Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices

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We study a bilevel noncooperative game-theoretic model of restructured electricity markets, with locational marginal prices. Each player in this game faces a bilevel optimization problem that we remodel as a mathematical program with equilibrium constraints, MPEC. The corresponding game is an example of an EPEC, equilibrium problem with equilibrium constraints. We establish sufficient conditions for existence of pure strategy Nash equilibria for this class of bilevel games and give some applications. We show by examples the effect of network transmission limits, i.e. congestion, on existence of equilibria. Then we study, for more general EPECs, the weaker pure strategy concepts of local Nash and Nash stationary equilibria. We pose the latter as solutions of mixed complementarity problems, CPs, and show their equivalence with the former in some cases. Finally, we present numerical examples of methods that attempt to find local Nash equilibria or Nash stationary points of randomly generated electricity market games. The CP solver PATH is found to be rather effective in identifying Nash stationary points.

Keywords: electricity market, bilevel game, MPEC, EPEC, Nash stationary point, equilibrium constraints, complementarity problem

JEL classification: C61, C62, C72, Q40

1 Introduction

Game-theoretic models are employed to investigate strategic behavior in restructured or deregulated electricity markets — see Berry *et al.* (1999), Cardell *et al.* (1997), Hobbs *et al.* (2000), Nasser (1998), Oren (1997), Seeley *et al.* (2000), Stoft (1999) to mention just a few references — and, of course, more general markets. Such models depend on the structure and rules of the market, and, for electricity and other distributed markets, the way network constraints are handled. Using the bilevel electricity market game of Berry *et al.* (1999) as a jumping-off point, this paper seeks more general modeling paradigms of bilevel games. The thrust of our investigation is twofold. First, in the context of Berry *et al.* (1999), we seek to understand when pure strategy Nash equilibria exist and when they may not. Second, for more general bilevel games, by modeling each player's problem as a mathematical program with equilibrium constraints, MPEC, we recast bilevel games as equilibrium problems with equilibrium constraints, EPECs. Stationarity theory for MPECs allows us to introduce Nash stationary points, a weakening of the Nash equilibrium concept, for EPECs and show that the standard mixed complementarity problem (CP) format captures such points. This opens the way to analysis and algorithms that seem new to bilevel games.

A particular motivation is the area of noncooperative games that model electricity markets over a network of generators and consumers in the style of Berry *et al.* (1999), see also Backerman *et al.* (2000), Cardell *et al.* (1997), Hobbs *et al.* (2000); and also Hu *et al.* (2004) for a brief review of these and related market models. A coordinator or central operator, called an independent system operator (ISO), schedules the dispatch quantities, prices and transmission of electricity. To do so the ISO solves an optimization problem that relies on the configuration and electrical properties of the network, and on bids from generators and consumers (e.g. retailers) that propose how price should vary with quantity. This results in a game that has a bilevel structure; in particular, the ISO's optimization problem, which lies on the lower level, gives rise to non-differentiability and non-quasiconcavity of an individual participant's profit function on the upper level.

In broader terms, we study games in which each player faces a bilevel maximization problem that can be modeled as an MPEC. Such problems have nonconvex constraints and therefore may have multiple local maxima. Thus the corresponding game, an EPEC, may not have any (pure strategy) Nash equilibria in pathological instances, e.g., Example 12 in §3.3. See also Berry *et al.* (1999, Footnote 8, p.143) and Weber and Overbye (1999). This motivates development of alternatives to the Nash equilibrium concept, aimed at model formats that are suitable for application of powerful mathematical programming solvers. Our computational approach is to seek points such that each player's strategy satisfies the stationary conditions for his or her MPEC.

The meaning of the situation when a Nash equilibrium does not exist is, by some leagues, beyond the scope of this paper. Without a Nash equilibrium, the questions "Is the model deficient"? and "Is the market unstable?" point to future investigations that might study the gap between models and their application in policy, commerce, and regulation.

While the emphasis of this paper is on methodology, a companion paper Hu *et al.* (2004) gives a comprehensive treatment of the economic implications of numerical results for the electricity market model of Berry *et al.* (1999), e.g. comparisons of market designs. Both papers are derived from the first author's PhD dissertation, Hu (2003). A preliminary version of this paper was presented in Hu and Ralph (2001).

An alternative complementarity problem approach to EPECs in electricity markets is given in Xian *et al.* (2004). The paper Leyffer and Munson (2005) gives various complementarity and MPEC reformulations of EPECs in multi-leader multi-follower games. See also the PhD dissertation Ehrenmann (2004a); various applications in economics such as Ehrenmann (2004b), Ehrenmann and Neuhoff (2003), Hu *et al.* (2004), Murphy and Smeers (2002), Su (2004a), Yao *et al.* (2005); and the algorithm investigation Su (2004b).

The paper is organized as follows. Section 2 briefly reviews the formulation of the ISO's problem and the associated bilevel game of Berry *et al.* (1999). Section 3 is mainly a study of sufficient conditions for existence of pure strategy Nash equilibria for the game scenario in which only the linear part of the cost/benefit function is bid, the so-called bid linear-only or bid-a-only scenario. It also gives examples showing how transmission limits may (not) affect existence of Nash equilibria. In this section the network and associated pricing and dispatch problem are implicit, which simplifies the format of the game at the expense of requiring analysis of nonsmooth payoff functions. Section 4 studies more general bilevel games which are reformulated as EPECs. It proposes local Nash equilibria and Nash stationary equilibria as weaker alternatives to Nash equilibria, and develops a CP formulation of Nash stationary points to which standard software can be applied. Finally, in a return to electricity market games, it is shown that Nash stationary points are actually local Nash equilibria for the bid-a-only game scenario in some circumstances. Section 5 presents numerical examples of two approaches to finding equilibria. The first approach is to apply the solver PATH (Dirkse and Ferris, 1995; Ferris and Munson, 1999) to the CP formulation of Nash stationary conditions, and the second uses a kind of fixed-point iteration, called diagonalization, that is implemented using the nonlinear programming solver SNOPT (Gill *et al.*, 2002). PATH is found to be particularly fast and robust regarding finding Nash stationary points — that turn out to be local Nash equilibria — for a test set of randomly generated EPECs. Section 6 concludes the paper.

2 Bilevel game formulation

2.1 Pricing and dispatch by optimal power flow model

As many others do, we work with a lossless direct current network, which consists of nodes where generators and consumers are located, and (electricity) lines or links connecting nodes; see Chao and

Peck (1997). For simplicity of notation assume there are N nodes with one generator at each node $i = 1, \ldots, s$ and one consumer at each node $i = s+1, \ldots, N$. This ordering of players does not play any role in the mathematical analysis other than simplifying notation and does not restrict us from multiple players of the same type at any node. However in this paper we do not model an interesting situation where one generator may own several generating units at different nodes. For further simplicity, only two types of network constraints (power balance and line flow constraints) are included in the ISO's social cost minimization problem.

The market works as follows. Generators/consumers bid nominal cost/utility functions to the ISO. To dispatch generation and consumption, the ISO determines the quantities of generation/consumption for generators/consumers by minimizing the nominal social cost, the difference between the total cost and the total utility, subject to network constraints. (Equivalently, the ISO could maximize nominal social welfare.) The prices at each node are set to the nominal marginal cost/utility at a given node, reflecting the fact that higher prices may be needed to curb demand at network nodes to which transmission constraints limit delivery of electricity. For further development of the locational or nodal pricing idea, see Chao and Peck (1997), Hogan (1992).

Various forms of cost/benefit functions are assumed in literature with or without start-up costs for generators, and in practical markets the functions are step functions. In this paper, we use a quadratic form without a start-up cost as in Berry *et al.* (1999), Cardell *et al.* (1997), Weber and Overbye (1999). To this end, the cost function of a generator at node i has the form $c_i(q_i) = a_i q_i + b_i q_i^2$ with $q_i \geq 0$ and the utility function of a consumer at node j has the form $-c_j(q_j) = -a_j q_j - b_j q_j^2$ with $q_j \leq 0$. Let $Q = Q(b) = \text{diag}(b_1, \cdots, b_N)$, $a = (a_1, \ldots, a_N)^T$, $q = (q_1, \ldots, q_N)^T$. A link from node i to node j is denoted ij and its transmission limit is C_{ij}. The vector of transmission limits is C_{max}. Then, the optimal power flow problem faced by the ISO can be written, e.g. Wu and Varaiya (1997), as:

$$
\begin{aligned}
\underset{q}{\text{minimize}} \quad & q^T Q q + a^T q \\
\text{subject to} \quad & q_1 + \cdots + q_N = 0 \\
& -C_{max} \leq \Phi q \leq C_{max} \\
& q_i \geq 0, \quad i = 1, \ldots, s, \quad q_i \leq 0, \quad i = s+1, \ldots, N
\end{aligned}
\tag{1}
$$

where Φ is a $K \times N$ matrix of distribution factors (Wood and Wollenberg, 1996) or injection shift factors (Liu and Gross, 2004); and K denotes the number of transmission lines in the network. The distribution factors give linear approximations of the first order sensitivities of power flows with respect to changes of net nodal injections. Distribution factors are determined by three sets of parameters: a reference node (always labelled node 0 in this paper), the topology of the network (connections between nodes by lines), and the electrical properties of lines such as susceptance. It worth pointing out that market outcomes are independent of the choice of the reference node.

Observe that the optimal power flow problem is a quadratic program with a nonempty feasible set; and has a strictly convex objective if each $b_i > 0$.

The next result is well known, e.g. (i) by discussion in Luo *et al.* (1996, pp.169-171) and (ii) by the more general results surveyed in Bonnans and Shapiro (1998, Section 5):

Lemma 1 Let C_{max} be a positive vector.

(i) Fix $b > 0$, and consider a as a parameter vector in (1). Then problem (1) is solvable for any a, its optimal solution $q^* = q^*(a)$ is unique, and $q^*(a)$ is a piecewise linear mapping in a.

(ii) Consider a and b as parameters in (1). For any $b > 0$ and any a, (1) is solvable, the optimal solution $q^* = q^*(a, b)$ is unique, and $q^*(a, b)$ is a locally Lipschitz mapping with respect to (a, b).

2.2 A bilevel game

Now, the question is how profit-seeking generators and consumers behave under such a market mechanism. We present a model from Berry *et al.* (1999), which is also considered in Backerman *et al.* (2000), that is designed to address this question.

Let $A_i q_i + B_i q_i^2$ be the actual cost incurred by generator i to generate $q_i \geq 0$ units of electricity. Similarly, $-A_j q_j - B_j q_j^2 = A_j(-q_j) - B_j(-q_j)^2$ is the actual utility received by consumer j when consuming $-q_j \geq 0$ units of electricity. Given participant i's bid (a_i, b_i) and quantity q_i, the market price at node i is the marginal price, i.e. the derivative $a_i + 2b_i q_i$ of its bid function at q_i. Thus a generator at node i receives revenue $(a_i + 2b_i q_i)q_i$, and its profit maximization problem is:

$$\begin{array}{ll} \underset{a_i, b_i, q}{\text{maximize}} & (a_i + 2b_i q_i)q_i - (A_i q_i + B_i q_i^2) \\ \text{subject to} & \underline{A}_i \leq a_i \leq \overline{A}_i \\ & \underline{B}_i \leq b_i \leq \overline{B}_i \\ & q = (q_1, \ldots, q_i, \ldots, q_N) \text{ solves (1) given } (a_{-i}, b_{-i}) \end{array} \tag{2}$$

where $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$, $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_N)$; and $\underline{A}_i$ ($\underline{B}_i$) and $\overline{A}_i$ ($\overline{B}_i$) are the lower bound and upper bound on a_i (b_i), respectively. For the rest of this paper, we assume that $0 \leq \underline{A}_i \leq A_i \leq \overline{A}_i$ and $0 < \underline{B}_i \leq B_i \leq \overline{B}_i$.

Formally, (2) is called a bilevel program, where the lower-level program is the parametric problem (1); thus $\{(2)\}_{i=1}^N$ is a bilevel game. In addition, since the lower-level problem is convex, we can represent it equivalently via its stationary conditions, which form an "equilibrium" system, as explained in §4.2. This converts the game to an EPEC; see Outrata (2003) for general formulation of EPECs.

3 Nash equilibria in the electricity market model

As suggested in Berry *et al.* (1999), and shown explicitly in Example 12 to follow, existence of pure strategy Nash equilibria is not guaranteed for the electricity market game $\{(2)\}_{i=1}^N$. However, given Lemma 1, the existence of mixed strategy Nash equilibria for $\{(2)\}_{i=1}^N$ is a straightforward consequence of a classical result Balder (1995, Proposition 2.1) for games with finitely many players, each of whom has a continuous payoff function and compact set of pure strategies.

Our main interest is the existence of (pure strategy) Nash equilibria of the game $\{(2)\}_{i=1}^{N}$: see Berry *et al.* (1999), Younes and Ilic (1999), Hobbs *et al.* (2000), where generators and consumers bid only a_i while leaving $b_i = B_i > 0$ fixed, or, equivalently, each player's bid has the form (a_i, B_i). This is referred to as the bid-a-only or bid linear-only scenario. The focus on bid-a-only games is for two main reasons. First, the bid-a-b game, in which each player bids (a_i, b_i), has multiple equilbria in general, which makes its use in economic comparisons fraught because it is not apparent which equilibrium to select, whereas the bid-a-only game and its bid-b-only counterpart both tend to have unique equilibria when these exist; see Hu (2003, Chapter 5). Second, the theoretical analysis of the bid-a-only game, to follow, is a reasonable surrogate for that of the bid-b-only game which is analysed in detail in Hu (2003, Chapter 5, 6). Nevertheless we do attempt to solve all three types of games in Section 5 because the modeling and computational framework makes it convenient to do so, and the bid-a-b games have a different numerical flavor from the others.

While computational methods or policy issues are the topics in papers cited above, we look at basic properties of the market model that relate to existence of Nash equilibria. We set up our notation in §3.1 and give sufficient conditions for existence of Nash equilibria in §3.2. In §3.3 we give examples of existence of Nash equilibria, or lack thereof.

3.1 Notation for the bid linear-only game scenario

For $i = 1, \ldots, N$, $b_i = B_i$ is fixed. That is, we assume B_i is known to the ISO and all players and, as a result, the quadratic coefficients are not strategic variables. The multi-strategy vector a may be written (a_i, a_{-i}) when considering player i. For example, the optimal dispatch referred to in Lemma 1 will be denoted by $q^*(a)$ or $q^*(a_i, a_{-i})$ in the context of player i, whose profit function is

$$f_i(a_i, a_{-i}) = (a_i - A_i)q_i^*(a_i, a_{-i}) + B_i q_i^*(a_i, a_{-i})^2. \tag{3}$$

This mapping is piecewise smooth on $\mathcal{A}$ by Lemma 1, where

$$\mathcal{A}_i = [\underline{A}_i, \overline{A}_i],\ \mathcal{A} = \prod_{i=1}^{N} \mathcal{A}_i.$$

The ith player's profit or utility maximization problem is

$$\underset{a_i}{\text{maximize}} \quad f_i(a_i, a_{-i}) \quad \text{subject to} \quad a_i \in \mathcal{A}_i. \tag{4}$$

In this paper we focus on games of complete information, that is, each player knows not only its own payoff functions, but other players' payoff functions including their strategy spaces as well. It is worth noting that the above game $\{(4)\}_{i=1}^{N}$ has a classical format in that each player's strategy set is convex and compact. In particular, the network and the associated pricing and dispatch problem are implicit rather than explicit, which allows a simple format of the game although the payoff functions are not smooth but piecewise smooth. We also note, for reference in the next subsection, that whatever bids a_{-i} are made by other players, Player i's strategy can always be chosen to make his or her profit nonnegative: $A_i \in \mathcal{A}_i$ from our assumptions in subsection 2.2, and (3) gives $f_i(A_i, a_{-i}) \geq 0$.

3.2 Existence of Nash equilibria

An often-cited sufficient condition for the existence of (pure strategy) Nash equilibria is quasi-concavity of players' profit functions on their *whole* strategy spaces. However, Theorem 2 below assumes, for each player, nonemptiness and convexity of the subset of its strategies with *nonnegative* profit, and quasi-concavity of that player's profit function over this subset. We omit the proof which is almost identical to that of the standard existence theorem; see, for example Myerson (1991, pp. 138-140).

Theorem 2 Given a game with (a finite number) N of players for which the ith player has the payoff function $f_i(x_i, x_{-i})$ and the strategy set X_i, assume the following conditions hold:

(i) the strategy space $X_i \subseteq \mathbb{R}^{n_i}$ is a nonempty compact set;

(ii) for each x_{-i} in $X_{-i} = \prod_{j:j\neq i} X_j$, the set $X_i(x_{-i}) := \{x_i \in X_i : f_i(x_i, x_{-i}) \geq 0\}$ is nonempty and convex;

(iii) for each $x_{-i} \in X_{-i}$, the function $f_i(\cdot, x_{-i})$ is quasi-concave on $X_i(x_{-i})$; and

(iv) $f_i(\cdot)$ is continuous in $\prod_{i=1}^{N} X_i$.

Then the game has at least one Nash equilibrium such that every participant's profit is nonnegative.

We begin the analysis of the game $\{(4)\}_{i=1}^{N}$ with some notation. Consider the mapping $a_i \mapsto q_i^*(a_i, a_{-i})$ which we denote $q_i^*(\cdot, a_{-i})$. Given a_{-i}, let $\partial_i q_i^*(a_i, a_{-i})$ denote the generalized gradient of Clarke (1993) of this mapping. (As an aside, we give the definition of generalized gradient $\partial f(x^0)$ of an arbitrary locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at $x^0 \in \mathbb{R}^n$, using the classical result that the function is differentiable almost everywhere; see Clarke (1993) for details. The generalized gradient can be defined as the convex hull of limit points of sequences $\{\nabla f(x^k)\}$ over all sequences $\{x^k\}$ converging to x^0 such that $\nabla f(x^k)$ exists for each k. The local Lipschitz property ensures that the generalized gradient is nonempty and compact.) Our first result is technical. Its proof can be given from first principles; see the online Appendix.

Proposition 3 For any participant $i = 1, \ldots, N$, its dispatch function $q_i^*(\cdot, a_{-i})$ is a non-increasing function of its own bid a_i given others' bid vector a_{-i}. Moreover, the directional derivative of $q_i^*(\cdot, a_{-i})$ along any direction $u \in \mathbb{R}$ satisfies

$$|q_i^{*\prime}(a_i, a_{-i}; u)| \;\leq\; |u|/(2B_i).$$

As a result, $\partial_i q_i^*(a_i, a_{-i}) \;\subseteq\; [-1/(2B_i), 0]$.

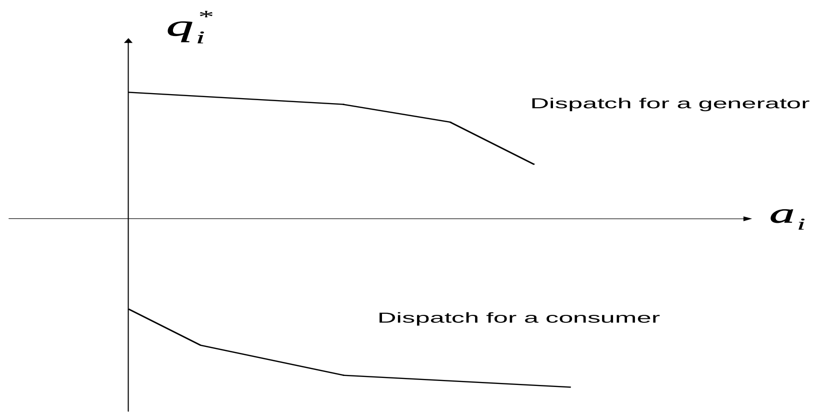

Figure 1: Sketch of piecewise linear dispatch for generators and consumers

By Lemma 1, there is a number W (depending on the number of pieces of the KKT system of (1)) such that, given a_{-i}, the derivative $q_i^{*\prime}(a_i, a_{-i})$ of $q_i^*(\cdot, a_{-i})$ exists except at the breakpoints $0 < w_1 = w_1(a_{-i}) \le w_2 = w_2(a_{-i}) \le \cdots \le w_W = w_W(a_{-i})$. Note that some of the points w_j may coincide for certain a_{-i} for which there are fewer than W breakpoints. We sketch the dispatch as a piecewise linear function of its bid in Figure 1 and we will fix these notations for the rest of this subsection. As an application of Proposition 3, Lemma 1, and Theorem 2, we have the following theorem, whose proof is given in the online Appendix.

Theorem 4 Suppose player i's dispatch $q_i^*(\cdot, a_{-i})$ satisfies the following conditions for each a_{-i} and breakpoint w_j with $f_i(w_j, a_{-i}) > 0$:

$$\text{if } i \text{ is a generator:} \quad \lim_{v \uparrow w_j} q_i^{*\prime}(v, a_{-i}) \ \ge \ \lim_{v \downarrow w_j} q_i^{*\prime}(v, a_{-i}), \tag{5}$$

$$\text{if } i \text{ is a consumer:} \quad \lim_{v \uparrow w_j} q_i^{*\prime}(v, a_{-i}) \ \le \ \lim_{v \downarrow w_j} q_i^{*\prime}(v, a_{-i}), \tag{6}$$

where $v \uparrow w_j$ means $v \to w_j$, $v < w_j$; and $v \downarrow w_j$ is similarly defined. Then the game $\{(4)\}_{i=1}^N$ has at least one Nash equilibrium such that each generator's profit and consumer's utility is nonnegative.

The next three results are applications of Theorem 4. In each case, the conditions (5) and (6) can be established, yielding existence of a Nash equilibrium. Such a Nash equilibrium may not be unique but we can nevertheless show that, at a given Nash equilibrium, each generator with positive profit has a unique optimal response to the other player's equilibrium strategies.

First, we apply Theorem 4 to uncongested networks where all players participate in the market.

Proposition 5 If, for any $a \in \mathcal{A}$, the ISO's dispatch yields uncongested flows and is nonzero for each player, then there exists a Nash equilibrium for $\{(4)\}_{i=1}^N$. Moreover, given other players' strategies at such a Nash equilibrium, the optimal strategy of each player is unique.

Proof Since for any bid a, every generator or consumer has nonzero dispatch, the KKT conditions of the ISO's problem (1) contains the following linear system:

$$a_1 + 2B_1 q_1 + \lambda = 0, \quad \cdots, \quad a_N + 2B_N q_N + \lambda = 0, \quad q_1 + \cdots + q_N = 0$$

where λ is the Lagrange multiplier corresponding to the constraint $q_1 + \cdots + q_N = 0$. As a result, $q_i^*(a_i, a_{-i})$ is a linear function for $(a_i, a_{-i}) \in \mathcal{A}$, so (5) and (6) hold trivially. Therefore, by Theorem 4, there is at least one Nash equilibrium. Uniqueness of each player's best response to the others' strategies follows if we have strict concavity of profit functions. In fact, q_i^* is strictly decreasing in a_i (see (8) for its formula), therefore, by Proposition 3, we have $-1/(2B_i) \leq q_i^{*\prime} < 0$. Strict concavity now results from (3). ■

Second, we consider the case where consumers are symmetric (identical) and are price takers (i.e. do not bid, rather their utility functions are known by all players and the ISO), but generators may have different generation costs. This is a weakening of a more typical assumption that generation costs are symmetric, e.g. Green and Newbery (1992), Klemperer and Meyer (1989). Here some generators may be out of the market, unlike Proposition 5. Using symmetric consumer utilities ensures that all consumers have nonzero consumption, or all have zero consumption, which makes the mathematical analysis tractable.

Proposition 6 Consider a network with capacities of all lines large enough so that there is no potential congestion. Assume that consumers have the same benefit function (i.e., they have the same A_i, B_i) which is known to the ISO and all players. Then there exists at least one Nash equilibrium for the game $\{(4)\}_{i=1}^N$. Moreover, with respect to any such Nash equilibrium, the optimal strategy of each generator with a positive profit is unique.

Proof It suffices to establish (5) for a given generator i by fixing a_{-i} and a breakpoint $w = w_j$ of $q_i^*(\cdot, a_{-i})$ such that $q_i^*(w, a_{-i}) > 0$, and considering the derivative $q_i^{*\prime}(v, a_{-i})$ for v near w but $v \neq w$.

In general, the vector (q, λ, μ) of optimal dispatches and the corresponding Lagrange multipliers satisfies the following KKT conditions of (1) without line capacity constraints:

$$\begin{aligned}
&a_1 + 2B_1 q_1 + \lambda + \mu_1 = 0 \cdots,\ a_N + 2B_N q_N + \lambda + \mu_N = 0,\ \ q_1 + \cdots + q_N = 0,\\
&q_k \mu_k = 0,\ q_k \geq 0,\ \mu_k \leq 0,\ \ \text{k: generator},\\
&q_k \mu_k = 0,\ q_k \leq 0,\ \mu_k \geq 0,\ \ \text{k: consumer}.
\end{aligned}$$

The KKT system can be used to derive the market price, which is uniform across the market, as

$$-\lambda(a_i, a_{-i}) \ = \ \Big(\sum_{k \in K} \frac{a_k}{2B_k}\Big)/\Big(\sum_{k \in K} \frac{1}{2B_k}\Big),$$

where $K = K(a_i, a_{-i}) = \{k : q_k^*(a_i, a_{-i}) \neq 0\}$, since $\mu_k = 0$ for $k \in K$. The KKT system then gives

$$-\lambda(a_i, a_{-i}) = a_k + 2B_k q_k^*(a_i, a_{-i}) \quad \text{for any } k \in K. \tag{7}$$

It follows that $q_i^*(\cdot, a_{-i})$ is differentiable at a_i if $i \in K$ and $K(v, a_{-i}) = K$ for v near a_i, with

$$q_i^{*\prime}(a_i, a_{-i}) = -\big(1 + L_i\big)/(2B_i) \text{ where } -1/L_i \ = \ \sum_{k \in K} B_i/B_k. \tag{8}$$

We need to establish some monotonicity properties. Suppose the generator i has $i \in K = K(a_i, a_{-i})$. From (7) with $k = i$, the generalized gradient of the market price with respect to a_i is the set $1 + 2B_i \partial_i q_i^*(a_i, a_{-i})$, which contains only nonnegative values by Proposition 3. That is, price is nondecreasing in a_i. Next consider a generator $k \notin K$, i.e. $q_k^*(a_i, a_{-i}) = 0$. We claim for $v < a_i$ that $q_k^*(v, a_{-i}) = 0$. For if $q_k^*(v, a_{-i}) > 0$ then (7) at $a_i = v$ and the monotonicity of price imply $q_k^*(v, a_{-i})$ is nondecreasing as $v \uparrow a_i$; this can only happen, given continuity of $q_k^*(\cdot, a_{-i})$ at a_i, if $q_k^*(v, a_{-i}) = 0$.

Now let w be a breakpoint as above. Then $q_i^*(\cdot, a_{-i})$ is differentiable at v near w with $v \neq w$. Suppose that generator i unilaterally raises its linear bid by a small amount from w to $v > w$. By continuity of the vector $q^*(\cdot, a_{-i})$, each quantity $q_k^*(v, a_{-i})$ stays positive for $k \in K(w, a_{-i})$, hence

$$K(v, a_{-i}) \;\supseteq\; K(w, a_{-i}) \;\text{ for } v > w,\ v \text{ near } w.$$

Suppose instead that player i unilaterally lowers its linear bid by a small amount to $v < w$. As before, we see that $K(v, a_{-i})$ contains $K(w, a_{-i})$, but we also claim the converse, namely

$$K(v, a_{-i}) \;=\; K(w, a_{-i}) \;\text{ for } v < w,\ v \text{ near } w.$$

To see this, first note, since consumers are symmetric and total production is positive for v near w, that each consumer k belongs to $K(v, a_{-i})$. Second, for a generator k with $q_k^*(w, a_{-i}) = 0$ we have already seen that $q_k^*(v, a_{-i}) = 0$ when $v < w$. The claimed equality follows. In light of these relationships and (8), with $a_i = v$ near but not equal to w and $K = K(v, a_{-i})$, the inequality (5) can be easily verified.

The uniqueness claim follows from the fact that all these participants' profit functions are strictly concave when their strategies are restricted to those with strictly positive dispatch. ■

Finally, we apply Theorem 4 to a two-node network with multiple generators and consumers at each node and a transmission capacity limit C on the only line between the two nodes. The line may be congested for some dispatches. This result has relevance, for example, in the New Zealand electricity market, which is a nodal pricing market where the South Island has 41% of the market generation capacity and 36% of market demand, while the North Island has the remainder of generation capacity and demand, and the two islands are connected by a single high voltage direct current link. See David Butchers & Associates (2001) for more details for the New Zealand market.

Proposition 7 Consider a network with two nodes and one line linking the two nodes with a transmission capacity limit $C > 0$ on the line. At each node, there may be multiple generators and/or consumers. If all consumers have the same benefit function, which is known to the ISO and all generators, and they do not bid strategically, then there exists at least one Nash equilibrium for the game $\{(4)\}_{i=1}^N$. Moreover, with respect to any such Nash equilibrium, the optimal strategy of each generator with a positive profit is unique.

Proof The proof is almost the same as that of Proposition 6. For example, let i be a generator, a_{-i} be fixed, and $\mathcal{S} = \mathcal{S}(a_i, a_{-i})$ denote the set of generators k such that $q_k^*(a_i, a_{-i}) > 0$. The KKT conditions for this problem yield

$$-\lambda(a_i, a_{-i}) \;=\; \Big(\sum_{k\in\mathcal{S}} \frac{a_k}{2B_k} + C\Big)/\Big(\sum_{k\in\mathcal{S}} \frac{1}{2B_k}\Big),$$

and $-\lambda(a_i, a_{-i}) = a_i + 2B_i q_i^*(a_i, a_{-i})$ if $i \in \mathcal{S}$. The proof can be completed as an exercise. ■

3.3 Effects of transmission constraints on the set of Nash equilibria

In this section, we present two simple networks (see Figure 2) to see the effect of congestion on the existence of Nash equilibria of the bid-a-only game $\{(4)\}_{i=1}^N$. In the two-node case, the transmission limit leads to a continuum of Nash equilibria, while the transmission limit in a three-node network may result in non-existence of any Nash equilibrium.

The calculation of Nash equilibria for these two networks follows the classical idea of finding the intersection of every individual's best response curves. The best response function of participant i is a possibly set-valued mapping which maps a_{-i} to the set $\text{argmax}_{\,a_i}\{f_i(a_i, a_{-i}) \,:\, a_i \in \mathcal{A}_i\}$. In the following examples, $\underline{A}_i = 0, \overline{A}_i = 100$.

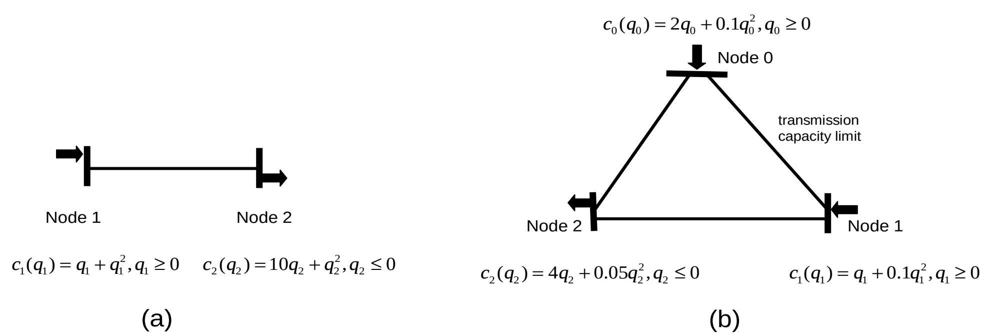

Figure 2: Two networks, location of generators and consumers and their cost/utility functions. Arrows show extraction or injection of electricity.

Example 8 (**two-node network, uncongested**) Consider a two-node network with one generator and one consumer and their true cost/benefit functions shown in Figure 2(a).

When there is no capacity limit on the line, Proposition 5 tells that there exists a Nash equilibrium and the equilibrium point is $(a_1^*, a_2^*) = (3.25, 7.75)$.

Example 9 (**two-node network, congested**) Now we consider the previous example with a capacity limit $C_{12} = 1$ on the line. Given a bid (a_1, a_2), the dispatches (a solution to problem (1)) are

$$q_1^* = -q_2^* = \frac{a_2 - a_1}{4} \text{ when } 0 \leq a_2 - a_1 \leq 4, \quad \text{or} \quad q_1^* = -q_2^* = 1, \text{ when } a_2 - a_1 \geq 4$$

or $q_1^* = -q_2^* = 0$ otherwise.

In fact, it can be shown that the best response mappings for the generator and the consumer are

$$a_1^*(a_2) = \begin{cases} [a_2, 100] & \text{if } 0 \le a_2 \le 1, \\ 0.3333a_2 + 0.6667, & \text{if } 1 \le a_2 \le 7, \\ a_2 - 4, & \text{if } a_2 \ge 7, \end{cases}$$

and

$$a_2^*(a_1) = \begin{cases} a_1 + 4, & \text{if } 0 \le a_1 \le 4, \\ 0.3333a_1 + 6.6667, & \text{if } 4 \le a_1 \le 10, \\ [0, a_1], & \text{if } a_1 \ge 10, \end{cases}$$

respectively. The intersection of the two best response mappings gives the set of Nash equilibria for the game, which is $\{(a_1, a_2) \in \mathbb{R}^2 : a_2 - a_1 = 4 \text{ and } 7 \le a_2 \le 8\}$.

Generally speaking, congested lines may lead to multiplicity of local Nash equilibria. The multiplicity may cause ambiguity of economic conclusions based on the equilibria. See debates on this between Stoft (1999) and Oren (1997) for a Cournot-Nash model representing an electricity market with two-node and three-node networks.

In general, we have the following corollary, which is different from Proposition 7 in that the consumer in Corollary 10 participates in strategic bidding and multiple generators are allowed in Proposition 7.

Corollary 10 Consider any two-node, one-line network with only one generator and one consumer located at different nodes. There always exists at least one Nash equilibrium for $\{(4)\}_{i=1}^2$ regardless of capacity limit on the line.

Proof If there is no capacity limit on the line, the corollary is true by Proposition 5. Now assume that there is a capacity limit C on the line. Consider condition (6) for the consumer (a similar proof holds for the generator). Fix the generator's bid $a_1 \ge 0$, if the capacity limit on the line is just reached by a dispatch given bid (a_1, w), i.e. $q_1^* = -q_2^* = C$ when $a_2 \ge w$ and $q_1^* = -q_2^* < C$ when $a_2 < w$, where w is a bid by the consumer. So $q_2^{*\prime} = 0$ for $a_2 > w$ since $q_2^*(a_1, a_2) = -C$ for $a_2 \ge w$. Similarly $q_2^{*\prime} \le 0$ for $a_2 < w$ (see also Proposition 3). Therefore (6) holds for the consumer. Hence the corollary follows from Theorem 4. ■

Example 11 (**three-node network, uncongested**) Consider a three-node network given in Figure 2(b). There are two competing generators and one consumer who does not compete but rather provides a passive demand function by revealing her true benefit function to the ISO and the two generators. The true cost and benefit functions are shown in Figure 2(b). Node 2 is chosen as the reference node and therefore its distribution factors to all three lines are zero. The distribution factors of node 0 to lines $01, 02, 12$ are $-0.25, -0.75, -0.25$ respectively, and those of node 1 to above lines are $0.25, -0.25, -0.75$.

Without any transmission limits on the three lines, we know by Proposition 6 that $\{(4)\}_{i=1}^2$ has a Nash equilibrium, which is $(a_1^*, a_2^*) = (2.2321, 1.4821)$ shown in Figure 3.

Example 12 (**three-node network, congested**) Next, assume that there is a capacity limit $C_{01} = 0.9$ on the line between node 0 and node 1.

The dispatches (only two possible cases in the KKT system of problem (1) are presented here) are

$$q_0^* = 10 - 3.75a_0 + 1.25a_1, q_1^* = 10 + 1.25a_0 - 3.75a_1, q_2^* = -20 + 2.5a_0 + 2.5a_1$$

when $a_0 - a_1 \leq 0.72$ (uncongested situation) or

$$q_0^* = 8.2 - 1.25a_0 - 1.25a_1, q_1^* = 11.8 - 1.25a_0 - 1.25a_1, q_2^* = -20 + 2.5a_0 + 2.5a_1$$

when $a_0 - a_1 \geq 0.72$ (congested situation).

It is easy to see that condition (5) in Theorem 4 is not true for generator 0:

$$q_0^{*\prime}(v, a_1) \mid_{v<a_1+0.72} = -3.75 \not\geq -1.25 = q_0^{*\prime}(v, a_1) \mid_{v>a_1+0.72} .$$

Therefore, Theorem 4 cannot apply to the game in this example.

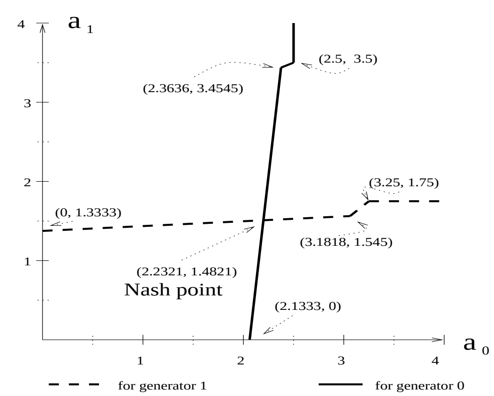

Figure 3: Best response curves without transmission limit

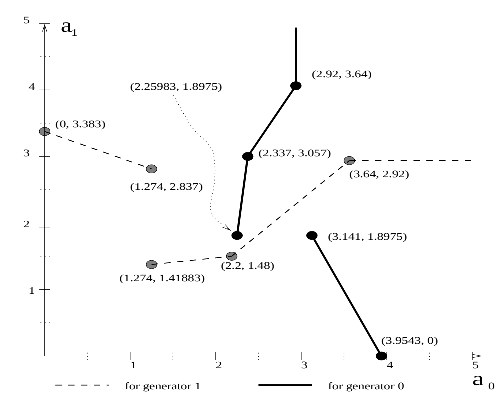

Figure 4: Best response curves with one transmission limit

In fact, the best response functions for the two generators are

$$a_0^*(a_1) = \begin{cases} 3.9543 - 0.4286a_1, & \text{if } 0 \leq a_1 \leq 1.8975, \\ 0.0667a_1 + 2.1333, & \text{if } 1.8975 \leq a_1 \leq 3.057, \\ a_1 - 0.72, & \text{if } 3.057 \leq a_1 \leq 3.64, \\ 2.92, & \text{if } a_1 \geq 3.64, \end{cases}$$

and

$$a_1^*(a_0) = \begin{cases} 3.383 - 0.4286a_0, & \text{if } 0 \leq a_0 \leq 1.274, \\ 0.0667a_0 + 1.3333, & \text{if } 1.274 \leq a_0 \leq 2.2, \\ a_0 - 0.72, & \text{if } 2.2 \leq a_0 \leq 3.64, \\ 2.92, & \text{if } a_0 \geq 3.64. \end{cases}$$

It can be clearly seen that there is no Nash equilibrium for the game in this example from Figure 4, because there is no intersection point for their best response curves.

4 EPEC formulations of bilevel games

In this section we start with a bilevel game model that generalizes the above electricity market models and, by writing each player's problem as an MPEC, reformulate the game as an EPEC; see §4.1. In §4.2 we propose *local Nash* and *Nash stationary* equilibria as more tractable concepts than the standard Nash equilibria. Nash stationary equilibria can be posed as solutions of a mixed complementarity problem, which is the subject of §4.3. Finally, §4.4 returns to the electricity market game $\{(4)\}_{i=1}^N$ to give conditions under which Nash stationary points are, in fact, local Nash equilibria.

4.1 A general bilevel game and its EPEC formulation

Given x_{-i}, player i's problem in the general bilevel game is

$$\begin{array}{ll} \underset{x_i}{\text{maximize}} & \phi_i(x,y) \\ \text{subject to} & g_i(x_i,y) \geq 0, \qquad h_i(x_i,y) = 0 \\ & y \text{ solves } \left\{ \begin{array}{ll} \underset{y}{\text{minimize}} & F(x,y) \\ \text{subject to} & H(x,y)=0,\ G(x,y) \geq 0 \end{array} \right. \end{array} \tag{9}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^{n_1+\cdots+n_N}$, $y = (y_1, \ldots, y_N) \in \mathbb{R}^{m_1+\cdots+m_N}$; the scalar-valued functions ϕ_i, F are smooth, and g_i, h_i, G and H are smooth vector functions of dimensions w_i, v_i, s and t, respectively for $i = 1, \ldots, N$. Obviously, (9) covers bilevel games (2) and (4) in previous sections.

Given x, it is to be expected that a solution y of the lower-level problem in (9) is a stationary point of the lower-level problem, i.e. solves the associated Karush-Kuhn-Tucker (KKT) system which involves multipliers $\lambda \in \mathbb{R}^s$ and $\mu \in \mathbb{R}^t$:

$$\begin{array}{l} \nabla_y F(x,y) - \nabla_y G(x,y)^T \mu + \nabla_y H(x,y)^T \lambda = 0 \\ H(x,y) = 0,\ \ 0 \leq G(x,y) \ \perp \ \mu \geq 0 \end{array} \tag{10}$$

where $\perp$ denotes orthogonality: $G(x,y)^T\mu = 0$. A constraint qualification is a sufficient condition for a solution of the lower-level problem to have KKT multipliers. One possible constraint qualification requires linearity of the constraint mappings G and H, which is the case in the optimal power flow problem (1). Conversely if, given x, the lower-level problem is convex (as (1) is), i.e. F and each component function of G are convex in y and H is affine in y, then any KKT solution (y, λ, μ) of (10) gives a global minimizer y of the lower-level problem of (9).

Within the bilevel program (9), replacing the lower-level optimization problem by (10) creates an MPEC, indeed a mathematical program with complementarity constraints or MPCC:

$$\begin{array}{ll} \underset{x_i,y,\mu,\lambda}{\text{maximize}} & \phi_i(x,y) \\ \text{subject to} & g_i(x_i,y) \geq 0, \qquad h_i(x_i,y) = 0 \\ & (y, \lambda, \mu) \text{ solves (10).} \end{array} \tag{11}$$

The corresponding EPEC is the game $\{(11)\}_{i=1}^{N}$; it well might be called an equilibrium program with complementarity constraints or EPCC.

When comparing the bilevel program (9) to the MPEC (11) it is helpful to suppose, for each x, that the lower-level KKT system (10) has a unique solution $(y(x), \lambda(x), \mu(x))$. This makes it obvious that there is an equivalence between the associated bilevel game and EPEC, cf. Proposition 16 in the next subsection. Uniqueness of $y(x)$ follows if the lower-level optimization problem is a convex problem with a nonempty feasible set (in y given x) and a strictly convex objective function $F(\cdot, x)$. Uniqueness of the multipliers corresponding to $y(x)$ follows if the standard linear independence constraint qualification, LICQ, holds for the lower-level optimization problem. The LICQ requires that the active constraints of $H(x, y) = 0$, $G(x, y) \geq 0$ have linearly independent gradients with respect to y at $y(x)$.

Even with uniqueness of $y(x)$, a difficulty previously discussed is that this mapping is generally nonsmooth, cf. Lemma 1, and may induce nonconcavity in the composite objective function $x_i \mapsto \phi_i(x, y(x))$, hence lead to lack of Nash equilibria. Our computational strategy, rather than attempting to demonstrate or disprove existence of a Nash equilibrium, will be to apply robust mathematical programming software as a heuristic for finding points that are likely to be Nash equilibria. That is, we will search for joint strategies x such that, for each i, x_i is stationary for MPEC (11) and, if so, then check whether x_i is a local solution of the MPEC.

4.2 Local Nash equilibria and Nash stationary equilibria

A local solution of (11), given $x_{-i} = x_{-i}^*$, is a point $(x_i^*, y^*, \lambda^*, \mu^*)$ such that for some neighborhood $\mathcal{N}_i$ of this point, $(x_i^*, y^*, \lambda^*, \mu^*)$ is optimal for (11) with the extra constraint $(x_i, y, \lambda, \mu) \in \mathcal{N}_i$.

Definition 13 A strategy vector $(x^*, y^*, \lambda^*, \mu^*)$ is called a *local Nash equilibrium* for the EPEC $\{(11)\}_{i=1}^{N}$ if, for each i, $(x_i^*, y^*, \lambda^*, \mu^*)$ is locally optimal for the MPEC (11) when $x_{-i} = x_{-i}^*$.

We may refer to a global Nash equilibrium, meaning a Nash equilibrium, to distinguish it from a local Nash equilibrium. While local Nash equilibria seem deficient relative to their global counterparts, we argue that they have some meaning. Local optimality may be sufficient for the satisfaction of players given that global optima of nonconcave maximization problems are difficult to identify. Putting it differently, limits to rationality or knowledge of players may lead to meaningful local Nash equilibria. For instance, generators may only optimize their bids locally (via small adjustments) due to operational constraints or general conservativeness. Another limit to rationality is implicit in the simple nature of the model $\{(11)\}_{i=1}^{N}$ explored in Section 3, e.g., a DC model of an electricity network is an approximation derived by linearizing a nonlinear AC power flow model in a steady state. Since very different bids may be better studied using different DC approximations, it is not clear how much more attractive a global Nash equilibrium would be relative to a local Nash equilibrium of this model.

A further challenge, from the computational point of view, is how to find a local optimizer for a single player. We adopt the classical strategy from nonlinear optimization in which a system of stationary conditions, that is necessarily satisfied at a locally optimal point, provides the main criterion for recognizing local optimality. That is, our computational approach to EPECs will be to search for points that satisfy stationary conditions for each player, and then to verify local optimality if possible. Therefore we further relax the equilibrium solution concept to allow for a strategy vector $(x^*, y^*, \lambda^*, \mu^*)$ such that each $(x_i^*, y^*, \lambda^*, \mu^*)$ is stationary, in a sense we define next, for player i's MPEC.

Given x_{-i}, the MPEC (11) is equivalent to a nonlinear program, NLP,

$$
\begin{array}{lll}
\underset{x_i,y,\lambda,\mu}{\text{maximize}} & \phi_i(x,y) & \\
\text{subject to} & g_i(x_i,y) \geq 0 & : \xi_i^g \geq 0 \\
 & h_i(x_i,y) = 0 & : \xi_i^h \\
 & \nabla_y F(x,y) - \nabla_y G(x,y)^T \mu + \nabla_y H(x,y)^T \lambda = 0 & : \eta_i \\
 & H(x,y) = 0 & : \xi_i^H \\
 & G(x,y) \geq 0 & : \xi_i^G \geq 0 \\
 & \mu \geq 0 & : w_i \geq 0 \\
 & G(x,y) * \mu = 0 & : \zeta_i
\end{array}
\tag{12}
$$

where $*$ denotes the Hadamard or componentwise product of vectors, i.e. $G(x,y) * \mu = (G_1(x,y)\mu_1, \ldots, G_s(x,y)\mu_s)$. The vectors at right, ξ_i^g etc., are multipliers that will be used later.

Thanks to recent developments on MPECs, we will see that the following stationary condition, which is developed further in §4.3, is sensible for the EPEC $\{(11)\}_{i=1}^N$. In the definition we use the term "point" when we could equally use "strategy" or "equilibrium".

Definition 14 A strategy $(x^*, y^*, \lambda^*, \mu^*)$ is called a *Nash stationary point* of the EPEC $\{(11)\}_{i=1}^N$ if, for each $i = 1, \ldots, N$, the point $(x_i^*, y^*, \lambda^*, \mu^*)$ is stationary for (12) with $x_{-i} = x_{-i}^*$.

The definition of Nash stationary points is justified by Proposition 16, to follow, under the following constraint qualification. First recall, given a feasible point of a system of equality and inequality constraints, that an active constraint is a scalar-valued constraint that is satisfied as an equality.

Definition 15 Let (x, y, λ, μ) be feasible for the EPEC $\{(11)\}_{i=1}^N$.

(i) Given a player index i, the *MPEC active constraints* at (x_i, y, λ, μ) are the active constraints of (11) ignoring the orthogonality condition denoted $\perp$ (equivalently, the active constraints of (12) excluding the last block-equation $G(x,y) * \mu = 0$).

(ii) *MPEC linear independence constraint qualification*, MPEC-LICQ, holds for (11) at (x_i, y, λ, μ) if the gradients, with respect to these variables, of the MPEC active constraints at the point (x, y, λ, μ) are linearly independent.

(iii) We say that the MPEC-LICQ holds for $\{(11)\}_{i=1}^N$ at (x, y, λ, μ) if, for each player i, the MPEC-LICQ holds for (11) at (x_i, y, λ, μ).

Proposition 16 Let $(x^*, y^*, \lambda^*, \mu^*)$ be feasible for the EPEC $\{(11)\}_{i=1}^N$ and consider the following statements:

(i) (x^*, y^*) is a local Nash equilibrium of the bilevel game $\{(9)\}_{i=1}^N$.

(ii) $(x^*, y^*, \lambda^*, \mu^*)$ is a local Nash equilibrium of the EPEC $\{(11)\}_{i=1}^N$.

(iii) $(x^*, y^*, \lambda^*, \mu^*)$ is a Nash stationary point of the EPEC $\{(11)\}_{i=1}^N$.

Statement (i) implies statement (ii). Statement (ii) implies statement (iii) if the MPEC-LICQ holds for the EPEC at $(x^*, y^*, \lambda^*, \mu^*)$.

The proof of this proposition is available in the online appendix of this paper.

Remark 17

(i) NLPs that are formulated from MPECs, as above, are unusually difficult in that they do not satisfy constraint qualifications that are standard in NLP theory. This difficulty is alleviated by the MPEC-LICQ. Without the MPEC-LICQ, however, it is possible to have a local minimum of (12) that is not stationary, hence a local or global Nash equilibrium of the associated EPEC $\{(11)\}_{i=1}^N$ that is not Nash stationary.

(ii) According to Scholtes and Stöhr (2001), the MPEC-LICQ holds generically for MPECs, that is, at all feasible points of "most" problems. This result combined with the above proposition does not promise, but does give us reasonable hope of finding Nash points, if they exist, by identifying Nash stationary points.

(iii) Stationarity of $(x_i^*, y^*, \lambda^*, \mu^*)$ for (12), where $x_{-i} = x_{-i}^*$, is equivalent by Anitescu (2000) to a kind of MPEC stationarity called *strong stationarity*. See Fletcher *et al.* (2002), Scheel and Scholtes (2000) for related discussions of stationarity conditions in MPECs.

4.3 Mixed complementarity problem formulation

We turn to a (mixed) CP formulation of the Nash stationary point conditions of an EPEC. This formulation is an aggregation of the KKT systems of (12) for each player. We call this the *complementarity problem (CP) formulation of the EPEC* $\{(11)\}_{i=1}^N$; it is referred to in Hu (2003), Hu *et al.* (2004) as the *All-KKT system*.

The Lagrangian function for player i's MPEC (12) is, omitting the arguments (x, y) from the functions on the right:

$$
\begin{aligned}
&L_i(x, y, \lambda, \mu, \xi_i^g, \xi_i^h, \eta_i, \xi_i^G, \xi_i^H, \zeta_i) \\
&\quad = \phi_i + g_i^T \xi_i^g + h_i^T \xi_i^h + (\nabla_y F - \nabla_y G^T \mu + \nabla_y H^T \lambda)^T \eta_i \\
&\qquad + \mu^T w_i + H^T \xi_i^H + G^T \xi_i^G + (G * \mu)^T \zeta_i.
\end{aligned}
$$

The CP formulation of the EPEC $\{(11)\}_{i=1}^N$ is:

$$\left.\begin{array}{rcl} 0 & = & \nabla_{(x_i,\,y,\lambda,\mu)} L_i \\ 0 & \leq & \xi_i^g \perp g_i(x_i, y) \geq 0, \\ 0 & = & h_i(x_i, y) \\ 0 & \leq & \xi_i^G \perp G(x, y) \geq 0 \\ 0 & \leq & \mu \perp w_i \geq 0 \end{array}\right\} \quad \forall i \tag{13.1}$$

$$\left.\begin{array}{rcl} 0 & = & \nabla_y F(x, y) - \nabla_y G(x, y)^T \mu + \nabla_y H(x, y)\lambda \\ 0 & = & H(x, y), \\ 0 & = & G(x, y) * \mu. \end{array}\right\} \tag{13.2}$$

where $\xi_i^g, \xi_i^h, \xi_i^H, \xi_i^G, \eta_i, w_i, \zeta_i$ are the Lagrange multiplier vectors that are listed in the rightmost column in (12). All players share the second group of constraints, (13.2); there is no need for it to appear more than once in the CP formulation.

Note that by ordering the vectors

$$(x_i, \eta_i, \xi_i^H, \zeta_i), \xi_i^g, \xi_i^h, \xi_i^G, w_i, \quad \forall i$$

we match their respective dimensions to the number of equations or complementarity relations of the blocks of (13.1), and likewise, y, λ, μ match the blocks of (13.2). That is, equations (13.1-2) define a mixed complementarity problem to which a standard solver like PATH (Dirkse and Ferris, 1995; Ferris and Munson, 1999) can be applied. See Ehrenmann (2004a) for an alternative CP formulation.

Applying PATH to the CP formulation of an EPEC is suggested by the success that some standard NLP methods have had in solving MPECs written as NLPs (like (12)). The sequential quadratic programming method has proven to be especially effective (Anitescu, 2000; Fletcher and Leyffer, 2004; Fletcher *et al.*, 2002). This particularly motivates the application of PATH because its underlying methodology (Ralph, 1994; Robinson, 1994) is the analog, for CPs, of the sequential quadratic programming method.

4.4 Local Nash equilibria of bid-a-only games

Given a Nash stationary point, one would like to check a further optimality condition to see if it is actually a (local) Nash equilibrium. For instance, a stationary point of (12) is an isolated local minimizer if a second-order sufficient condition holds at that point; see Bonnans and Shapiro (1998). Here we revisit electricity market games, namely the bid-a scenario $\{(4)\}_{i=1}^N$, to show that Nash stationary points are local Nash points under reasonable conditions.

For brevity we will not address the bid-b-only scenario, and instead refer the reader to Hu (2003), which shows how NLP sensitivity analysis can be applied to check second-order sufficient optimality conditions via second-order parabolic directional derivatives, see Ben-Tal and Zowe (1982), of the objective function, and preliminary numerical results for which this checking usually verifies local optimality of each player's equilibrium bid.

To speak about Nash stationary equilibria for the bid-a-only game, we need to define the MPEC faced by player i which is derived from (2) by setting $b_i = B_i$:

$$
\begin{array}{ll}
\underset{a_i, q, \lambda, \underline{\mu}, \overline{\mu}, \nu}{\text{maximize}} & (a_i - A_i)q_i + B_i q_i^2 \\
\text{subject to} & \underline{A}_i \le a_i \le \overline{A}_i \\
& q_1 + \cdots + q_N = 0 \\
& \left.\begin{array}{l}
a_j + 2B_j q_j + \lambda + \displaystyle\sum_{\ell \in \mathcal{L}} \Phi_{\ell,j}(\overline{\mu}_\ell - \underline{\mu}_\ell) - \nu_j = 0 \\
0 \le q_j \;\perp\; \nu_j \ge 0, \quad \text{if player } j \text{ is a generator} \\
0 \ge q_j \;\perp\; \nu_j \le 0, \quad \text{if player } j \text{ is a consumer}
\end{array}\right\} j = 1, \ldots, N \\
& \left.\begin{array}{l}
0 \le C_\ell + \sum_{k=1}^N \Phi_{\ell,k} q_k \;\perp\; \underline{\mu}_\ell \ge 0 \\
0 \le C_\ell - \sum_{k=1}^N \Phi_{\ell,k} q_k \;\perp\; \overline{\mu}_\ell \ge 0
\end{array}\right\} \ell \in \mathcal{L}.
\end{array}
\tag{13}
$$

Denote by $\underline{\mu}, \overline{\mu}, q$, and ν the vectors whose respective components are $\underline{\mu}_\ell, \overline{\mu}_\ell, q_j$, and ν_j.

Theorem 18 Let $(a^*, q^*, \lambda^*, \underline{\mu}^*, \overline{\mu}^*, \nu^*)$ be a Nash stationary point for the EPEC $\{(13)\}_{i=1}^N$. If the LICQ holds for the optimal power flow problem (1) at $q = q^*$, with $a = a^*$, then

(i) $(a^*, q^*, \lambda^*, \underline{\mu}^*, \overline{\mu}^*, \nu^*)$ is a local Nash equilibrium of the EPEC $\{(13)\}_{i=1}^N$;

(ii) a^* is a local Nash equilibrium for the electricity market game $\{(4)\}_{i=1}^N$; and

(iii) each generator's profit or consumer's utility $f_i(a_i^*, a_{-i}^*)$ is nonnegative.

The proof of this result, given in the appendix uses optimality conditions for nonsmooth problems like (4) that we will sketch below. It will be convenient to develop our results starting from the general bilevel profit maximization (9) under some simplifying assumptions. The objective ϕ_i is required to be smooth as previously and we further assume that

(a) each of the upper-level constraint mappings g_i and h_i is an affine function of x_i only;

(b) F is quadratic in (x, y) and strictly convex in y for each fixed x; and

(c) G and H are affine.

A consequence of assumptions (b) and (c) is that the lower-level solution y is uniquely defined by x as a piecewise linear function $y(x)$, cf. Lemma 1. (If, in addition, the LICQ holds analogous to Theorem 18, then the multipliers $\lambda(x)$, $\mu(x)$ exist uniquely as piecewise smooth functions of x near x^*.) Similar to our formulation for the bid-a-only game in §3.1, uniqueness of $y(x)$ allows us to reformulate each player's problem (9) as an *implicit program* as discussed in Luo *et al.* (1996):

$$\begin{array}{ll} \underset{x_i}{\text{maximize}} & f_i(x_i, x_{-i}) \\ \text{subject to} & g_i(x_i) \geq 0, \qquad h_i(x_i) = 0 \end{array} \tag{14}$$

where

$$f_i(x_i, x_{-i}) = \phi_i(x, y(x)). \tag{15}$$

The constraints are now linear or polyhedral. The objective function is the composition of a smooth with a piecewise linear mapping, hence it is piecewise smooth and both locally Lipschitz and directionally differentiable, i.e. B-differentiable. We denote its partial directional derivative with respect to x_i in a direction d_i (of the same dimension n_i) by $f_i'(x_i, x_{-i}; d_i)$.

Suppose x_{-i} is given as x_{-i}^* and x_i^* is feasible for (14). A tangent direction to the feasible set at x_i^* is a vector d_i that can be written as the limit, as $k \to \infty$, of a convergent sequence $(x_i^k - x_i^*)/\tau_k$ where each x^k is feasible and each τ_k is a positive scalar. (Since the feasible set is polyhedral, the tangent directions are precisely positive scalings of vectors $x_i - x_i^*$ where x_i is feasible and near x_i^*.) If x_i^* is a local maximum of (14), it is well known and easy to check (Shapiro, 1990) that $f_i'(x_i^*, x_{-i}^*; d_i) \leq 0$ for each tangent direction d_i described. This necessary optimality condition is called *B-stationarity* of x_i^*. B-stationarity is also a term applied to MPECs, see Scheel and Scholtes (2000).

We present a relationship between stationary points of MPECs of the form (11) and B-stationary points of their related implicit programs (14). Its proof is technical and in the spirit of the analysis of implicit programs given in Luo *et al.* (1996, Chapter 4) and is available in the online appendix of this paper. As above, the stationarity analysis of Anitescu (2000) for MPECs written as NLPs is used to connect MPEC strong stationarity of Scheel and Scholtes (2000) of each player's profit maximization problem to the conditions defining Nash stationary points.

Proposition 19 Let the above assumptions (a)-(c) hold for the general EPEC $\{(11)\}_{i=1}^N$ and $(x^*, y^*, \lambda^*, \mu^*)$ be a Nash stationary point of this game. If the LICQ holds with respect to y at $y = y^*$ for the constraints $H(x^*, y) = 0$ and $G(x^*, y) \geq 0$, then, for each i, x_i^* is B-stationary for the implicit program (14) with $x_{-i} = x_{-i}^*$.

If ϕ_i is twice continuously differentiable then f_i also has second directional derivatives (Ben-Tal and Zowe, 1982), including the following,

$$f_i''(x_i, x_{-i}; d_i, d_i) \;=\; \lim_{t \to 0^+} \frac{f_i(x_i + td_i + t^2 d_i, x_{-i}) - f_i(x) - tf_i'(x_i, x_{-i}; d_i)}{t^2}$$

which yields a Taylor-type expansion:

$$f_i(x_i + td_i, x_{-i}) \;=\; f_i(x_i, x_{-i}) + tf_i'(x_i, x_{-i}; d_i) + t^2 f_i''(x_i, x_{-i}; d_i, d_i) + o(t^2).$$

Suppose f_i is piecewise quadratic, as it is in the particular case of the bid-a-only game. Then the above Taylor expansion holds exactly without the $o(t^2)$ error term. This means that x_i is a local maximum of (14) if it is B-stationary and satisfies the second-order condition that $f_i''(x_i, x_{-i}; d_i, d_i) \leq 0$ for each feasible direction d_i with $f_i'(x_i, x_{-i}; d_i) = 0$. This fact is used in the proof which appears in the Online Appendix.

5 Numerical methods and examples

We first describe the diagonalization method, which is familiar in the computational economics literature from the days of PIES energy model, Ahn and Hogan (1982), and, more recently, in Cardell *et al.* (1997). Then we give computational results for randomly generated electricity market games using both the diagonalization framework, which is implemented via standard nonlinear programming methods, and the CP formulation of §4.3 which requires a complementarity solver.

5.1 Diagonalization method

Diagonalization is a kind of fixed-point iteration in which players cyclically or in parallel update their strategies while treating other players' strategies as fixed.

Diagonalization method for the game $\{(12)\}_{i=1}^N$

(i) We are given a starting point $x^{(0)} = (x_1^{(0)}, \ldots, x_N^{(0)})$, the maximum number of iterations K, and a convergence tolerance $\varepsilon \geq 0$.

(ii) We are given the current iterate $x^{(k)}$. *For* $i = 1$ *to* N*:*
Player i solves problem (12) while holding $x_j = z_j$ for $j < i$ and $x_j = x_j^{(k)}$ for $j > i$. Denote the x_i-part of the optimal solution by z_i.

(iii) Let $x^{(k+1)} = (z_1, \ldots, z_N)$. *Check convergence and stopping condition*: if $\|x_i^{(k)} - x_i^{(k-1)}\| < \varepsilon$ for $i = 1, \ldots, N$, then accept the point and *stop*; else if $k < K$, then increase k by one and *go to* (ii) and *repeat* the process; else if $k = K$ *stop* and output "no equilibrium point found".

Note that the diagonalization method may fail to find a Nash equilibrium even if there exists such a point.

In step (ii), we need an algorithm to solve the MPEC (12). There has been a lot of recent progress in numerical methods for solving MPECs, e.g. Anitescu (2000), Fletcher *et al.* (2002), Hu and Ralph (2004) and references therein, resulting in several different but apparently effective approaches. We could simply apply an off-the-shelf NLP method to an MPEC formulated as an NLP such as (12). An alternative is to use an iterative scheme like the regularization method, see Scholtes (2001). Regularization, like the penalty (Hu and Ralph, 2004) and smoothing (Fukushima and Pang, 2000) methods, re-models an MPEC as a one-parameter family of better-behaved nonlinear programs, and solves a sequence of the latter while driving the parameter to zero, at which point the NLP becomes equivalent to the MPEC. This leads us to call each Diagonalization iteration an outer iteration; and each iteration within step (ii) of Diagonalization an inner iteration.

Step (ii) by regularization method (*inner iteration* for ith player in step (ii))

(ii-1) We are given $x^{(k)}$, z_j for $j < i$, and a sequence of (typically, decreasing) regularization parameters $\rho_0^{(k)}, \rho_1^{(k)}, \ldots, \rho_M^{(k)}$, i.e. a fixed number of iterations to execute, where $\rho_m^{(k)}$ bounds the total violation of complementarity in sub-iteration m. Let $m = 0$, $\xi_i^0 = x_i^{(k)}$.

(ii-2) Given the regularization parameter ρ_m, $x_j = z_j$ for $j < i$ and $x_j = x_j^{(k)}$ for $j > i$, and the starting point $x_i = \xi_i^m$, solve the following nonlinear program

$$\begin{array}{ll} \underset{x_i, y, \lambda, \mu}{\text{maximize}} & f_i(x, y) \\ \text{subject to} & g_i(x_i, y) \geq 0, h_i(x_i, y) = 0 \\ & \nabla_y F(x, y) - \nabla_y G(x, y)^T \mu + \nabla_y H(x, y)^T \lambda = 0 \\ & H(x, y) = 0 \\ & G(x, y) \geq 0, \quad \mu \geq 0, \quad G_j(x, y)\mu_j \leq \rho_m^{(k)}, j = 1, \ldots, s. \end{array} \tag{16}$$

If $m < M$, let ξ_i^{m+1} be the x_i part of the solution, let $m = m + 1$, and repeat **(ii-2)**.
Else let $x_i^{(k+1)}$ be the x_i part of the solution, and stop.

For $m > 1$, the solution values of y, λ and μ at sub-iteration m can be used, in addition to $x_i = \xi_i^m$, to initialize the corresponding variables in sub-iteration $m + 1$. We focus on x_i however because the other variables are assumed to be defined implicitly by it.

We give a convergence result for the diagonalization method where, at each iteration, step (ii) is carried out using regularization. This result is essentially a corollary of the regularization convergence analysis of Scholtes (2001) which requires that the "solution" of (16), in (ii-2), is a stationary point that satisfies a second-order necessary condition as described in Scholtes (2001). Assuming the diagonalization method produces a convergent subsequence, with limit $(x_i^*, y^*, \lambda_i^*, \mu_i^*)$ and limiting multipliers ξ_i^g, ξ_i^h, etc as laid out in (12), a technical condition called *Upper-Level Strict Complementarity*, ULSC, is needed: for each component j of μ_i^* and $G(x^*, y^*)$ such that $(\mu_i^*)_j = G_j(x^*, y^*) = 0$, neither $(\xi_i^G)_j$ nor $(\xi_i^\mu)_j$ is zero.

Proposition 20 Let M be fixed in (ii-1)–(ii-2) for each outer iteration k of the diagonalization method. Assume $0 < \rho_M^{(k)} \to 0$. Suppose $x^{(m)} = (x_i^{(m)})_{i=1}^N$, satisfies the following conditions:

(i) there exist $Y^{(m)} = (y_i^{(m)})_i$, $\lambda^{(m)} = (\lambda_i^{(m)})_i$ and $\mu^{(m)} = (\mu_i^{(m)})_i$ such that, for each $i = 1, \ldots, N$, the tuple $(x_i^{(m)}, y_i^{(m)}, \lambda_i^{(m)}, \mu_i^{(m)})$ is a stationary point satisfying a second-order necessary condition for (16) with $x_{-i} = (x_1^{(m-1)}, \ldots, x_{i-1}^{(m-1)}, x_{i+1}^{(m-1)}, \ldots, x_n^{(m-1)})$;

(ii) the sequence $(x^{(m)}, Y^{(m)}, \lambda^{(m)}, \mu^{(m)})$ has a limit point $(x^*, Y^*, \lambda^*, \mu^*)$ as $m \to \infty$, where $x^* = (x_i^*)_i$, $Y^* = (y_i^*)_i$ etc.; and

(iii) for each i, the MPEC-LICQ and the upper level strict complementarity condition hold at $(x_i^*, y_i^*, \lambda_i^*, \mu_i^*)$ for (12).

If $y_1^* = \ldots = y_n^*$ then the point x^* is a Nash stationary point of game $\{(9)\}_{i=1}^N$.

We mention that the diagonalization method presented above is a Gauss-Seidel-type scheme in that each player updates its optimal strategy in turn, and the next player takes this into account. An alternative is a Jacobi scheme in which, given the kth iterate $x^{(k)}$, each player i updates its strategy x_i based only on $x^{(k)}$; these updates can therefore be carried out in parallel. We describe the Jacobi version, which would use a different step (ii).

Jacobi (ii) We are given the current iterate $x^{(k)}$. *Each player i executes the following:*

Solve problem (12) while holding $x_{-i} = x_{-i}^{(k)}$. Denote the x_i-part of the optimal solution of (12) by $x_i^{(k+1)}$.

A Jacobi scheme and its convergence was discussed in Hu (2003). Further developments, including more details of the convergence analysis supporting the above proposition, and numerical comparisons of the Gauss-Seidel and Jacobi approaches to EPECs that favor the former, appear in Su (2004b).

5.2 Numerical examples

We give numerical results for randomly generated electricity market models in the form (2). We solve these as EPECs in the complementarity problem formulation, see §4.3, and by applying the Gauss-Seidel diagonalization approaches of §5.1. Comprehensive numerical testing is a subject for future work and seems to require definition of a reasonably broad test set of problems.

We have modeled the CP formulation and two versions of the diagonalization methods using GAMS (Brooke *et al.*, 1992) for the bilevel game (2) over a triangular network with three bidding scenarios: bid-a-only, bid-b-only and bid-a-b, respectively. Our interest is in numerical behavior; see Hu *et al.* (2004) for an investigation of the economic implications of equilibria computed for bid-a and bid-b games.

The PATH solver (Dirkse and Ferris, 1995; Ferris and Munson, 1999) is used for the CP formulation. If PATH terminates successfully, we then check the second-order sufficient condition for each player's MPEC. In all our successful CP tests below, we were able to establish numerically that a solution found by PATH was indeed a local Nash equilibrium.

By Diag we mean the Gauss-Seidel diagonalization method in which each MPEC in step (ii) of the outer iteration is formulated as an NLP, cf. (11), and solved using SNOPT (Gill *et al.*, 2002). By Diag/Reg we mean the Gauss-Seidel scheme using the inner iteration (ii-1)-(ii-2) described above; each inner iteration requires solution of an NLP which is carried out using SNOPT.

The computer used to perform the computations has two 300 MHz UltraSparc II processors and 768 MB memory, and runs the Solaris operating system.

5.2.1 Test problems

We generate 30 random test problems based on a three-node network. The participants are one consumer at node 1, two generators at node 2, and one generator at node 3. However, only the generators optimize their bids while the consumer reveals its true utility function to the ISO. In this network all lines have the same physical characteristics, which determine power flows over the network given the injections and withdrawal of power at each node, but possibly different capacities. Node 1 is chosen as the reference node and therefore its distribution factors are zero. The distribution factors of node 2 to lines $12, 13, 23$ are $-2/3, -1/3, 1/3$ respectively, and those of node 3 are $-1/3, -2/3, -1/3$ respectively. The transmission limits are $C_{12} = C_{13} = 800.0$, and $C_{23} = 5.0$.

The analysis of Section 3 suggests that equilibria are likely to exist when the lines are either largely congested, over all feasible bids, or largely uncongested. Our numerical experience seems to confirm this at least anecdotally. Looking ahead to Tables 1-3 we see that at least 27 out of 30 games for each of the bid-a-only, bid-b-only and bid-a-b cases were solved by one or more of the methods tested; looking behind these results we identified around 25 out of 30 games, for each of the three cases, in which congestion occurred in line 23 at the solution point identified. For the discussion on the economic effects of congestion, we refer the readers to Hu *et al.* (2004).

With regard to the cost coefficients A_i and B_i we note that, while quadratic cost functions are merely stylized representations of the cost profiles faced by generators, it is accepted (Saadat, 1999) that B_i is much smaller than A_i in the units that are commonly employed for electricity market models. In (2), we set the bounds for the a_i and b_i bids as $\underline{A}_i = 0$, $\overline{A}_i = 200.0$, $\underline{B}_i = 0.00001$ and $\overline{B}_i = 6.0$. Our test problems are generated by random selection of A_i and B_i as follows.

We use GAMS's pseudo-random number generator (with the default seed = 3141) to generate 30 sets of uniformly distributed coefficients of generation/utility functions. The intervals over which the random selection is made differ for each node: for the consumer at node 1, A_i and B_i are chosen in $[90.0, 120.0]$ and $[0.2, 0.6]$ respectively; for the generators at node 2 the intervals are $[5.0, 10.0]$ and $[0.01, 0.08]$ respectively; and, for the generator at node 3, $[5.0, 20.0]$ and $[0.01, 0.04]$ respectively. For example, the consumer at node 1 has utility function $A_1 q_1 + B_1 q_1^2$ where A_1 and B_1 are chosen by sampling uniform distributions on $[90.0, 120.0]$ and $[0.2, 0.6]$ respectively. However the two generators at node 2 are provided with the same cost functions. The code checks every equilibrium outcome to see whether both players have the same strategies; in our experiments, symmetric strategies at node 2 are verified in all cases (at least where the code converges).

The true generation cost functions are used as the starting points for all solvers.

The maximal number of outer iterations for both Diag and Diag/Reg is set to be $K = 400$ and the convergence tolerance $\varepsilon = 10^{-5}$ for all the problems. The norm used to check convergence of the diagonalization schemes at iteration k is the maximum, over all player indices i, of $|a_i^k - a_i^{k-1}| + |b_i^k - b_i^{k-1}|$. In the bid-$a$-only (bid-$b$-only respectively) situation, this simplifies to the maximum over i of

$|a_i^k - a_i^{k-1}|$ ($|b_i^k - b_i^{k-1}|$ respectively).

In the inner iteration process of Diag/Reg, given any outer iteration number k, we start with $\rho_0^{(k)} = 10^{-3}$, solve the regularized problem (16) and then divide the regularization parameter by 10. This is repeated five times, for a total of 6 calls to SNOPT. At the end of the inner iteration process, the maximum violation of the products of pairs of complementary scalars is 10^{-9}.

We also introduce a tolerance for checking Nash stationarity of solutions, $\varepsilon^{\text{stat}} = 10^{-3}$. For the CP formulation, each successful termination of PATH is deemed to be Nash stationary, i.e., the violation of the complementarity system is below 10^{-6} measured in the ∞-norm, which is a PATH default. Checking Nash stationarity for solutions provided by Diag and Diag/Reg is performed by applying PATH to the CP formulation using the solution provided as a starting point; if PATH terminates at a point within ∞-distance $\varepsilon^{\text{stat}}$ of the solution provided, then the latter is deemed Nash stationary.

Setting the value of $\varepsilon^{\text{stat}}$ to be orders of magnitude larger than the PATH default stopping tolerance allows for considerable inaccuracy in the solution provided by either of the diagonalization methods. This is a heuristic that allows us to check the claimed success of Diag or Diag/Reg given that simple fixed-point iteration generally does not converge faster than linearly, and therefore the termination point has relatively low accuracy. There is, of course, the possibility that Diag or Diag/Reg may identify a local Nash point that is not Nash stationary in which case PATH cannot confirm success.

5.2.2 Numerical results

Tables 1-3 summarize numerical results for solving 30 randomly generated models. Each model is solved for three different game scenarios — bid-a-only, bid-b-only and bid-a-b — so we attempt to solve 90 EPECs altogether. The title of each row of data in these tables is explained next. A "convergent problem" refers to a problem for which the algorithm in question successfully converged.

no. converged (stationary) For the CP method, this is the number of successful terminations of PATH using its default settings. This number is repeated in the brackets since each successful termination of PATH is associated with a Nash stationary point.

For the diagonalization methods, this is the number of problems for which fewer than $K = 400$ outer iterations were needed to achieve convergence of the bid sequence. The number in brackets is the number of problems for which PATH verified Nash stationarity of the solution provided (to within tolerance $\varepsilon^{\text{stat}}$ explained above).

avg. CPU time The average CPU time is calculated as the total CPU time used by the solver for all convergent problems divided by the number of such problems. For Diag and Diag/Reg the average CPU time includes the time required for all outer and, if applicable, inner iterations (for all convergent problems).

avg. no. outer iterations This is defined only for Diag and Diag/Reg. It is the total number of outer iterations k over all convergent problems divided by the number of convergent problems.

avg. no. solver iterations This is the total number of major iterations over all convergent problems of either PATH, for the CP formulation, or SNOPT, for the diagonalization formulations, divided by the number of convergent problems.

In Tables 1-2, one can see both Diag and Diag/Reg successfully converge on all of the bid-a-only and bid-b-only games, and, with only two exceptions out of 60 problems, their termination points are verified as being approximately Nash-stationary points. For these games, applying Path to the CP gives a solution to all but 7 problems, which is slightly less reliable than the diagonalization methods. By contrast Table 3 shows that the bid-a-b games are very difficult for the diagonalization methods whereas PATH is seen to be quite robust in solving all but 3 out of the 30 problem instances.

We stress that the difficulty experienced by the diagonalization methods in bid-a-b games was not in convergence, since for the vast majority of problems these methods successfully terminated, but in convergence to a Nash stationary point. This leaves open the question of whether or not the diagonalization methods usually find local Nash points. If indeed a local Nash point is not Nash stationary, then, by implication of Proposition 16, the MPEC-LICQ cannot hold there. In any event, difficulties with most computational approaches to bid-a-b games are to be expected if, as we believe, in contrast to the bid-a-only and bid-b-only games, the bid-a-b games have solutions that are not locally unique; see Hu (2003, Chapter 5) for discussion on nonuniqueness of solutions. Why PATH performs so much better than the diagonalization schemes in finding Nash stationary points of bid-a-b games is a question for future work.

It is clear that average CPU time for the CP formulation was much less than for the diagonalization methods, and was less affected by the type of game than the other methods. A similar statement holds for the average number of solver iterations, shown in the last row of the tables.

	CP	Diag	Diag/Reg
no. converged (stationary)	27 (27)	30 (28)	30 (30)
avg. CPU time	0.37	1.27	8.02
avg. no. outer iterations	N/A	8	8
avg. no. solver iterations	1116	599	3652

Table 1: Comparisons of the three solution formulations for bid-a-only

	CP	Diag	Diag/Reg
no. converged (stationary)	26 (26)	30 (30)	30 (30)
avg. CPU time	0.75	4.69	26.82
avg. no. outer iterations	N/A	27	26
avg. no. solver iterations	324	2376	12476

Table 2: Comparisons of the three solution formulations for bid-b-only

	CP	Diag	Diag/Reg
no. converged (stationary)	27 (27)	27 (0)	29 (0)
avg. CPU time	0.14	5.86	21.38
avg. no. outer iterations	N/A	27	20
avg. no. solver iterations	390	2525	9753

Table 3: Comparisons of the three solution formulations for bid-a-b

6 Conclusion

Bilevel games and EPECs, unlike classical games with nonempty convex compact strategy spaces, need not admit existence of (pure strategy) Nash equilibria under convenient general conditions. Nevertheless we have provided several situations in which the bilevel games considered by Berry *et al.* (1999), and their EPEC counterparts, have Nash equilibria. More generally we have shown that the stationary conditions of an EPEC can be phrased as a mixed complementarity problem whose solutions characterize Nash stationary points. Computational viability of finding Nash stationary points is shown for a class of randomly generated EPECs in the format of Berry *et al.* (1999). Of note is the relatively robust and fast behavior of the PATH solver when applied to the complementarity formulation.

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On line appendix

Proof of Proposition 3 Let $\mathcal{C}$ denote the (closed convex) feasible set of (1). Lemma 1 gives the existence of the solution $q^*(a)$ of (1). The usual optimality conditions for convex programs say that the gradient of the objective function of (1) at $q^*(a)$ satisfies

$$[a + 2Qq^*(a)]^T(c - q^*(a)) \ \geq \ 0, \quad \text{for all } c \in \mathcal{C}.$$

For another value of a, call it a', the optimal q is $q^*(a') \in \mathcal{C}$. Taking $c = q^*(a')$ above gives

$$[a + 2Qq^*(a)]^T(q^*(a') - q^*(a)) \ \geq \ 0.$$

Exchanging a and a' by symmetry,

$$[a' + 2Qq^*(a')]^T(q^*(a) - q^*(a')) \ \geq \ 0.$$

Adding terms in the last two inequalities, we have

$$[a - a' + 2Q(q^*(a) - q^*(a'))]^T(q^*(a') - q^*(a)) \ \geq \ 0.$$

Hence

$$(a - a')^T(q^*(a') - q^*(a)) \quad \geq \quad 2(q^*(a) - q^*(a'))^T Q(q^*(a) - q^*(a')) \geq 0 \tag{17}$$

where nonnegativity follows because Q is a diagonal matrix with positive diagonal entries (or, more generally, Q is positive definite).

If $a'_j = a_j$ for each $j \neq i$, then

$$(a - a')^T(q^*(a') - q^*(a)) = (a_i - a'_i)(q^*_i(a') - q^*_i(a)). \tag{18}$$

So, (17) implies that $(a_i - a'_i)(q^*_i(a') - q^*_i(a)) \geq 0$, which means that $q^*_i(\cdot, a_{-i})$ is a non-increasing function in a_i.

In addition, note that Q is a diagonal matrix with ith diagonal element $B_i > 0$, hence $(q^*(a) - q^*(a'))^T Q(q^*(a) - q^*(a')) \geq (q^*_i(a) - q^*_i(a'))^2 B_i$. From (17) and (18) we get

$$|q^*_i(a) - q^*_i(a')| \ \leq \ |a_i - a'_i|/2B_i,$$

i.e., $q^*(\cdot, a_{-i})$ is Lipschitz of modulus $1/2B_i$. The required bound on the directional derivative follows. And the estimate of the set of generalized gradients follows directly from its definition. ■

Proof of Theorem 4 Define $X(a_{-i}) = \{a_i \in \mathcal{A}_i : f_i(a_i, a_{-i}) \geq 0\}$, $X^+(a_{-i}) = \{a_i \in \mathcal{A}_i : f_i(a_i, a_{-i}) > 0\}$, and $X^-(a_{-i}) = \{a_i \in \mathcal{A}_i : f_i(a_i, a_{-i}) < 0\}$. Let participant i be a generator in the proof; a similar proof holds for a consumer node. Let a_{-i} be fixed. Recall the payoff function $f_i(a_i) = [a_i - A_i + B_i q^*_i(a_i, a_{-i})]q^*_i(a_i, a_{-i})$ from (3). Using Proposition 3, we see that the directional derivative of the mapping $a_i \mapsto a_i - A_i + B_i q^*_i(a_i, a_{-i})$ in the direction $u = 1$ is bounded below by 1/2, i.e. the mapping is increasing in a_i. As $q_i^*(a_i, a_{-i}) \geq 0$ is a non-increasing function in a_i by Proposition 3, the set $X^-(a_{-i})$ is an intersection of two open intervals in $\mathbb{R}$. So the set $X(a_{-i}) = \mathcal{A}_i \setminus X^-(a_{-i})$ is a closed interval of $\mathcal{A}_i$. It is nonempty since A_i is a feasible bid by assumption and $f_i(A_i) = B_i q_i^*(A_i, a_{-i})^2 \geq 0$. Similarly, the set $X^+(a_{-i})$ is an open, possibly empty, interval in $\mathcal{A}_i$.

If $X^+(a_{-i}) = \emptyset$, then $f_i(a_i, a_{-i}) = 0$ for any $a_i \in X(a_{-i})$. In this case, $f_i(a_i, a_{-i})$ is concave in $a_i \in X(a_{-i})$. So, assume that $X^+(a_{-i}) \neq \emptyset$.

Let $w_1, \ldots, w_W$ be the breakpoints of the derivative of $q_i^*(\cdot, a_{-i})$. The concavity of $f_i(\cdot, a_{-i})$ in (w_j, w_{j+1}) follows from the negative semidefiniteness of its Hessian, that is,

$$f_i{}'' \;=\; 2(1 + B_i q_i^*{}')q_i^*{}' \leq 0, \tag{19}$$

where $f_i{}''$ is the second derivative of $f_i(\cdot, a_{-i})$, which follows from Proposition 3. Note that (19) implies that the derivative $f_i{}'$ of $f_i(\cdot, a_{-i})$ is nonincreasing between any two consecutive breakpoints. Now we check the concavity of $f_i(\cdot, a_{-i})$ near a breakpoint w_j such that $f_i(w_j, a_{-i}) > 0$. By inequality (5), we have the following inequality:

$$\begin{aligned} \lim_{v \uparrow w_j} f_i{}'(v, a_{-i}) \;&=\; \lim_{v \uparrow w_j} \left[q_i^*(v, a_{-i}) + \big(v - A_i + 2B_i q_i^*(v, a_{-i})\big) q_i^*{}'(v, a_{-i})\right] \\ \geq\; & \lim_{v \downarrow w_j} [q_i^*(v, a_{-i}) + \big(v - A_i + 2B_i q_i^*(v, a_{-i})\big) q_i^*{}'(v, a_{-i})] = \lim_{v \downarrow w_j} f_i{}'(v, a_{-i}) \end{aligned} \tag{20}$$

since $w_j - A_i + 2B_i q_i^*(w_j, a_{-i}) > w_j - A_i + B_i q_i^*(w_j, a_{-i}) > 0$ by the positive profit condition.

Inequality (20) implies that the (generalized) gradient of $f_i(\cdot, a_{-i})$ is non-increasing near w_j. Therefore, $f_i(\cdot, a_{-i})$ is concave on $X^+(a_{-i})$ and is quasi-concave on the nonempty closed interval $X(a_{-i})$. By Theorem 2, the game $\{(4)\}_{i=1}^N$ has at least one Nash equilibrium. ■

Proof of Proposition 16 Statement (i) ⇒ statement (ii): Write $\pi = (\mu, \lambda)$ and $\pi^* = (\mu^*, \lambda^*)$. Fix i and $x_{-i} = x_{-i}^*$. For any neighborhood U_i of (x_i^*, y^*) there exists a neighborhood V_i of (x_i^*, y^*, π^*) such that $(x_i, y, \pi) \in V_i$ implies $(x_i, y) \in U_i$, and the corresponding value of objective function of the MPEC (11) coincides with that of the bilevel game (9). It is therefore clear that (x^*, y^*, π^*) is a local Nash point of the EPEC if (x^*, y^*) is a local Nash point of the bilevel program.

Statement (ii) ⇒ statement (iii): This relies on a combination of standard MPEC stationary conditions using MPEC-LICQ, e.g., Scheel and Scholtes (2000), and work relating these to stationary conditions for NLP formulations of MPECs, Anitescu (2000). ■

Proof of Theorem 18 Recall we are dealing with the implicit programming formulation $\{(4)\}_{i=1}^N$ of the bid-linear-only electricity market game, with EPEC reformulation $\{(13)\}_{i=1}^N$. From Proposition 16, the LICQ condition for (1) at $q = q^*$ with $a = a^*$ means that a local Nash equilibrium of the former game is equivalent to a local Nash equilibrium of the latter EPEC. That is, statements (i) and (ii) of the theorem are equivalent.

Let f_i be player i's objective function, (3). From Proposition 19 we have that a_i^* is B-stationary for Player i's problem (4). Furthermore, since $f_i(\cdot, a_{-i}^*)$ is piecewise quadratic, from previous remarks we know a sufficient condition for local optimality of a_i^* is that for any feasible direction α_i with $f_i'(a_i^*, a_{-i}^*; \alpha_i) = 0$, we must have $f_i''(a_i^*, a_{-i}^*; \alpha_i, \alpha_i) \leq 0$. We will prove that this sufficient condition holds; this yields statement (ii). Then we'll show that B-stationary points have nonnegative payoffs, statement (iii). We'll refer to $q_i(a_i^*, a_{-i}^*)$ as q_i^* but will drop the $*$ superscript for a_i and a_{-i}.

Let a_{-i} be given. Using piecewise linearity of $q_i(\cdot, a_{-i})$, we can show (details given after this proof) that

$$f_i''(a_i, a_{-i}; \alpha_i, \alpha_i) = f_i'(a_i, a_{-i}; \alpha_i) + q_i'(a_i, a_{-i}; \alpha_i)\alpha_i + B_i(q_i'(a_i, a_{-i}; \alpha_i))^2.$$

Let α_i be a feasible direction at a_i such that $f_i'(a_i, a_{-i}; \alpha_i) = 0$; assume $\alpha_i \neq 0$ without loss of generality. We have

$$f_i''(a_i, a_{-i}; \alpha_i, \alpha_i) = [\alpha_i + B_i q_i'(a_i, a_{-i}; \alpha_i)]q_i'(a_i, a_{-i}; \alpha_i)$$

and by Proposition 3,

$$-\frac{\alpha_i}{2B_i} \leq q_i'(a_i, a_{-i}; \alpha_i) \leq 0 \ \text{ if } \alpha_i > 0, \quad -\frac{\alpha_i}{2B_i} \geq q_i'(a_i, a_{-i}; \alpha_i) \geq 0 \ \text{ if } \alpha_i < 0.$$

Therefore, in both cases, we have $f_i''(a_i, a_{-i}; \alpha_i, \alpha_i) \leq 0$, hence a_i is a local maximum for Player i.

Finally, we shall show that $f_i(a_i, a_{-i}) \geq 0$. We assume without loss of generality that $q_i^* > 0$ and $a_i < \overline{A}_i$. Therefore $\alpha_i = 1$ is a feasible direction, hence B-stationarity of a_i yields

$$0 \ \geq \ f_i'(a_i, a_{-i}; 1) \ = \ q_i^*[1 + 2B_i q_i'(a_i, a_{-i}; 1)] + (a_i - A_i)q_i'(a_i, a_{-i}; 1).$$

Proposition 3 tells us that the first term is nonnegative, whence $(a_i - A_i)q_i'(a_i, a_{-i}; 1) \leq 0$; and that $q_i'(a_i, a_{-i}; 1) \leq 0$, whence either $a_i \geq A_i$ or $q_i'(a_i, a_{-i}; 1) = 0$. If $q_i'(a_i, a_{-i}; 1) = 0$ then $f_i'(a_i, a_{-i}; 1) = q_i^* > 0$, which is not possible. Instead, we have $a_i \geq A_i$ and it follows from the definition of f_i that $f_i(a_i, a_{-i}) > 0$. ■

Derivation of the second order directional derivative in the proof of Theorem 18

We derive the formula for an arbitrary player. Its index, strategies of all other players and all subscripts and superscripts are dropped here to simplify notations. First for a piecewise linear function $g(x)$ of the form $g(x) = g_0 + k(x - x_0)$ with $x \geq x_0$, we have for $h > 0$

$$g'(x_0; h) = k\,h, \quad g''(x_0; h, h) = k\,h.$$

Now for any player, its profit function takes the form of $f(a) = (a - A)q(a) + Bq(a)^2$ and noting that $q(a)$ is piecewise linear in a as shown before, we have

$$\begin{aligned}
f'(a; h) &= \lim_{t \downarrow 0^+} [f(a + th) - f(a)]/t \\
&= \lim_{t \downarrow 0^+} [(a - A)\frac{q(a + th) - q(a)}{t} + hq(a + th) + B(q(a + th) + q(a))\frac{q(a + th) - q(a)}{t}] \\
&= (a - A)q'(a; h) + hq(a) + 2Bq'(a; h).
\end{aligned}$$

Therefore the second order directional derivative is given by

$$\begin{aligned}
f''(a;h,h) &= \lim_{t\downarrow 0^+}[f(a+th+t^2h)-f(a)-tf'(a;h)]/t^2 \\
&= \lim_{t\downarrow 0^+}[(a-A)\frac{q(a+th+t^2h)-q(a)-tq'(a;h)}{t^2}+h\frac{q(a+th)-q(a)}{t}+hq(a+th+t^2h) \\
&\quad + B\frac{q(a+th+t^2h)^2-q(a)^2-2tq(a)q'(a;h)}{t^2}] \\
&= (a-A)q'(a;h)+hq'(a;h)+hq(a) \\
&\quad + B\lim_{t\downarrow 0^+}(q(a+th+t^2h)+q(a))\frac{q(a+th+t^2h)-q(a)-tq'(a;h)}{t^2} \\
&\quad + Bq'(a;h)\lim_{t\downarrow 0^+}\frac{q(a+th+t^2h)-q(a)}{t} \\
&= (a-A+2B)q'(a;h)+q(a)h+q'(a;h)h+Bq'(a;h)^2 \\
&= f'(a;h)+q'(a;h)h+Bq'(a;h)^2
\end{aligned}$$

which establishes the formula for the second order parabolic directional derivative of f_i used in the proof of Theorem 18. ■

Proof of Proposition 19 Let $(x^*, y^*, \lambda^*, \mu^*)$ be Nash stationary with the LICQ holding as described above. Consider player i's MPEC with $x_{-i} = x^*_{-i}$. According to Anitescu (2000), since $(x^*_i, y^*, \lambda^*, \mu^*)$ is stationary for the corresponding NLP (12), then it is strongly stationary for the MPEC (11) in the terminology of Scheel and Scholtes (2000). Strong stationarity implies what is called B-stationarity of the MPEC which is expressed in terms of a linearization of the problem about the point $(x^*_i, y^*, \lambda^*, \mu^*)$; again see Scheel and Scholtes (2000). Since the constraint functions of the MPEC are already affine, linearizing these functions is redundant and we can express B-stationarity using the tangent cone T to the feasible set at $(x^*_i, y^*, \lambda^*, \mu^*)$, that is

$$\nabla_{x_i,y}\phi_i(x^*, y^*)(d_i, d_y) \le 0 \text{ for each } d = (d_i, d_y, d_\lambda, d_\mu) \in T. \tag{21}$$

This is a basic stationarity condition for the MPEC discussed in generality in Luo *et al.* (1996, §3.1); it is implied by B-stationarity under smooth rather than affine constraint functions, and is equivalent to B-stationarity under linear constraint functions or other constraint qualifications.

Next suppose d_i lies in the tangent cone of the feasible set $\{x_i : g_i(x_i) \ge 0, h_i(x_i) = 0\}$ of the implicit program (14) at x^*_i. This feasible set is exactly the upper-level feasible set for (11) and is polyhedral. This means $x^*_i + \tau d_i$ is upper-level feasible for all small $\tau > 0$. Now use piecewise smoothness of $(y(x), \lambda(x), \mu(x))$ to obtain the following partial directional derivatives:

$$\big(y'_i(x^*; d_i), \lambda'_i(x^*; d_i), \mu'_i(x^*; d_i)\big) = \lim_{\tau\downarrow 0}\frac{\big(y(x^*_i+\tau d_i), \lambda(x^*_i+\tau d_i), \mu(x^*_i+\tau d_i)\big) - (y^*, \lambda^*, \mu^*)}{\tau}.$$

That is, $(d_i, y'_i(x^*; d_i), \lambda'_i(x^*; d_i), \mu'_i(x^*; d_i))$ belongs to T.

Finally we have for any d_i, from first principles or by using a chain rule Shapiro (1990), that

$$f'_i(x^*_i, x^*_{-i}; d_i) = \nabla_{x_i,y}\phi_i(x^*, y^*)(d_i, y'_i(x^*; d_i)).$$

With this, the above tangent cone relationship and the MPEC B-stationary condition (21), the result follows. ■