k}}) \Big(H - L - L(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) \Big) \\ & + V_{\bar{k}}(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) \Big(H - Lf(V_{\bar{k}-1})(1 - f(V_{\bar{k}})) - (1 - f(V_{\bar{k}-1}))f(V_{\bar{k}-2}) \Big) \end{aligned}$$

   and

$$\begin{aligned} W_2 = {} & f(V_{\bar{k}}) \Big(H - L - L(1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) \Big) \\ & + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1}) \Big(H - Lf(V_{\bar{k}-1})(1 - f(V_{\bar{k}})) - (1 - f(V_{\bar{k}-1}))f(V_{\bar{k}-2}) \Big) \end{aligned}$$

2. When $p \leq p^*_{12}$, by choosing low support L, the expected payoff is given by:

$$U_i(L|p \leq p^*_{12}) = L \times \Big\{ f(V_{\bar{k}})(V_{\bar{k}+1} - p) + [1 - f(V_{\bar{k}})] \times \Big(f(V_{\bar{k}-1}) \Big\{ \lambda(V_{\bar{k}} - p)$$
$$(1 - \lambda) \Big[f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big] \Big\} + (1 - f(V_{\bar{k}-1}))f(V_{\bar{k}-2})f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big) \Big\}$$

   Equivalently, given $p \leq p^*_{12}$, agent i of type $\theta_i = H$ will choose H only when

$$p \leq p^*_{24} := \frac{W_3}{W_4}$$

where

$$W_3 = V_{\bar{k}+1} f(V_{\bar{k}}) \Big(H - L - L(1 - f(V_{\bar{k}}))(1-\lambda) \Big) + V_{\bar{k}} (1 - f(V_{\bar{k}})) f(V_{\bar{k}-1}) \\ \times \Big(H - L\big[\lambda + (1-\lambda)(1 - f(V_{\bar{k}})) f(V_{\bar{k}-1})\big] - L(1 - f(V_{\bar{k}-1})) f(V_{\bar{k}-2}) \Big)$$

and

$$W_4 = f(V_{\bar{k}}) \Big(H - L - L(1 - f(V_{\bar{k}}))(1-\lambda) \Big) + (1 - f(V_{\bar{k}})) f(V_{\bar{k}-1}) \\ \times \Big(H - L\big[\lambda + (1-\lambda)(1 - f(V_{\bar{k}})) f(V_{\bar{k}-1})\big] - L(1 - f(V_{\bar{k}-1})) f(V_{\bar{k}-2}) \Big)$$

Now, (53) summarizes the discussion above, which concludes the analysis for case 2. ■

Case 3: $k_{i-1} = \bar{k}(p) - 2$**,** $A_{i-1} = T - H - L$ **(and** $x_i = 1$**).**

Case 3.1: $N = i$**.** It is impossible to reach AON. Agent i can choose any action from $\{0, L, H\}$.

Case 3.2: $N = i + 1$**.** In this case, only when $x_{i+1} = 1$ occurs, the threshold can be reached with the belief given by $k_{i+1} = 1$. Since $V_{\bar{k}} - p$ is non-negative, it is optimal to choose H and the payoff is given by

$$U_i(H) = H f(V_{\bar{k}-1})(V_{\bar{k}} - p)$$

In contrast, if agents i chooses L, then the project will only be implemented when $x_{i+1} = 1$ and $\theta_{i+1} = H$ (which occurs with probability λ). Thus, the payoff of choosing L is given by

$$U_i(L) = L\lambda f(V_{\bar{k}-1})(V_{\bar{k}} - p)$$

Henceforth, it is always weakly dominant to choose full support (i.e., $a_i^* = H$).

Case 3.3: $N = i + 2$**.**

- By choosing rejection, the payoff is 0.
- By choosing H, then $k_i = \bar{k} - 1$ and $A_i = A_{i-1} + H = T - L$. Threshold implementation is not triggered and thus project implementation depends further on future signals in period $i + 1$.
    1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}-1})$, which implies that $k_{i+1} = \bar{k}$ and thus $A_{i+1} \geq A_i + L = T$. Under this event, the return per dollar invested is $(V_{\bar{k}} - p)$.
    2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. In this case, $k_{i+1} = \bar{k} - 2$ and thus $A_{i+1} = A_i = T - L$. Now, there is DOWN cascade since even low support L will trigger threshold implementation, but the belief is bounded above by $\bar{k} - 1$ and thus impossible to break even. In short, the return per dollar invested is 0.

To summarize, the expected payoff from choosing H is given by

$$H\left\{f(V_{\bar{k}-1})(V_{\bar{k}}-p)+(1-f(V_{\bar{k}-1}))*0\right\}=f(V_{\bar{k}-1})(V_{\bar{k}}-p)H$$

- By choosing L, then $k_i=\bar{k}-1$ and $A_i=A_{i-1}+L=T-H$.

  1. $x_{i+1}=1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+1}=\bar{k}$. Note the independence between realization of signals and the type of the agent.

     - $\theta_{i+1}=H$ occurs with probability λ. If agent $i+1$ chooses high support H, then $k_{i+1}=\bar{k}$ and $A_{i+1}=A_i+H=T$. The return per dollar invested is $(V_{\bar{k}}-p)$. If agent $i+1$ chooses low support L, then $k_{i+1}=\bar{k}$ and $A_{i+1}=A_i+L=T-H+L$. In this case, we need to consider the signal x_{i+2}.

       (a) $x_{i+2}=1$ occurs with probability $f(V_{\bar{k}})$. Now, the posterior $k_{i+2}=k_{i+1}+1=\bar{k}+1$, which triggers UP cascade. Hence, the return per dollar invested is $(V_{\bar{k}+1}-p)$.

       (b) $x_{i+2}=-1$ occurs with probability $1-f(V_{\bar{k}})$. Now, the posterior $k_{i+2}=k_{i+1}-1=\bar{k}-1$, and $A_{i+2}=A_{i+1}=T-H+L$. The proposal is abandoned and thus the return is 0.

     - $\theta_{i+1}=L$ occurs with probability $1-\lambda$. Now, $k_{i+1}=\bar{k}$ and $A_{i+1}=A_i+L=T-H+L$.

       (a) $x_{i+2}=1$ occurs with probability $f(V_{\bar{k}})$. Then, $k_{i+2}=\bar{k}+1$, which implies UP cascade. The return per dollar invested is $V_{\bar{k}+1}-p$.

       (b) $x_{i+2}=-1$ occurs with probability $1-f(V_{\bar{k}})$. Then, $k_{i+2}=\bar{k}-1$ and $A_{i+2}=A_{i+1}=T-H+L$.

  2. $x_{i+1}=-1$ occurs with probability $1-f(V_{\bar{k}-1})$. Now, $k_{i+1}=\bar{k}-2$ and $A_{i+1}=A_i=T-H$. Since it is impossible to break even and the belief is bounded above by $\bar{k}-1$ even if $x_{i+2}=1$. Hence, the project is abandoned and the return is 0 under this history.

To summarize, when $p>p^*_{11}$ (i.e., agent $i+1$ of type $\theta_{i+1}=H$ chooses L after $x_{i+1}=1$), then the payoff of choosing L for agent i is given by

$$U_i(L|p>p^*_{11})=Lf(V_{\bar{k}-1})f(V_{\bar{k}})(V_{\bar{k}+1}-p)$$

This implies that agent i of type $\theta_i=H$ after signal $x_i=1$ will choose H when

$$p\leq\frac{HV_{\bar{k}}-Lf(V_{\bar{k}})V_{\bar{k}+1}}{H-Lf(V_{\bar{k}})}:=p^*_{31}=p^*_{11}$$

Otherwise, when $p\leq p^*_{11}$ (i.e., agent $i+1$ of type $\theta_{i+1}=H$ chooses H after $x_{i+1}=1$), then the payoff of choosing L is given by

$$U_i(L|p \leq p_{11}^*) = Lf(V_{\bar{k}-1})\Big[\lambda(V_{\bar{k}} - p) + (1-\lambda)f(V_{\bar{k}})(V_{\bar{k}+1} - p)\Big]$$

This implies that agent i of type $\theta_i = H$ after signal $x_i = 1$ will choose H if

$$p \leq \frac{HV_{\bar{k}} - \lambda LV_{\bar{k}} - (1-\lambda)Lf(V_{\bar{k}})V_{\bar{k}+1}}{H - \lambda L - (1-\lambda)Lf(V_{\bar{k}})} := p_{32}^*$$

Actually, we can verify that $p_{32}^* > p_{11}^*$.[34] Hence, $a_i^* = H$ for any $p \leq p_{11}^*$.

Case 3.4: $N \geq i+3$.

- By choosing rejection, the payoff is 0.
- By choosing H, then $k_i = \bar{k} - 1$ and $A_i = A_{i-1} + H = T - L$. Threshold implementation is not triggered and thus project implementation depends further on future signals in period $i+1$.

   1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}-1})$, which implies that $k_{i+1} = \bar{k}$ and thus $A_{i+1} \geq A_i + L = T$. Under this event, the return per dollar invested is $(V_{\bar{k}} - p)$.
   2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. In this case, $k_{i+1} = \bar{k} - 2$ and thus $A_{i+1} = A_i = T - L$. Now, there is DOWN cascade since even low support L will trigger threshold implementation, but the belief is bounded above by $\bar{k} - 1$ and thus impossible to break even. In short, the return per dollar invested is 0.

   To summarize, the expected payoff from choosing H is given by

$$H\Big\{f(V_{\bar{k}-1})(V_{\bar{k}} - p) + (1 - f(V_{\bar{k}-1})) * 0\Big\} = f(V_{\bar{k}-1})(V_{\bar{k}} - p)H$$

- By choosing L, then $k_i = \bar{k} - 1$ and $A_i = A_{i-1} + L = T - H$. From the perspective of agent $i+1$, it reduces to Case 1 discussed above (depending on how many agents left). By the same token, project implementation depends on future signals, including those from period $i+1$ to $i+3$.

   1. $x_{i+1} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Now, $k_{i+1} = \bar{k}$. Note the independence between realization of signals and the type of the agent.
      - $\theta_{i+1} = H$ occurs with probability λ. If agent $i+1$ chooses high support H, then $k_{i+1} = \bar{k}$ and $A_{i+1} = A_i + H = T$. The return per dollar invested is $(V_{\bar{k}} - p)$. If

[34] The proof is omitted and available upon request.

agent $i+1$ chooses low support L, then $k_{i+1} = \bar{k}$ and $A_{i+1} = A_i + L = T - H + L$. In this case, we need to consider two more rounds as below.

(a) $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}})$. Now, the posterior $k_{i+2} = k_{i+1} + 1 = \bar{k} + 1$, which triggers UP cascade. Hence, the return per dollar invested is $(V_{\bar{k}+1} - p)$.

(b) $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Now, the posterior $k_{i+2} = k_{i+1} - 1 = \bar{k} - 1$, and $A_{i+2} = A_{i+1} = T - H + L$.

* $x_{i+3} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Then, $k_{i+3} = \bar{k}$ and $A_{i+3} \geq A_{i+2} + L \geq T$. The return per dollar invested is $(V_{\bar{k}} - p)$.
* $x_{i+3} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Then, $k_{i+3} = \bar{k} - 2$ and $A_{i+3} = A_{i+2} = T - H + L$. We have DOWN cascade under this history since the belief is bounded above by $\bar{k} - 1$ when the threshold implementation is triggered. Hence, the return per dollar is 0.

– $\theta_{i+1} = L$ occurs with probability $1 - \lambda$. Now, $k_{i+1} = \bar{k}$ and $A_{i+1} = A_i + L = T - H + L$.

(a) $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}})$. Then, $k_{i+2} = \bar{k} + 1$, which implies UP cascade. The return per dollar invested is $V_{\bar{k}+1} - p$.

(b) $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}})$. Then, $k_{i+2} = \bar{k} - 1$ and $A_{i+2} = A_{i+1} = T - H + L$. By the same token, we consider one more round.

* $x_{i+3} = 1$ occurs with probability $f(V_{\bar{k}-1})$. Then, $k_{i+3} = \bar{k}$ and $A_{i+3} \geq A_{i+2} + L \geq T$. The return per dollar invested is $(V_{\bar{k}} - p)$.
* $x_{i+3} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Then, $k_{i+3} = \bar{k} - 2$ and $A_{i+3} = A_{i+2} = T - H + L$. We have DOWN cascade under this history since the belief is bounded above by $\bar{k} - 1$ when the threshold implementation is triggered. Hence, the return per dollar is 0.

2. $x_{i+1} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+1} = \bar{k} - 2$ and $A_{i+1} = A_i = T - H$.

– $x_{i+2} = -1$ occurs with probability $1 - f(V_{\bar{k}-2})$. Now, $k_{i+2} = \bar{k} - 3$. Under this history, DOWN cascade happens eventually since there is no enough room to push the belief to break even without triggering project implementation. Hence, the return under this case is 0.

– $x_{i+2} = 1$ occurs with probability $f(V_{\bar{k}-2})$. Now, $k_{i+2} = \bar{k} - 1$. Given the independence of agent type and signal realization,

* $\theta_{i+2} = H$ occurs with probability λ. First, note that it is suboptimal to choose H. Otherwise, we have $k_{i+2} = \bar{k} - 1$ and $A_{i+2} = A_{i+1} + H = T$. The project

is implemented with a strict loss. Now, we know for sure that $a_{i+2} = L$, then $k_{i+2} = \bar{k} - 1$ and $A_{i+2} = T - H + L$. Since project implementation is undetermined, we need to consider one more round.

(a) $x_{i+3} = 1$ occurs with probability $f(V_{\bar{k}-1})$. In this case, $k_{i+3} = \bar{k}$ and $A_{i+3} \geq A_{i+2} + L = T - H + 2L > T$. The return per dollar invested is $(V_{\bar{k}} - p)$.

(b) $x_{i+3} = -1$ occurs with probability $1 - f(V_{\bar{k}-1})$. Now, $k_{i+3} = \bar{k} - 2$ and $A_{i+3} = A_{i+2} = T - H + L$, which triggers DOWN cascade. The return per dollar is 0.

* $\theta_{i+2} = L$ occurs with probability $(1 - \lambda)$. The analysis is almost identical with the discussion when $\theta_{i+2} = H$ because the rich type will not choose H.

Hence, given that $x_{i+2} = 1$, the realization of type θ_{i+2} is irrelevant. Thus, with probability $f(V_{\bar{k}-1})$, $x_{i+3} = 1$, the return per dollar invested is $(V_{\bar{k}} - p)$; With complementary probability $1 - f(V_{\bar{k}-1})$, $x_{i+3} = -1$, the return is 0 due to the DOWN cascade.

To summarize, when $p > p^*_{12}$ (i.e., agent $i+1$ of type $\theta_{i+1} = H$ chooses L after $x_{i+1} = 1$), then the payoff of choosing L for agent i is given by

$$\begin{aligned} U_i(L|p > p^*_{12}) = L \times \Big\{ & f(V_{\bar{k}-1}) \Big[f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big] \\ & + (1 - f(V_{\bar{k}-1}))f(V_{\bar{k}-2})f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big\} \end{aligned}$$

Otherwise, when $p \leq p^*_{12}$ (i.e., agent $i+1$ of type $\theta_{i+1} = H$ chooses H after $x_{i+1} = 1$), then the payoff of choosing L is given by

$$\begin{aligned} U_i(L|p \leq p^*_{12}) = L \times \Big\{ & f(V_{\bar{k}-1}) \Big[f(V_{\bar{k}})(V_{\bar{k}+1} - p) + (1 - f(V_{\bar{k}}))f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big] \\ & + (1 - f(V_{\bar{k}-1}))f(V_{\bar{k}-2})f(V_{\bar{k}-1})(V_{\bar{k}} - p) \Big\} \end{aligned}$$

Henceforth, given that $p \in (p^*_{12}, V_{\bar{k}}]$, agent i of type $\theta_i = H$ will choose full support H only when

$$U_i(H) \geq U_i(L|p > p^*_{12})$$

or equivalently,

$$p \leq \frac{V_{\bar{k}} \Big(H - Lf(V_{\bar{k}-1})[1 - f(V_{\bar{k}})] - L[1 - f(V_{\bar{k}-1})]f(V_{\bar{k}-2}) \Big) - Lf(V_{\bar{k}})V_{\bar{k}+1}}{H - Lf(V_{\bar{k}}) - L[1 - f(V_{\bar{k}})]f(V_{\bar{k}-1}) - L[1 - f(V_{\bar{k}-1})]f(V_{\bar{k}-2})} \equiv p^*_{33} \tag{59}$$

Actually, we can verify that $p^*_{33} < p^*_{12}$.[35] Hence, for all $p \in (p^*_{12}, V_{\bar{k}}]$, agent i of type $\theta_i = H$ will choose L.

Moreover, given that $p \leq p^*_{12}$, agent i of type $\theta_i = H$ will choose full support H if $U_i(H) \geq U_i(L|p \leq p^*_{12})$, or equivalently,

$$p \leq \frac{V_{\bar{k}}\Big(H - [1-f(V_{\bar{k}})]f(V_{\bar{k}-1})L - [1-f(V_{\bar{k}-1})]f(V_{\bar{k}-2})L\Big) - LV_{\bar{k}+1}f(V_{\bar{k}})}{H - [1-f(V_{\bar{k}})]f(V_{\bar{k}-1})L - [1-f(V_{\bar{k}-1})]f(V_{\bar{k}-2})L - Lf(V_{\bar{k}})} \equiv p^*_{34}. \tag{60}$$

We can verify that $p^*_{34} < p^*_{12}$. Hence, for any $p \leq p^*_{12}$, agent i of type $\theta_i = H$ will choose H when $p \leq p^*_{34}$ and choose L when $p \in (p^*_{34}, p^*_{12}]$.

Now, (54) summarizes the discussion above, which concludes the analysis for case 3. ■

Case 4: $k_{i-1} = \bar{k}(p) - 1$, $A_{i-1} > T - H$ (given that $x_i = 1$ and $\theta_i = H$). We divide this case into two sub-cases.

Case 4.1: $A_{i-1} \geq T - L$. This case is simple. Even choosing low support $a_i = L$ will trigger threshold implementation, thus agent i of type $\theta_i = H$ has an incentive to choose H as long as the return is non-negative.

Case 4.2: $A_{i-1} \in (T-H, T-L)$. Basically, it coincides with Case 1 under the assumption that $H \in (L, 2L)$, since choosing H will instantly implement the project, while choosing L allows for one more period of learning. We omit the analysis here and just characterize the optimal decision rule.

Now, (55) summarizes the discussion above, which concludes the analysis for case 4. ■

We then prove the existence of the optimal AoN design and pricing when they are endogenous. Note that Proposition 9 features a rent linked to the incentive compatibility conditions. This is very intuitive. To get full rent extraction, the proposer needs to set a price such that all agents are indifferent between support and rejection. Now, this is problematic since if the price is so high (i.e., $p \to V_{\bar{k}}$, which is a necessary condition for full surplus extraction), then the agent will switch from high support to low support so as to accommodate more periods of trials. Under such a strategy, there exist some signal sequences such that an UP cascade is more likely to happen, which generates a positive payoff.[36]

[35] The calculation for this claim is available upon request.

[36] Note that the no DOWN cascade before approaching the threshold is a by-product of Assumption 1 in the baseline model. The same situation arises here. Consider the "prolonged learning" episode where $A_{i-1} > T - H - (\bar{k}(p) - k_{i-1} - 1)L$. When $T - A_{i-1} < (\bar{k}(p) - k_{i-1})L$ is satisfied (which is consistent with the case here), it is impossible to achieve the break-even belief without triggering threshold implementation, so every agent should abstain from supporting the project. However, the agents still choose to (partially) support, because the decision is delegated to the agent with $A_{i-1} \geq T - L$. This prevents the DOWN cascade before approaching the threshold. One caveat is such that this cannot be supported by the equilibrium refinement by lowering the price p by a tiny amount, unless the break-even belief is changed.

Finally, the next proposition discusses the existence of an optimal pair (p^*, T^*) that maximizes the proposer's expected revenue and provides a numerical procedure for finding them.

Given agents' equilibrium strategies specified in Proposition A.15, there exists a pair (p^*, T^*) that maximizes the proposer's expected revenue. Specifically, there exists $k^* \in \{-1, 0, \cdots, N\}$ such that $p^* \in \{p^*_{11}(k^*), p^*_{12}(k^*), p^*_{21}(k^*), p^*_{22}(k^*), p^*_{23}(k^*), p^*_{24}(k^*), p^*_{34}(k^*), V_{k^*}\}$ and $T = m^*L + n^*H$ for some $0 \leq m^*, n^* \leq N$ and $m^* + n^* \leq N$.

Proof. The proof consists of two steps.

i) we show that $T^* \in \mathbb{T} := \{mH + nL : m \in \mathbb{Z}_+, m, n \leq N\}$, where $\mathbb{Z}_+$ is the set of all non-negative integers. Rearranging the set $\mathbb{T}$ in an increasing order and keep only one copy if there exists duplicates. For instance, if $m_1 * H + n_1 * L = m_2 * H + n_2 * L = T_0$, then we just need to keep T_0. Recall that the proposer's expected utility is

$$E\left[(p-\nu)A_N \mathbb{1}_{A_N \geq T} \middle| \{a^*_i\}_{i=1,\cdots,N}\right]$$

Fix any two successive T's, say, $T_i, T_{i+1} \in \mathbb{T}$. Consider any $T \in (T_i, T_{i+1}]$. It induces the same equilibrium outcome at that under $T^* = T_{i+1}$. Since $p > \nu$, any threshold $T \in (T_i, T_{i+1})$ is strictly dominated by T_{i+1}. This implies the optimal threshold $T^* \in \mathbb{T}$. Note that the cardinality $|\mathbb{T}| \leq (N+1)^2$.

ii) We show that $p^* \in \mathbb{W}$, where

$$\begin{aligned}
\mathbb{W} := \bigcup_{k=-1}^{N} \Big\{&p^*_{11}(k), p^*_{12}(k), p^*_{11}(k) \wedge p^*_{22}(k), p^*_{11}(k) \vee p^*_{21}(k), p^*_{12}(k) \wedge p^*_{24}(k), \\
&p^*_{12}(k) \wedge p^*_{23}(k), p^*_{34}(k), V_k\Big\} \\
\subseteq \bigcup_{k=-1}^{N} &\Big\{p^*_{11}(k), p^*_{12}(k), p^*_{21}(k), p^*_{22}(k), p^*_{23}(k), p^*_{24}(k), p^*_{34}(k), V_k\Big\}
\end{aligned}$$

First, note that $p < V_{-1}$ is suboptimal since it induces UP cascade from the very begining. Meanwhile, $p > V_N$ is also suboptimal, because it is impossible to get enough support after a sequence of all positive signals. Hence, $p \in [V_{-1}, V_k]$.

Second, consider any $p \in (V_{k-1}, V_k]$, $\forall k \in \{0, \cdots, N\}$. Note that, there are only eight discrete points defined as in the set $\mathbb{W}$. Note that the decision rules defined in lemma 4 only depend on $\bar{k}(p)$. Since any $p \in (V_{k-1}, V_k)$ induces the same $\bar{k}(p)$ and is dominated by $p = V_k$. This implies the optimum must be achieved at $p = V_k$ when we only consider lemma 4.

Similarly, we can consider the decision rules define in equations (55), (56), (57), and (58). For instance, in equation (55), when $N = i + 1$, H is optimal when $p \leq p^*_{11}$, and L is optimal when $p > p^*_{11}$ (and $p \leq V_k$). In the case, $p^* \in \{p^*_{11}, V_k\}$. To see it, if H generates a higher payoff for the proposer, then we should choose $p^* = p^*_{11}$, because any other $p \in (V_{k-1}, p^*_{11})$ induces the same outcome as that under $p = p_{11}^*$. In contrast, if L generates a higher payoff for the proposer, then $p^* = V_k$ is optimal, since any other $p \in (p_{11}^*, V_k)$ induces the same outcome as that under $p = V_k$. Thus, in light of the decision rule for case 1 in equation (55), we have three discrete points $\{p_{11}^*, p_{12}^*, V_k\}$, which we need to check for optimality. By the same token, for case 2, 3, and 4, we need to check the cutoff points in the respective decision rule. In total, we need to check $8*(N+1)+1$ discrete points in equations (56), (56), (57), and (58), because we only need to check V_{-1} when $k = -1$.

To summarize, we need to check $|\mathbb{T}| \times |\mathbb{W}| \leq (N+1)^2 * (8N+9)$ discrete points, which ensures the existence of the optimal solution (p^*, T^*). The proof concludes. □

Finally, we describe the numerical procedure for searching for the optimal design, i.e.,the global optimum for (p^*, T^*):

1. Set the values for all parameters, including N, λ, ν, and q.

2. Generate the set $\mathbb{T} = \{mH + nL : m \in \mathbb{Z}_+, m, n \leq N\}$, where $\mathbb{Z}_+$ are all non-negative integers. Rearrange $\mathbb{T}$ in an increasing order, that is, $\mathbb{T} = \{T_l \big| T_l < T_{l+1}, \forall l\}$.

3. Generate the set $\mathbb{W}$, where

$$\mathbb{W} = \bigcup_{k=-1}^{N} \{p_{11}^*(k), p_{12}^*(k), p_{11}^*(k) \wedge p_{22}^*(k), p_{11}^*(k) \vee p_{21}^*(k), p_{12}^*(k) \wedge p_{24}^*(k), \\ p_{12}^*(k) \wedge p_{23}^*(k), p_{34}^*(k), V_k\}$$

   For each k, delete any points $p_j^*(k) \notin (V_{k-1}, V_k]$, for $j \in \{11, 12, 21, 22, 23, 24, 34\}$.

   Rearranging $\mathbb{W}$ in an increasing order such that $\mathbb{W} = \{p_j | p_j < p_{j+1}\}$.

4. Generate all the possible sequence of signals x_i and types θ_i, for $i \in \{1, ..., N\}$.

5. Fix a pair $(T_l, p_j) \in \mathbb{T} \times \mathbb{W} := \{(T_l, p_j) : 1 \leq l \leq |\mathbb{T}|, 1 \leq j \leq |\mathbb{W}|\}$.

   - For each sequence, find out the decision s by each $i \in \{1, ..., N\}$ and then calculate A_N and check whether $A_N \geq T_l$. Return $A_N(p - \nu)\mathbb{1}(A_N \geq T_l)$.
   - Aggregate $A_N(p - \nu)\mathbb{1}(A_N \geq T_l)$ to form the expected payoff $EU_{T_l,p_j} = E[A_N(p - \nu)\mathbb{1}(A_N \geq T_l)|\{T_l, p_j\}]$.

6. Return $(T_l^*, p_j^*) \in \underset{(T_l,p_j)\in\mathbb{T}\times\mathbb{W}}{\arg\max}\, EU_{T_l,p_j}$, and return $EU_{T_l^*,p_j^*}$.

□

A.16 Proof for Proposition 10

Proof. Clearly, for equilibria with only one free-rider, one can simply view the case to be an informer equilibrium with $N-1$ agents. As $N \to \infty$, the resulting equilibria converge. Therefore, we only need to focus on free-rider equilibria with at least two free-riders.

As discussed in the proof of Lemma 3, in particular Eq.(48), the unique equilibrium is an informer equilibrium when $p \in \{V_k, k = -1, 0, \ldots N\}$. Then, the proposer's per-investor profit $\frac{1}{N}\pi(p_N, T_N, N)$ should be at least $\frac{1}{N}\pi(p_N^*, T_N^*, N)$, which is the profit when the proposer is restricted to choose the price from $\mathbb{Z} \cup \{-1\}$.

Step 3 in the proof for Proposition 4 implies that any optimal path of $\{(p_N, T_N)\}_{N=1}^{\infty}$ satisfies $\lim_{N\to\infty} \frac{1}{N}\pi(p_N, T_N, N) = \frac{1}{2}(1-\nu)$. In other words, project implementation efficiency is achieved and the proposer extracts all the surplus as N goes to infinity. This condition requires the price p_N converges to one and both error probabilities go to zero, as shown in Proposition 4.

Next, let the number of informers in a sub-game equilibrium E be $Z_N^E(p_N, T_N)$ when the proposer's endogenous design is (p_N, T_N), then for any positive integer l, as N goes to infinity, $Pr\left(Z_N^E(p_N, T_N) < l\right) \to 0$.

To see this, again we only need to consider free-rider equilibria with at least two free-riders. For a given N, the corresponding proposal (p_N, T_N), sub-game equilibrium E, and a sequence of signals x, define $Z_N^E(x; (p_N, T_N))$ as the total number of informers for sequence $x \in X$. Hence, we need to show $Pr(Z_N^E(x; (p_N, T_N) < l) \to 0$, where the probability is taken over $x \in X$.

Consider the contrary and suppose for some $\varepsilon > 0$, $Pr(Z^E(x; (p_N, T_N)) < l) > \varepsilon$ for infinite values of N. For such an N, we have:

$$\frac{\pi^E(p_N, T_N)}{N} < \frac{1}{2Np_N}\pi_1^E(p_N, T_N) < \frac{1}{2Np_N}(1-\varepsilon(1-q)^l)Np_N(1-\nu) = \frac{1-\varepsilon(1-q)^l}{2}(1-\nu),$$

where $\pi_1^E(p_N, T_N)$ denotes the proposer's profit when the project is good. The first inequality holds because an agent should assign at least probability p_N on the project being good to be willing to pay p_N. Since it holds for all the agents and after all the histories, the gross proposer's revenue when $V = 0$ cannot exceed $\frac{1-p_N}{p_N}$ times that of when $V = 1$. Therefore, $\frac{1-p_N}{2Np_N}(\pi_1^E(p_N, T_N) + \nu) \geq \frac{1}{2N}(\pi_0^E(p_N, T_N) + \nu)$, which further can be simplified and get the first inequality, since $\nu \geq 0$.

The second inequality follows from the fact that the proposal is not accepted when all the informers receive a low signal. But in the proof of Proposition 10, we showed that $\frac{\pi^{E_N}(p_N, T_N)}{N}$ goes to $\frac{1}{2}(1-\nu)$ in this sequence of numbers, which is a contradiction.

With this, we are ready to prove that in any free-rider equilibrium with endogenous $\{p_N, T_N\}_{N=1}^{\infty}$ and a sub-game equilibrium that is Pareto-undominated, $\lim_{N\to\infty} \mathcal{P}_N^I = 0$, $\lim_{N\to\infty} \mathcal{P}_N^{II} = 0$, and $\mathbb{E}[V|\mathcal{H}_N] \xrightarrow{prob.} V$ as $N \to \infty$. In Proposition 10 we have shown that both error probabilities go to zero. The convergence of $\mathbb{E}[V|\mathcal{H}_N]$ in probability can be shown similarly as in the proof of Proposition 5. $\square$

Appendix B. Asymmetric Signal Structure

In this extension, we consider the case with asymmetric private signals:

$$\Pr(x_i = 1|V = 1) = q \in (1/2, 1) \quad \text{and} \quad \Pr(x_i = -1|V = 0) = \tilde{q} \in (1/2, 1). \tag{61}$$

We specified $q > \tilde{q}$ here without loss of generality. If $\tilde{q} = q$, then it reduces to the benchmark case.

Unlike Proposition 1, the gap between the inferred positive signals and negative signals is no longer a sufficient statistics for the equilibrium strategy. Essentially, we need to keep track of both the number of inferred positive signals and that of inferred negative signals. As in most information cascade models, the additional dimension of the state variable often renders the problem intractable. That is the case for explicit derivations of the endogenous pricing and threshold implementation. However, we can still characterize the equilibria to a good extent and distill the intuition.

First, the basic insights and results from Proposition 1 all go through, as summarized in the next proposition. The proof is almost identical to that of Proposition 1 and is thus omitted here. Define $\bar{k}(p,l) = \min\left\{k : p \leq \frac{q^k}{q^k + \left(\frac{\tilde{q}(1-\tilde{q})}{q(1-q)}\right)^l (1-\tilde{q})^k}\right\}$. We have:

Proposition 11. *Given (p, T), there exists an essentially unique informer equilibrium with $a_i^\star(x_i, \mathcal{H}_{i-1}) \equiv a^\star(x_i, k(\mathcal{H}_{i-1}), l(\mathcal{H}_{i-1}), A(\mathcal{H}_{i-1}))$ and posteriors $P(V = 1|\mathcal{H}_i) = V_{\{k^*(\mathcal{H}_i), l^\star(\mathcal{H}_i)\}}$, where:*

$$a_i^\star(x_i, k_{i-1}, l_{i-1}, A_{i-1}) = \begin{cases} x_i & \text{if} \quad A_{i-1} < T-1 \;\&\; k_{i-1} \leq \bar{k}(p, l_{i-1}) \\ 1 & \text{if} \quad k_{i-1} > \bar{k}(p, l_{i-1}) \\ -1 & \text{if} \quad A_{i-1} \geq T-1 \;\&\; k_{i-1} < \bar{k}(p, l_{i-1}) - 1 \\ x_i & \text{if} \quad A_{i-1} \geq T-1 \;\&\; k_{i-1} \in \{\bar{k}(p, l_{i-1}), \bar{k}(p, l_{i-1}) - 1\} \end{cases} \tag{62}$$

$$k_i^*(\mathcal{H}_i) = \begin{cases} k_{i-1} + a_i & \text{if} \quad A_{i-1} < T-1 \;\&\; k_{i-1} \leq \bar{k}(p, l_{i-1}) \\ k_{i-1} & \text{if} \quad k_{i-1} > \bar{k}(p, l_{i-1}) \\ k_{i-1} & \text{if} \quad A_{i-1} \geq T-1 \;\&\; k_{i-1} < \bar{k}(p, l_{i-1}) - 1 \\ k_{i-1} + a_i & \text{if} \quad A_{i-1} \geq T-1 \;\&\; k_{i-1} \in \{\bar{k}(p, l), \bar{k}(p, l_{i-1}) - 1\} \end{cases} \tag{63}$$

$$l_i^*(\mathcal{H}_i) = \begin{cases} l_{i-1} + \mathbf{1}(a_i = -1) & \text{if} \quad A_{i-1} < T-1 \;\&\; k_{i-1} \leq \bar{k}(p, l_{i-1}) \\ l_{i-1} & \text{if} \quad k_{i-1} > \bar{k}(p, l_{i-1}) \\ l_{i-1} & \text{if} \quad A_{i-1} \geq T-1 \;\&\; k_{i-1} < \bar{k}(p, l_{i-1}) - 1 \\ l_{i-1} + \mathbf{1}(a_i = -1) & \text{if} \quad A_{i-1} \geq T-1 \;\&\; k_{i-1} \in \{\bar{k}(p, l), \bar{k}(p, l_{i-1}) - 1\} \end{cases} \tag{64}$$

where $k_0 = 0$, $l_0 = 0$ and $A_0 = 0$.

This means all the main results with exogenous pricing and threshold implementation still hold under this asymmetric signal structure. When pricing and AoN thresholds are endogenous, the equilibrium exists featuring efficient implementation and information aggregation. To see the existence, note that the arguments for existence in Proposition 2 goes through via a global search over a finite grid of triplets of (k, l, T), instead of pairs (k, T) as in the symmetric signal case.[37] Moreover, the law of large numbers implies that the efficient project implementation and information aggregation would ensue.

Specifically, on the one hand, under any equilibrium the proposer's profit should not exceed $\frac{1}{2}(1-\nu)*N$. This is because, the total output from the project is $N*E[V] = \frac{1}{2}N$, and the total cost is $\frac{1}{2}\nu N$. This implies that the total surplus is $\frac{1}{2}(1-\nu)*N$. However, to incentivize the investors to participate in the project, the expected payoff cannot be negative. Hence, the expected profit for the proposer cannot exceed the net total surplus, i.e.,

$$\frac{1}{N}\pi(p, T, N) \leq \frac{1}{2}(1-\nu).$$

On the other hand, the maximum profit $\frac{1}{2}(1-\nu)*N$ is achievable. To see it, given the true state is $V = 1$, then when N is sufficiently large, the total positive signal minus the total negative signals would be close to $[q-(1-q)]*N = (2q-1)*N$.[38] This implies that if we take $T_N = k(N) = m*N$ such that $m = \frac{1}{2}(2q-1)$, the project can always be implemented under AoN when $V = 1$.[39] In contrast, when $V = 0$, if we take N large enough, the total positive signals minus negative signals will be close to $(1-2\tilde{q})*N < 0 \ll k(N)$, which means that it is almost surely impossible to implement the project under $V = 0$ when $T_N = k(N)$. Thus, the expected profit for the proposer satisfies

$$\lim_{N\to\infty}\frac{1}{N}\pi(p_N^*, T_N^*, N) \geq \lim_{N\to\infty}\frac{1}{N}\pi(V_{k(N)}, T_N, N) = \lim_{N\to\infty}\frac{1}{2}(V_{k(N)} - \nu) = \frac{1}{2}(1-\nu)$$

Hence, we have $\lim_{N\to\infty}\frac{1}{N}\pi(p_N^*, T_N^*, N) = \frac{1}{2}(1-\nu)$.

Now, we can show that $\lim_{N\to\infty}\mathcal{P}_N^{II} = 0$ and $\lim_{N\to\infty}\mathcal{P}_N^{I} = 0$.

To see it, note that the investors' expected payoff is given by

$$\begin{aligned}\frac{1}{2}(1-\mathcal{P}_N^I)(1-p_N^*) + \frac{1}{2}\mathcal{P}_N^{II}(0-p_N^*) &= \frac{1}{2}(1-\mathcal{P}_N^I)(1-p_N^*) - \frac{1}{2}\mathcal{P}_N^{II}p_N^* \\ &= \frac{1}{2}(1-\mathcal{P}_N^I) - \frac{1}{2}p_N^*(1-\mathcal{P}_N^I+\mathcal{P}_N^{II})\end{aligned}$$

[37] Note that $k \in \{-N, \cdots, N\}$, $l \in \{0, \cdots, N\}$ and $T \in \{0, \cdots, N\}$.

[38] In fact, a "large deviation" result holds: $\frac{1}{N}\sum_{j=1}^N x_j = [q-(1-q)] + o(N^{-1/2}(\log N)^{1/2+\delta})$, almost surely, where δ is sufficiently small.

[39] Any $m > 0$ such that $m < (2q-1)$ works, or we can even take $T_N = \log N$, $\log\log N$, $\log\log\log N$, etc.

and that the proposer's expected profit is given by

$$\begin{aligned}\frac{1}{N}\pi(p_N^*, T_N^*, N) &= \frac{1}{2}(1-\mathcal{P}_N^I)(p_N^*-\nu) + \frac{1}{2}\mathcal{P}_N^{II}(p_N^*-\nu)\\ &= \frac{1}{2}(p_N^*-\nu)(1-\mathcal{P}_N^I+\mathcal{P}_N^{II})\end{aligned}$$

First, we show that $\lim_{N\to\infty} p_N^* = 1$. If not, suppose $\lim_{N\to\infty} p_N^* = a < 1$, then by the fact that $\lim_{N\to\infty} \frac{1}{N}\pi(p_N^*, T_N^*, N) = \frac{1}{2}(1-\nu)$, we have $\lim_{N\to\infty}(1-\mathcal{P}_N^I+\mathcal{P}_N^{II}) = \frac{1-\nu}{a-\nu}$.

We can plug this into the investor's expected payoff to get

$$\lim_{N\to\infty}(1-\mathcal{P}_N^I) = \lim_{N\to\infty} p_N^* \lim_{N\to\infty}(1-\mathcal{P}_N^I+\mathcal{P}_N^{II}) = \frac{a(1-\nu)}{a-\nu} > 1. \tag{65}$$

which is a contradiction.

Second, we show that $\mathcal{P}_N^{II} \to 0$ and $\mathcal{P}_N^I = 0$. Taking the limit as $N \to \infty$, and using the fact that $\lim_{N\to\infty} p_N^* = 1$, we know that the investor's limit expected payoff is given by

$$-\frac{1}{2}\lim_{N\to\infty}\mathcal{P}_N^{II} \geq 0$$

This is because, the participation constraint requires a non-negative payoff and thus the limit payoff is also non-negative. Thus, we have $\lim_{N\to\infty}\mathcal{P}_N^{II} = 0$.

Similarly, by the facts that $\lim_{N\to\infty} p_N^* = 1$ and that $\lim_{N\to\infty}\mathcal{P}_N^{II} = 0$, we get

$$\lim_{N\to\infty}\frac{1}{N}\pi(p_N^*, T_N^*, N) = \frac{1}{2}(1-\nu)(1-\lim_{N\to\infty}\mathcal{P}_N^I) = \frac{1}{2}(1-\nu)$$

which implies that $\lim_{N\to\infty}\mathcal{P}_N^I = 0$.