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The prisoner's dilemma in Cournot models: when endogenizing the level of competition leads to competitive behaviors.

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Ibrahim Abada and Andreas Ehrenmann*

Abstract In resource based economies, regulating the production and export activities have always been an important challenge. Examples in oil and gas show that different behaviors have been adopted ranging from the export monopoly to the complete opening of the export market. This paper tries to explain this multitude of solutions via strategic interactions. When modeling imperfect competition, players are separated in two categories: those who exert market power and those who are competitive and propose the good at their marginal supply cost. Letting a player freely choose whether it wants to exert market power or not when it optimizes its utility is not discussed in the literature. This paper addresses this issue by letting the players choose the level of competition they want to exert in the market. To do so, we analyze the behavior of two countries competing to supply a market with a homogeneous good in an imperfect competition setting. Each country decides the number of firms it authorizes to sell in the market. The interaction between the firms is of a Nash-Cournot type, where each one exerts market power and is in competition with all other firms allowed to sell, whether they belong to the same country or not. Each country optimizes its utility, that is the sum of the profits of its firms. We have studied four kinds of interaction between the countries. The first calculates the closed loop Nash equilibrium of the game between the countries. The second setup analyzes the cartel when the countries collude. The third focuses on the open loop Nash equilibrium and the fourth models a bi-level Stackelberg interaction where one country plays before the other. We demonstrate that in the closed loop Nash equilibrium, our setting leads to the prisoner's dilemma: the equilibrium occurs when both countries authorize all their firms to sell in the market. In other words, countries willingly chose not to exert market power. This result is at first sight similar to the Allaz & Vila (1993) result but is driven by a completely different economic

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reasoning. In the Stackelberg and coordinated solutions, the market is on the contrary very concentrated and the countries strongly reduce the number of firms that enter the market in order to fully exert market power and increase the price. The open loop result lies in between: the countries let all their firms sell but market power remains strong. These results suggest that the prisoner's dilemma outcome is due to the conjectural inconsistency of the Nash equilibrium. Finally, in the Stackelberg setting, we give countries the choice of being leader or follower and demonstrate that the counter-intuitive competitive outcome is very unlikely to occur in the market.

Keywords Imperfect competition, export oligopoly, open and closed loop Nash equilibrium

JEL Classification L13, L7

Contact andreas.ehrenmann@engie.com; ibrahim.abada@engie.com
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Ibrahim ABADA* & Andreas EHRENMANN†

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Abstract

In resource based economies, regulating the production and export activities have always been an important challenge. Examples in oil and gas show that different behaviors have been adopted ranging from the export monopoly to the complete opening of the export market. This paper tries to explain this multitude of solutions via strategic interactions. When modeling imperfect competition, players are separated in two categories: those who exert market power and those who are competitive and propose the good at their marginal supply cost. Letting a player freely choose whether it wants to exert market power or not when it optimizes its utility is not discussed in the literature. This paper addresses this issue by letting the players choose the level of competition they want to exert in the market. To do so, we analyze the behavior of two countries competing to supply a market with a homogeneous good in an imperfect competition setting. Each country decides the number of firms it authorizes to sell in the market. The interaction between the firms is of a Nash-Cournot type, where each one exerts market power and is in competition with all other firms allowed to sell, whether they belong to the same country or not. Each country optimizes its utility, that is the sum of the profits of its firms. We have studied four kinds of interaction between the countries. The first calculates the closed loop Nash equilibrium of the game between the countries. The second setup analyzes the cartel when the countries collude. The third focuses on the open loop Nash equilibrium and the fourth models a bi-level Stackelberg interaction where one country plays before the other. We demonstrate that in the closed loop Nash equilibrium, our setting leads to the prisoner's dilemma: the equilibrium occurs when both countries authorize all their firms to sell in the market. In other words, countries willingly chose not to exert market power. This result is at first sight similar to the Allaz & Vila (1993) result but is driven by a completely different economic reasoning. In the Stackelberg and coordinated solutions, the market is on the contrary very concentrated and the countries strongly reduce the number of firms that enter the market in order to fully exert market power and increase the price. The open loop result lies in between: the countries let all their firms sell but market power remains strong. These results suggest that the prisoner's dilemma outcome is due to the conjectural inconsistency of the Nash equilibrium. Finally, in the Stackelberg setting, we give countries the choice of being leader or follower and demonstrate that the counter-intuitive competitive outcome is very unlikely to occur in the market.

*ENGIE. ibrahim.abada@engie.com

†ENGIE. andreas.ehrenmann@engie.com

1 Introduction

In a resource based industry, the regulation of exports is crucial. Most oil or gas producing countries, like Saudi Arabia, United Arab Emirates or Russia, export via regulated monopolies. Other countries, like Norway and Canada, have decided to open their export activity to competition. In these countries, many firms can produce oil and directly export it to international spot markets where they are in competition. Some oil producing countries have recently experienced an economic shift toward export nationalization: The government of Argentina decided in 2012 to expropriate Repsol YPF and acquired more than 51% of the company. Other countries have carried out a shift in the other direction. As an example, Norway and the United Kingdom have increased competition in the production and export activity in the previous decades.

From the producing country's point of view, deciding the level of competition among its exporting firms ends up to arbitraging between price and volume. If market power is exerted by the exporters, increasing the number of exporting firms reduces the price, but increases the total sale of the country. The impact on the total payoff is not obvious and clearly depends on the number of exporters from the other producing countries and their strategic decisions. In an imperfect competition context, it is not straightforward that an export monopoly is optimal for a country. The intuition behind this assertion is the possibility of the other producers to free-ride: if country A is an export monopoly, country B can increase its export competition to a certain extent, in order to exploit a high price situation created by country A. Therefore, country A might not be willing to remain an export monopoly and might want to increase its sales too. These arguments pave the way to the use of the non-cooperative game theory to treat the subject.

As explained above, the oil industry provides many examples of different behaviors of the producing countries in the selling activity in terms of market concentration. The gas industry is another interesting field: as an example, Russia only authorizes Gazporm Export to sell gas to Europe. Algeria does the same with Sonatrach. On the contrary, Norway has opened its export market to firms other than Statoil. The main objective of this paper is to analyze the economic incentives that can lead a country to open or close its production/commercialization activity to different firms, at the risk of potentially reducing its market power exercise. Our analysis will focus on strategic interactions between different players and will study the robustness of the results with respect to the upstream cost structure.

Generally, analyzing the means to exert market power when competition is imperfect is tricky. In competitive markets, prices are the only instruments set by the participants that intervene in the market (whether they are competitive or distorted by different interventions). On the contrary, market power can be exerted using different instruments: prices, volumes, quality of the good, entry barriers, antisymmetric information, etc. Furthermore, the interaction between the strategic players can occur in single or multiple stages. Another difficulty arises with the possibility of some players anticipating the reaction function of other players, as is the case in the standard Stackelberg model. Therefore,the economic theory dedicated to imperfect competition (like the monopoly or oligopoly theory) is more concerned about the concepts of market power exercise and its instruments than the functioning of the markets themselves.

In energy economics Cournot models are widely used to mimic imperfect competition. For oil markets, Nash cournot models have been developed to analyze market power since the seventies. In oil markets, Salant (1976) proposed one of the first Cournot models that captures market power and the exhaustibility of the resource. In natural gas where the concentration of the supply justifies the possible exercise of market power, recent Cournot related works include the World Trade Gas Model of the Rice university (2004), Perner & Seeliger (2004), Lise & Hobbs (2008), Gabriel, Kiet & Zhuang (2005), Gabriel & Kiet (2005), Egging, Holz & Gabriel (2010) and Abada et al. (2013). Furthermore, power and coal markets models have also seen a growth in the use of Cournot approaches. As an example, one can cite Neuhoff et al. (2005) (for electricity) and Haftendorn & Holz (2010) (for coal). We refer to Huppmann (2013) for a detailed description of recent Cournot based energy models and their differences. The common feature between all these models is that the Cournot players (also called strategic, those who exert market power), and the competitive players (those who do not exert market power), are perfectly identified and their behavior is modeled accordingly. In other words, a player is *a priori* identified as being a Cournot or a competitive player[1]. Therefore, the players cannot choose, as a strategic decision, the level of competition they want to exert in the market, i.e. whether they want to be competitive or Cournot players. This is mainly due to the fact that the common belief is that if a market actor has the possibility to exert market power, he will do so because this would increase his payoff. This reasoning is obviously true in the case of a monopolist. However as explained above, when it comes to the oligopoly case, this conclusion is not straightforward, since any volume withholding in the aim to increase the price, can be compensated by some opponents who might see their payoffs increase accordingly.

Outside the energy domain there is a great amount of literature dedicated to the question of optimal behavior of a firm in a market. Singh & Vives (1984), analyze whether profit maximizing firms should set prices or quantities when competing and find that when goods are substitutes, firms should set volumes rather than prices (and *vice versa* if goods are complements). In Hamilton & Slutsky (1990), the authors analyze whether firms should play simultaneously or sequentially and conclude that if goods are substitutes, the firms should play simultaneously (and vice-versa is goods are complements). Other works examine the question from the angle of capacity constraints: Davidson & Deneckere (1986), Shubik & Levitan (1980) and Osborne & Pitchik (1986). When volume constraints are present, firms should better compete *a la Cournot* (quantity setting). On the contrary, without capacity constraints a Bertrand (price setting) competition is more likely to happen.

Using the Cournot approach to mimic imperfect competition has raised a certain number of controversies: In a seminal article, Ulph & Folie (1980) studies an oligopoly interacting with a competitive fringe in a Cournot and a Stackelberg settings. They draw to the counter-intuitive conclusion that in the Cournot situation, a firm earns more when it stops exerting market power and becomes competitive (i.e. offers its quantity at its marginal supply cost). This occurs because a Cournot player does not anticipate the fact that the fringe will replace part of the volume he withholds to try to force the price up.

Another important criticism addressed to the Cournot equilibrium is that it leads to inconsistent results with respect to the anticipation of the reaction functions by the players. In a standard Cournot setting, the reaction function of a player is calculated by assuming that the opponents have a constant supply and will not react to his actions. However, at the equilibrium, such a situation does not hold. Consistent Conjectures Equilibria (CCE) have been introduced to correct this feature (see Bresnahan (1981) and Perry (1982) for instance). These equilibria have been criticized in return: in Makowski (1987) and Lindh (1992), the authors argue that at the equilibrium, no player can actually know its rivals' reaction functions since by definition, at the equilibrium, no deviation is supposed to occur. Daughety (1985) uses an infinite regress-model applied to a duopoly to find conditions under which the Cournot equilibrium becomes consistent.

Another criticism of the Cournot framework lies in the fact that it might lead to counter-intuitive results when embedded in multi-stage models. In a seminal paper, Allaz & Vila (1993) model the interaction between two Cournot duopolists who can sign forward contracts before delivering in the spot market when the demand is revealed in the last stage. There are $N-1$ forward contract stages and one spot market stage. The spot market game is a classic

[1]With the exception of Egging (2010) where market power can be exerted in the range between the extreme cases of competitive and Cournot behaviors. However, the level of market power exercise is not endogenous and is also fixed *a priori* by the modeler.

Cournot game where no reaction function is anticipated. In the forward contract stages, both players know the reaction functions of the competitor in the spot market (because they observe how the equilibrium shifts), but not in the spot market. At the equilibrium, the result is quite surprising: the suppliers earn less when there are forward markets and the consumers' welfare is better off. Besides, the equilibrium converges toward the perfectly competitive outcome when the number of forward stages is infinite. In other words, embedding the Cournot paradigm in a multi-stage setting makes the market power disappear. This conclusion paved the way to a great amount of literature that advocates the use of contracts to mitigate the effect of market power.

The divisionalization literature is also very useful to study the exports regulation of a resource producing country. In that vein, we will refer to two important papers dedicated to the optimal choice of the division of a firm: Baye et al. (1996) and Corchon (1991), because their setting is close to the economic case we want to study in this paper. In Baye et al. (1996), a two-stage game is formulated to mimic the incentive of a firm to share its production among autonomous units that compete in the second stage as a Cournot oligopoly. One main finding is that with linear demand and costs, the actors reach the socially optimal welfare and too many units are chosen. Similarly, Corchon (1991) also solves a two-stage interaction between groups that can divide their production. Two cases are treated depending on the shape of the inverse-demand function. The paper drives to the same conclusion, namely that if the number of groups is high enough, the Subgame Perfect Nash Equilibrium is perfectly competitive. It is worthwhile noting that both Baye et al. (1996) and Corchon (1991) solve the game in closed loop where the second stage is solved first and its output is plugged into the first stage problem.

In our paper, we start by constructing a set of classical game theoretical models that differ in their representation of strategic interaction between the countries. We model a two-stage oligopoly *a la Cournot* providing a homogeneous good to a market where the price is set via an inverse-demand function and the production cost is quadratic. The players decide endogenously the level of competition they exert in the market. To do so, we assume that two big entities (we will consider them as countries) have many exporting firms and each country chooses in the first stage the number of firms it owns that can sell in the market. The market is solved in the second stage by analyzing the interaction between the firms that have been allowed to sell. These are in an imperfect competition and we will assume that they can all exert market power *a la Cournot*. Each firm allowed to sell will therefore be in competition with the other firms active in the market, whether they belong to its country or to the other one. This setting will be referred to as the standard model. Like Baye et al. (1996) and Corchon (1991), the standard model is solved using backward induction in closed loop: we first solve the market interaction between the firms allowed to sell in second stage (assuming that the countries have already chosen the firms to enter the market) and then, given this outcome, the Nash equilibrium of the game between the countries is calculated in first stage. Our main finding is that this setting leads to the prisoner's dilemma outcome: the Nash equilibrium is not Pareto-optimal and occurs when the countries have a competitive exporting behavior. In other words, if the countries were to choose the level of competition they want to exert the market, the equilibrium is reached when all the firms are allowed to enter. This finding is not new and is similar to what Baye et al. (1996) and Corchon (1991) have shown, but extended to the convex cost function case (both Baye et al. (1996) and Corchon (1991) consider a constant marginal production cost). We argue that this seemingly counter-intuitive result is due to the inconsistency of conjectural variations of the Nash equilibrium in the game between the countries. To challenge this explanation, we have extended our setting to three benchmarks for the interaction between the countries. The first is the situation, where the countries coordinate the number of firms in the market to optimize their payoff. This is not a classical cartel solution since the firms still compete with each other and there is no transfer between countries. The second situation models the same interaction between the countries as the standard model but the game is solved in open loop: the countries decide at the same time the number of firms to enter the market and how these should operate. The last benchmark represents a Stackelberg interaction, when one country plays before the other and anticipates its reaction function (bi-level game). Our results suggest that in the coordination between the countries and Stackelberg cases (which are consistent conjectural variations approaches), the counter-intuitive effect disappears: the market becomes very concentrated and market power is fully exerted by the countries. The open loop model gives more subtle results: the market is not concentrated but market power is still exerted by the countries. In summary, it turns out that the closed loop setting leads to the most competitive outcome among the different approaches to model a competitive environment where players choose their level of competition. Both closed and open loops representations lead to extreme market structures that are not observed in practice. The Stackelberg outcome is more realistic but contradicts the basic set up of symmetric players. Indeed by choosing a leader and a follower, the game is clearly not symmetric anymore.

As a final step we expand our setting to a new game theoretical design by giving the countries the additional possibility to choose if they want to be leader or follower and hence re-introduce symmetry. We find a mixed strategy equilibrium between the countries where the number of exporters will be very small and market power quite important, with a very high probability.

The remaining of this paper is structured as follows: Section 2 presents our standard model and the three benchmark cases with their main assumptions. The optimization programs of the countries are detailed and a particular attention is paid to the way the choice of the level of competition (i.e. the number of firms allowed to enter the market) by the countries is endogenized. Section 3 gives the main theoretical properties of the models and shows that there exists a unique solution to each one of them, which is calculated and presented for an illustrative example. Section 3 also analyzes the results and discusses the economic implications of the study. We show in particular that the market becomes very concentrated (and market power is effectively exerted) when the equilibrium holds consistent conjectural variations. In section 4 we amend the game by allowing countries to choose to be leader or follower and calculate the mixed strategy equilibrium. Finally, the last section concludes the paper and presents possible extensions for future work. Proofs are in the appendix.

2 The models

2.1 Assumptions and notation

Our basic setting is as follows: we restrict ourselves to the case of two countries, Country 1 and Country 2, that choose the number of firms which can export to the market. The countries will also be referred to as players or agents throughout the paper. We will assume that all the firms are symmetric and face the same cost structure. The marginal production cost is assumed to be linear and the cost function is denoted by $x \longrightarrow c_1 x + c_2 x^2$. This functional form of the cost accommodates the particular case of constant marginal cost, when $c_2 = 0$. For the sake of simplicity, we assume that the firms do not face production or transport capacity constraints. The consumers are represented by their inverse-demand function: if p is the market price and q the consumption, then $p = a - b.q$, with $a \geq 0$ and $b \geq 0$. In other words, we assume in our model that the consumers are passive and price takers.

We will assume that Country 1 (respectively Country 2) has $N \geq 1$ (respectively $M \geq 1$) producing firms on its territory. Hence, Country 1 has to decide the subset $I \subset \{1, 2...N\}$ of its firms allowed to export (same for Country 2 and $J \subset \{1, 2...M\}$). All the firms that have a right to sell in the market are assumed to be in an imperfect competition and act as a Cournot oligopoly.

To simplify the description, our models are static and we will study the market's outcome for one period. Nevertheless, our setting can easily be generalized to a dynamic interaction as long as no investment decisions are considered and no cross-temporal links represented. We assume that a country taxes in an uniform way all its exporting firms by taking a fixed percentage of their net profit. Therefore, a country maximizes its income when it maximizes the sum of the profits of all its firms. For the sake of simplicity, we will consider that the objective function of a country is equal to the sum of the profits of its exporting firms. The optimal choice of a country depends on what the other country decides, regarding the subset of its firms allowed to export (and how these behave in the market).

Variables $x_i, (i \in I)$ are the volumes of the firms belonging to Country 1 allowed to export. $y_j, (j \in J)$ are the volumes of the firms belonging to Country 2 allowed to export. The price in the market is therefore:

$$p = a - b\left(\sum_{i \in I} x_i + \sum_{j \in J} y_j\right) \tag{1}$$

Country 1's payoff is

$$\Pi_1(I, J) = p \sum_{i \in I} x_i - c_1 \sum_{i \in I} x_i - c_2 \sum_{i \in I} x_i^2 \tag{2}$$

Country 2's payoff is

$$\Pi_2(I, J) = p \sum_{j \in J} y_j - c_1 \sum_{j \in J} y_j - c_2 \sum_{j \in J} y_j^2 \tag{3}$$

The fact the Π_1 depends on the two subsets I and J is due to equation (1). It indicates that Country 1's profit depends on its choices, but also on the choice of Country 2. The same observation holds for Country 2.

It is important to highlight that we assume the firms to be in competition, even if they belong to the same country. In other words, no collusion is possible between the firms. Indeed, if this was the case, the economic theory tells us that to optimize its utility, each country should act as one player in the market and redistribute its profit between its firms[2].

2.2 The closed loop formulation

We present now our reference case where the model is solved using backward induction: first, we assume that the sets I and J have been chosen by the countries and solve the interaction between the firms accordingly. This allows to calculate the payoffs of both countries (which depend on I and J) that are optimized subsequently.

2.2.1 The market model

First, we assume that the sets I and J have been chosen by the countries and let us write the market equilibrium problem between the firms.

Each firm belonging to I solves the following:

$$\begin{array}{lll} \forall i \in I, & \text{Max} & \left(a - b\left(\sum_{i \in I} x_i + \sum_{j \in J} y_j\right)\right) x_i - c_1 x_i - c_2 x_i^2 \\ & \text{st} & x_i \geq 0 \end{array} \tag{4}$$

Each firm belonging to J solves the following:

$$\begin{array}{lll} \forall j \in J, & \text{Max} & \left(a - b\left(\sum_{i \in I} x_i + \sum_{j \in J} y_j\right)\right) y_j - c_1 y_j - c_2 y_j^2 \\ & \text{st} & y_j \geq 0 \end{array} \tag{5}$$

Following the Cournot paradigm, we assume here that each firm believes that the opponents will not deviate from the equilibrium when it optimizes its payoff. In other words,

[2]However, if the firms are not symmetric, it is not straightforward that the kernel of the game is non-empty.

each firm assumes that the other firms' output remain constant (which implies that the conjectures are not consistent, see Bresnahan (1981) for more details).
The market model (programs (4) and (5)) can be solved easily. The objective function of each firm is quadratic with respect to its decision variable and standard constraints qualification obviously hold. This implies that the first order conditions (or KKT conditions) are sufficient and necessary to characterize optimality. Hence, the market model can be solved as a complementarity problem: the KKT conditions of all the optimization programs are written and solved simultaneously, which gives the following problem:

$$\forall i \in I, \qquad 0 \leq x_i \qquad \perp \qquad a - c_1 - b\left(\sum_{i\in I} x_i + \sum_{j \in J} y_j\right) - 2cx_i - bx_i \leq 0 \tag{6a}$$

$$\forall j \in J, \qquad 0 \leq y_j \qquad \perp \qquad a - c_1 - b\left(\sum_{i\in I} x_i + \sum_{j \in J} y_j\right) - 2cy_j - by_j \leq 0 \tag{6b}$$

It is easy to demonstrate that problem (6a) and (6b) has a unique solution[3]. In our case, the solution can be found using a symmetry argument. Since the firms are symmetric, at the equilibrium their output is the same: if n is the number of elements of I and m of J, then (the $*$ denotes the equilibrium):

$$\forall i \in I, \forall j \in J,\ x_i^* = y_j^* = \frac{a - c_1}{b} . \frac{1}{(m + n + 1 + \frac{2c_2}{b})} \tag{7}$$

We deduce that at the equilibrium, the solution depends only on parameters a, b, c_1, c_2 and the number of firms n and m. Hence, we will index the solution by m and n (instead of I and J): $x_i^{n,m}$ and $y_j^{n,m}$.
The total quantity consumed is:

$$Q^{n,m} = (m+n)\frac{(a-c_1)}{b}\frac{1}{(m+n+1+\frac{2c_2}{b})} \tag{8}$$

and the market price is:

$$p^{n,m} = \frac{a\left(1 + \frac{2c_2}{b}\right) + (m+n)c_1}{(m+n+1+\frac{2c_2}{b})} \tag{9}$$

We also calculate the countries' profits as follows:

$$\Pi_1(I,J) = \Pi_1^m(n) = \frac{n}{(m+n+1+\frac{2c_2}{b})^2} \frac{(a-c_1)^2\left(1+\frac{c_2}{b}\right)}{b} \tag{10}$$

$$\Pi_2(I,J) = \Pi_2^n(m) = \frac{m}{(m+n+1+\frac{2c_2}{b})^2} \frac{(a-c_1)^2\left(1+\frac{c_2}{b}\right)}{b} \tag{11}$$

2.2.2 The optimization program of the countries

The novelty of this work is that an imperfect competition between the countries is added on top of the Cournot interaction of the firms. Each country anticipates the outcome of the market and knows how it depends on its decision, namely the number of firms it allows to enter the market. By choosing m and n, each country can decide the level of competition it wants to have in the market and optimizes its payoff accordingly. Therefore, the market equilibrium will be embedded in the joint optimization programs of the countries. This ends up to a double-layer market structure.

[3]See for example Harker & Pang (1990) for an in-depth analysis of complementarity problems and the ways to solve them.

Country 1 maximizes its payoff by tuning the number of its exporting firms n.

$$\begin{array}{lll} \text{Country 1} & \text{Max} & \Pi_1^m(n) \\ & \text{st} & n \in \{1, 2...N\} \end{array} \tag{12}$$

Country 2 maximizes its payoff by tuning the number of its exporting firms m.

$$\begin{array}{lll} \text{Country 2} & \text{Max} & \Pi_2^n(m) \\ & \text{st} & m \in \{1, 2...M\} \end{array} \tag{13}$$

Both programs (12) and (13) are non-convex integer optimization programs. Unlike what has been done previously for the firms' interaction, the countries' competition cannot be formulated in its complementarity form due to the presence of integer variables. However, the double structure we obtain can be treated by the standard non-cooperative game theory framework. The aim now is to calculate the Nash equilibrium of our problem (if it exists) and study the impact of this double layer competition on the market. A particular attention will be paid to the market concentration.
One can summarize the game between the countries by the bi-matrix shown in Figure 1. Country 1 (Country 2)'s strategy set is $\{1, 2...N\}$ ($\{1, 2...M\}$). The columns (rows) represent the possible choices of Country 1 (Country 2).

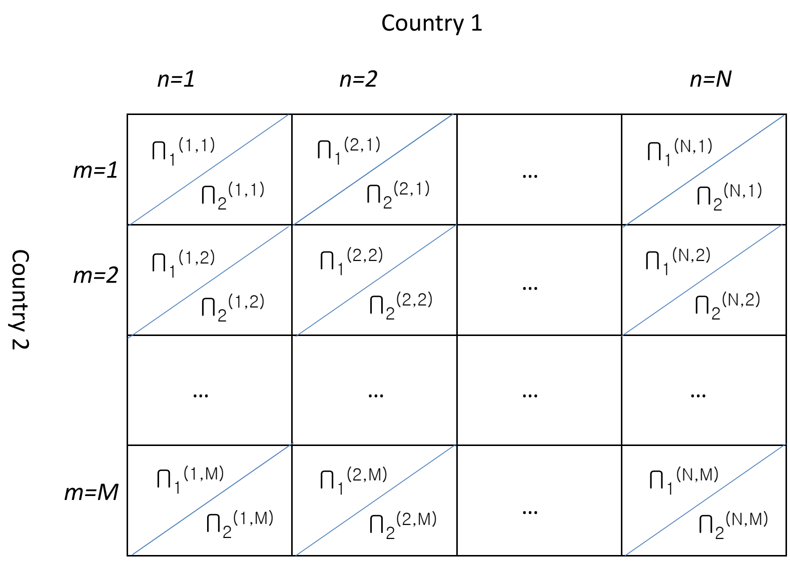

Figure 1:
The payoff bi-matrix

Next we introduce three benchmarks for the interactions between the countries: 1) the coordinated situation where the countries agree jointly on a number of firms, 2) the open loop game where the decisions about the number of firms and the production levels are taken at the same time and 3) the Stackelberg situation with one country playing first and anticipating the reaction of the other country.

2.3 The coordinated situation

In this setting, it is assumed that the countries collude. This means that they cooperate in setting the number of firms to optimize their common profit $\Pi = \Pi_1^m(n) + \Pi_1^n(m)$. But the firms continue to compete. This ends up to solving:

$$\begin{array}{ll} \text{Max} & \Pi = \Pi_1^m(n) + \Pi_2^n(m) = \frac{(a-c_1)^2}{b}\left(1+\frac{c_2}{b}\right)\frac{m+n}{\left(m+n+1+\frac{2c_2}{b}\right)^2} \\ \text{st} & n \in \{1,2...N\}, m \in \{1,2...M\} \end{array} \tag{14}$$

When the countries collude there is a cooperative game between them. The standard framework to treat collusion effects is the cooperative game theory that allows one to understand whether the outcome of the cartel is stable inasmuch as the parties agree on a satisfying sharing rule that prevents each of them from leaving the cartel. However, in this paper, we restrict ourselves to calculating the optimal payoff and the corresponding market concentration without studying the stability of the cartel.

The cartel program (45) is a non-convex integer optimization program. We will see in Section 3 that it has a unique solution.

2.4 The open loop formulation

In the open loop formulation, the countries decide at the same time the subset of firms to enter the market, I and J, and how these behave in the market x_i and y_j. Open loop games are used when players cannot observe the choices of the other agents hence in our setting countries do not know how many firms the other country would grant an export permission.

Country 1 maximizes its payoff by tuning at the same time the number of its exporting firms I and their operational variables $x_i,\ i \in I$.

$$\begin{array}{lll} \text{Country 1} & \text{Max} & \left(a - b\left(\sum_{i\in I} x_i + \sum_{j\in J} y_j\right)\right)\sum_{i\in I} x_i - c_1 \sum_{i\in I} x_i - c_2 \sum_{i\in I} x_i^2 \\ & \text{st} & I \subset \{1,2...N\} \\ & & \forall i \in I,\ x_i \geq 0 \end{array} \tag{15}$$

Country 2 maximizes its payoff by tuning at the same time the number of its exporting firms J and their operational variables $y_j,\ j \in J$.

$$\begin{array}{lll} \text{Country 2} & \text{Max} & \left(a - b\left(\sum_{i\in I} x_i + \sum_{j\in J} y_j\right)\right)\sum_{j\in J} y_j - c_1 \sum_{j\in J} y_j - c_2 \sum_{j\in J} y_j^2 \\ & \text{st} & J \subset \{1,2...M\} \\ & & \forall j \in J,\ y_j \geq 0 \end{array} \tag{16}$$

Both programs (15) and (16) are non-convex mixed integer optimization programs. Here again, they cannot be solved using the complementarity formulation.

2.5 The Stackelberg setting

Now we extend our setup to an imperfect competition where one leader, say Country 1, plays before the other. The leader anticipates the reaction of Country 2, the follower, and takes it into consideration in its optimization. In other words, the conjecture of the leader about the decision of the follower is consistent. The optimization programs of the countries are given below. First we start with the follower (backward induction):

Country 2 maximizes its payoff by tuning the number of its exporting firms m, given n.

$$\begin{array}{lll} \text{Country 2} & \text{Max} & \Pi_2^n(m) \\ & \text{st} & m \in \{1,2...M\} \end{array} \tag{17}$$

If we assume that Problem (17) has a unique solution (which will be shown in 1 in the following section), we can calculate the reaction function of Country 2: given n, Country 2 should allow $m = m^*(n)$ firms to enter the market in order to maximize its profit.

Country 1 anticipates the reaction of the follower, which means that it can consider that $m = m^*(n)$ in its optimization program:

$$\begin{array}{lll} \text{Country 1} & \text{Max} & \Pi_1^s = \Pi_1^{m^*(n)}(n) = \dfrac{n}{(m^*(n)+n+1+\frac{2c_2}{b})^2}\dfrac{(a-c_1)^2\left(1+\frac{c_2}{b}\right)}{b} \\ & \text{st} & n \in \{1, 2...N\} \end{array} \tag{18}$$

Both the close and open loop models do not provide consistent conjectural variations equilibria, since each player believes that the decisions of the other player are do not react to his own decision, which is obviously not true. On the contrary, the cartel and Stackelberg settings are consistent. This is straightforward for the cartel case since there is no game between the countries and only one optimization program is solved for both players. The Stackelberg case is (partially) consistent because the reaction function of one player (the follower) is correctly and exactly anticipated by the other player (the leader). We refer to Julien, Musy & Saidi (2011) and Figuières et al. (2004) for an overview of consistent conjectural equilibria and a demonstration of the conjectural consistency of the Stackelberg model.

3 Solving the models and economic considerations

In this section, we focus on the resolution of our models. We establish an order of the degree of market power in the different settings and provide an illustration.

3.1 The closed loop formulation

First, we demonstrate that each player faces an optimization program that gives a unique solution, considering that the other player's decision is constant. To simplify the calculations, we will assume that $\frac{2c_2}{b} \in \mathbb{N}$. All proofs are reported in Appendix 1.

Theorem 1. *When m is fixed, program (12) has a unique solution $n^*(m) = min\left(m+1+\frac{2c_2}{b}, N\right)$. When n is fixed, program (13) has a unique solution $m^*(n) = min\left(n+1+\frac{2c_2}{b}, M\right)$.*

Figure 2 shows the evolution of Country's 1 payoff Π_1^m with respect to n for a fixed m. The optimal payoff is

$$\Pi_1^m(n^*(m)) = \frac{(a-c_1)^2}{4b}\left(1+\frac{c_2}{b}\right)\frac{1}{m+1+\frac{2c_2}{b}}$$

The fact that each country faces an optimization program that has a unique solution (considering that the other country does not deviate) does not necessarily imply that there exists a Nash equilibrium to our problem. We recall that a Nash equilibrium is a game situation where each Country has reached an optimum, provided that the other country does not deviate.

As stated before, the equilibrium is usually calculated by solving the KKT conditions of the players simultaneously in a complementarity framework. Existence and uniqueness results can be drawn using the VI (Variational Inequality) formulation of the problem (see Facchinei & Pang (2003) for instance for a clear explanation on the use of the VI formulation to solve equilibrium problems). However, this cannot be done in our case since the players' optimization programs are not convex (in their objectives) and contain integer variables. However, we demonstrate in this subsection that our model has a unique equilibrium.

If it exists, the equilibrium has the interesting property to be extremal. Let us first define the concept of extremal point:

Definition 1. *We call an extremal point of the game a couple of strategies $(n, m) \in \{1, 2...N\} \times \{1, 2...M\}$ such that $n = N$ or $m = M$.*

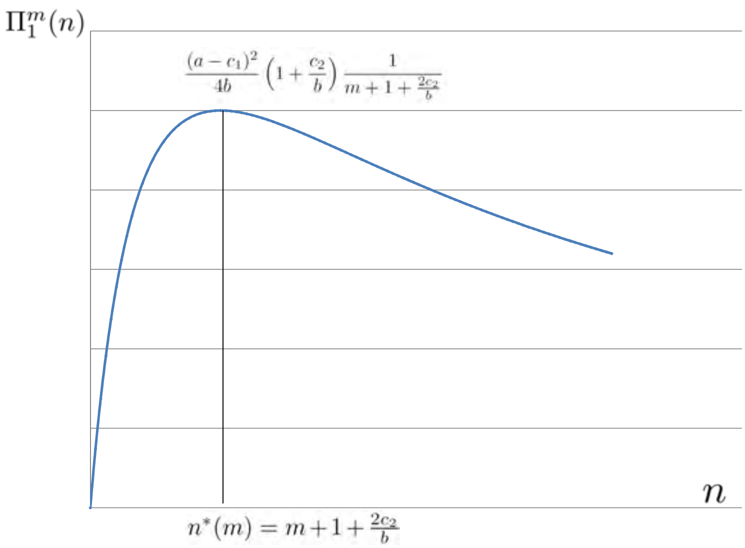

Figure 2:
Evolution of Country 1's payoff with respect to its decision n

As shown in figure 3, an extremal point makes the market "competitive" because at least one country allows all its firms to sell in the market, which reduces the price.

Theorem 2. *If it exists, the equilibrium is an extremal point.*

We state now our existence and uniqueness result.

Theorem 3. *Our problem (12) and (13) has a unique (closed loop) Nash equilibrium.*

As an example, if the countries are symmetric ($N = M$), then the closed loop Nash equilibrium is $(n_{cl}, m_{cl}) = (N, M)$, which makes the market the least concentrated.
Our model leads to a surprising result: a strategic interaction between the countries leads to a competitive outcome. If the countries are given the choice of firms they allow to export, then the equilibrium is reached when at least one of the countries authorizes all its firms to sell. Under a mild condition (see theorem 4) the closed loop Nash equilibrium also maximizes the social welfare. Given that in economic theory, pure and perfect competition maximizes the social welfare, the imperfect competition between the countries leads to a quasi-competitive outcome. Among all the possible strategies, the countries will select the one that maximizes the social welfare (however, this outcome is called quasi-competitive because the firms exert market power and as a consequence, the price will still be higher than the marginal supply cost) and the closed loop Nash equilibrium is the closest outcome to the pure and perfect competition. This result still holds in the case of constant marginal cost $c_2 = 0$.
The link between the closed loop Nash equilibrium and the optimality of the welfare is detailed in the coming theorem. For the sake of simplicity, we will assume that the countries are symmetric enough, in the sense that their numbers of potential exporting firms are not very different:

$$|N - M| \leq 1 + \frac{2c_2}{b} \tag{19}$$

Theorem 4. *If the countries are symmetric enough (assumption (19)), the closed loop Nash equilibrium maximizes the social welfare.*

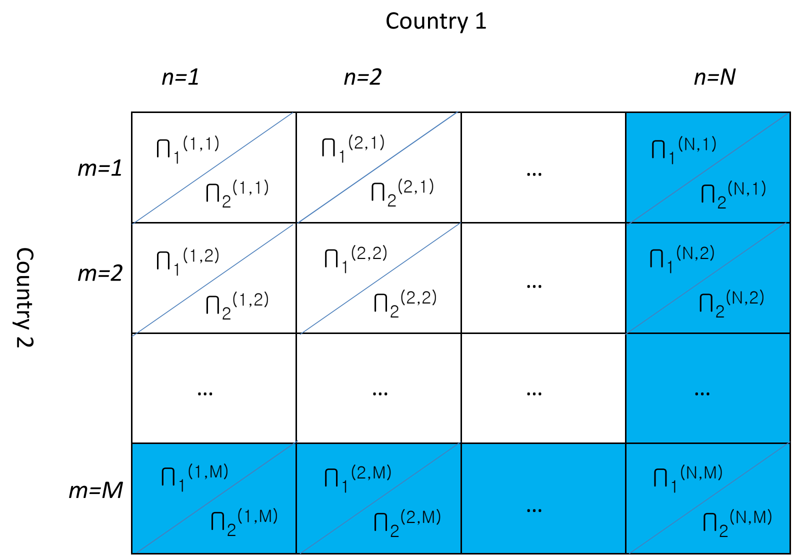

Figure 3:
Extremal strategies

Note that the previous result is an unexpected benefit of the (double) imperfect competition between the actors. Indeed, the countries strive to maximize their respective payoffs separately and do not consider at all the consumer's surplus in their decisions.

Given the closed loop equilibrium, we can easily calculate the offered volume to the consumers:

$$Q(closed\ loop) = (n_{cl} + m_{cl})\frac{(a - c_1)}{b}\frac{1}{n_{cl} + m_{cl} + 1 + \frac{2c_2}{b}} \tag{20}$$

This volume will be further discussed in Section 3.5, where we will compare between the different volumes obtained by our settings in order to quantify the degree of market power exercise.

To sum up, at first sight, we are facing a counter-intuitive result (that will be referred to as the closed loop Nash Cournot paradox). In the closed loop system, when the countries are left the possibility to choose the level of competition in the market, they will nonetheless go for the most competitive market structure that generates the lowest price. As pointed out above, this finding also applies when the marginal cost is constant: the competitiveness of the market is not due to the fact that a quadratic cost function makes the total production cost lower when the production is split among different firms. Both Baye et al. (1996) and Corchon (1991) arrived to the same conclusion in the constant marginal cost case in closed loop. We believe that this is due to the conjectural inconsistency of the closed loop Nash equilibrium: each country assumes that the reaction function of the opponent is constant, which is not true. However, as pointed out in Makowski (1987) and Lindh (1992), it is economically hard for a player to have a consistent conjecture about the opponent, since this requires him to know its cost structure and market behavior. We will see later on that if we nevertheless assume that a player anticipates the reaction of the other one in its economic calculation, the market becomes concentrated, market power becomes important and the counter-intuitive effect disappears.

3.2 The coordinated solution

Obviously, the closed loop result is at the opposite of what one expects from a coordinated behavior. If the countries are to collude, the economic theory suggests that they would reduce the number of actors in the market in order to increase prices and fully exploit market power. This can be replicated by our model thanks to the following theorem:

Theorem 5. *Under the assumption* $\frac{2c_2}{b} \leq 1$*, if the countries collude, the outcome is* $(n_c, m_c) = (1, 1)$*: each country leaves only one firm to export and the market becomes a duopoly. This outcome is Pareto-optimal for the game between the countries.*

As stated before, we do not discuss here the existence of a stable sharing rule of the total payoff between the countries. However, since all the firms are symmetric, it is easy to demonstrate that if the countries were to collude, the kernel of the cooperative game is non-empty and the cartel is stable (see for example Tijs & Driessen (1986) for a better understanding of cost allocation problems in game theory).
Note that assuming that $\frac{2c_2}{b} \leq 1$ is not restrictive and means that when the market is competitive, the intercept of the inverse demand curve a is superior than the production marginal cost plus the market price.
The equilibrium reached by the countries when they compete imperfectly in closed loop is much more competitive than the collusive outcome and is not Pareto-optimal (see for instance Pardalos, Migdalas & Pitsoulis (2008) for a formalization of Pareto-optimality in game theoretical models). This result is not new and is exactly the prisoner's dilemma issue. A particular country, say Country 1, knows that reducing the number of firms increases the price and the joint payoff. However, it also knows that if the opponent (Country 2) does reduce the number of its firms, a free-riding opportunity emerges since by increasing the number of Country 1's own firms, the volume increase effect will increase unilaterally its payoff and consequently decrease Country 2's one. Therefore, the closed loop Nash equilibrium cannot be Pareto-optimal.

Given the cartel outcome $(n_c, m_c) = (1, 1)$, we can easily calculate the offered volume to the consumers in this situation:

$$Q(cartel) = \frac{2(a - c_1)}{b} \frac{1}{3 + \frac{2c_2}{b}} \tag{21}$$

3.3 The open loop equilibrium

In the open loop case, the countries optimize the number of firms as well as their production at the same time.

Theorem 6. *The open loop Nash equilibrium of the game given in (15) and (16) leads to the following outcome:* $n_{ol} = N$*,* $m_{ol} = M$*.*

The corresponding consumption in the open loop game between the countries is calculated in Appendix 1:

$$Q(open\ loop) = \frac{2(a - c_1)}{3b} \frac{\left(MN + \frac{c_2}{b}(M + N)\right)}{\left(MN + \frac{4}{3}\frac{c_2}{b}(M + N) + \frac{4}{3}\frac{c_2^2}{b^2}\right)} \tag{22}$$

The open loop leads to an un-concentrated market: $n_{ol} = N$, $m_{ol} = M$. However, market power still remains important in the market. The total quantity consumed is similar to the one in the coordinated case but differs only by a term linked to the marginal cost (If we set $c_2 = 0$ then the quantities are the same). We will discuss this further in Section 3.5.

3.4 The Stackelberg equilibrium

The Stackelberg model is solved by backward induction. First we start with the follower:

Country 2 maximizes its payoff by setting the number of its exporting firms m, given the choice of the country 1 n.

$$\begin{array}{lll} \text{Country 2} & \text{Max} & \Pi_2^n(m) \\ & \text{st} & m \in \{1, 2...M\} \end{array} \tag{23}$$

Problem (23) has already been studied in Section 2 and using Theorem 1, one can already deduce that the optimum is reached for

$$m = m^*(n) = \min\left(n + 1 + \frac{2c_2}{b}, M\right)$$

The leader considers that $m = m^*(n)$ in his optimization program:

$$\begin{array}{lll} \text{Country 1} & \text{Max} & \Pi_1^s = \Pi_1^{m^*(n)}(n) = \dfrac{n}{(m^*(n) + n + 1 + \frac{2c_2}{b})^2} \dfrac{(a - c_1)^2 \left(1 + \frac{c_2}{b}\right)}{b} \\ & \text{st} & n \in \{1, 2...N\} \end{array} \tag{24}$$

To simplify the analysis, we will assume that the countries have the same number of firms: $N = M$. We will also make the previous assumption $\frac{2c_2}{b} \leq 1$. We will show that when the imperfect competition is of the Stackelberg form, the outcome is no more competitive. Figure 4 shows the evolution of the leader (Country 1)'s objective function with respect to n.

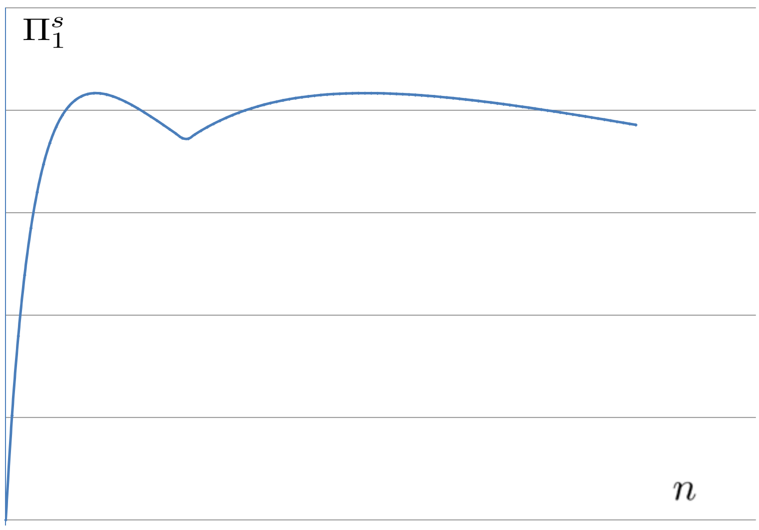

Figure 4:
The leader's objective function in the Stackelberg setting

The leader's profit function is not convex and might have multiple local optima. However, the following theorem provides a simple assumption on N (added to the assumptions $N = M$ and $\frac{2c_2}{b} \leq 1$) that ensures existence and uniqueness of the equilibrium.

Theorem 7. *If $M = N \geq 5$ and $\frac{2c_2}{b} \leq 1$, the Stackelberg game given in (23) and (24) leads to the following outcome: $n_s = 1 + \frac{2c_2}{b}$, $m_s = 2.(1 + \frac{2c_2}{b}) = 2n_s$*

Theorem 7 shows that in a Stackelberg (bi-level) game, if the countries are allowed to choose the level of competition they want in the market to optimize the payoff, the outcome is much less competitive that the closed loop Nash equilibria, but more competitive that the coordinated situation. Note that the leader is less competitive than the follower who leaves more firms to enter the market, which suggests that the leader exerts more market power than the follower.
The corresponding consumption in the Stackelberg case is given as follows:

$$Q(Stackelberg) = (n_s + m_s)\frac{(a - c_1)}{b}\frac{1}{n_s + m_s + 1 + \frac{2c_2}{b}} \tag{25}$$

3.5 Economic discussion and a numerical illustration

3.5.1 Economic implications

In order to analyze the degree of market power exercise, we compare the resulting consumption of each model, since less consumption means higher prices and more market power exercise. To benchmark our results, we also report the consumption of a monopoly in the market $Q(monopoly) = \frac{(a-c_1)}{2(b+c_2)}$, which corresponds to least competitive market situation. The comparison between the different levels of consumption is given in the following theorem:

Theorem 8. *Under our standard assumptions $\frac{2c_2}{b} \leq 1$ and $M = N \geq 5$, we have:*

$$Q(monopoly) \leq Q(cartel) \leq Q(open\ loop) \leq Q(Stackelberg) \leq Q(closed\ loop) \tag{26}$$

The demonstration of Theorem 8 is straightforward, given relations (20), (21), (22) and (25).

It comes with no surprise that the monopoly situation guaranties the highest upstream payoff and exacerbates market power. In term of market power effect, the cartel outcome comes next and differs from the monopoly by only one feature inherent to our representation: even if the countries collude, each one of them must let at least one of its firm enter the market: $n \geq 1$ and $m \geq 1$ (in other words, the coordinated solution imposes that both countries should be represented in the market). As a consequence, the cartel solution is equivalent to a Cournot duopoly with a high market concentration.

The closed loop Nash equilibrium is the most competitive and the least concentrated outcome. Furthermore, it is straightforward to notice from equation (20) that the theoretical pure and perfect competition (with quadratic cost) occurs if the number of firms $N + M$ goes toward infinity. This result is very similar to the Allaz & Vila (1993) finding, but the underlying economic reasons are completely different. In Allaz & Vila (1993), the duopolists interact on a conceptually infinite number of stages and see the reaction function of the opponent only in the spot market. In our setting, we stick to the Nash Cournot paradigm since no reaction function is anticipated. Our result is only driven by the free riding opportunity that occur when the agents play simultaneously: each player would prefer increasing his sales by allowing a little more firms to enter the market. Finally, one will conclude that the two-stage closed loop Nash Cournot model is equivalent to the Bertrand competition that leads to the same paradox: a lack of market power at the equilibrium.

The most surprising result maybe lies in the comparison between the closed and open loop cases. In open loop, the players are myopic inasmuch as they do not know the market outcome before selecting the firms. However, at the equilibrium, market power effect is stronger than in closed loop where agents are supposed to be more "clever". Both our closed loop interaction and the Allaz & Vila (1993) model share a lack of behavioral consistency: in Allaz & Vila (1993), the players see the reaction function of the opponent's contract decisions only in the spot market and besides, when the spot market is revealed, a classical Cournot game with no anticipation occurs. In our closed loop Nash equilibrium, none of the players see the reaction function of the opponent in the market.

The Stackelberg model is more consistent: the leader sees the reaction function of the follower and takes it into consideration in his profit maximization. Thus, at the equilibrium, the conjecture of the leader about the decision of the follower is consistent (this is obviously not true for the follower). We notice that the Stackelberg solution makes the market (much) more concentrated than the closed loop Nash equilibrium and the effect of market power is stronger. This last result suggests that the inconsistency of the Nash Cournot paradigm explains the paradox: when left with the possibility to exert market power, a Cournot player might prefer to be competitive.

As seen in the introduction, the Cournot paradigm has been very successful in analyzing concentrated markets in many sectors (energy, airlines, water, etc). It describes correctly a basic withholding strategy to increase prices and mimics the imperfection of the competition. However, the modeler should be cautious when it comes to analyzing the impact of market power on the profits of the agents, as these might not be consistent anymore with the assumption of market power exercise itself.

3.5.2 Illustration

To illustrate our finding, consider a situation where each country can split its exporting activity into five firms: $N = M = 5$. We give arbitrary values to the demand parameters: $a = 10$ and $b = 1$. The cost structure is such that $c_1 = 0$ and $c_2 = 0.1$. The following array summarizes the results in terms of market power exercise and market concentration.

	symbol	Monopoly	Cartel	Open loop	Stackelberg	Closed loop
Market concentration	$n+m$	1	2	10	3	10
Total supply	Q	4.5	6.2	6.5	7.1	8.9
Price	p	5.5	3.8	3.5	2.9	1.1

As a indicator for the degree of market power exercise we report the level of supply brought to the market. Not surprisingly, our numerical results are in line with Theorem 8.

As stated above, the monopoly case is the least competitive. Besides, the cartel exploits market power at nearly its maximum: the level of supply is exactly the one offered by the duopoly and the market is also very concentrated ($n + m = 2$). The closed lop Nash equilibrium is, as was noticed before, the outcome where market power is the least exploited and the market is the most concentrated ($n+m = 10$). The open loop Nash equilibrium has a strong market power exercise (very close to the cartel one). This result is quite intuitive because in open loop, the countries also operate the firms and hence will eventually end up acting as a duopoly. The fact that the market is not concentrated in open loop ($n + m = 10$) despite the strong market power exercise is simply due to the strictly convex cost structure: given a volume to be supplied, it is less costly to share it among many firms than make it produced by one firm. The Stackelberg outcome allows the countries to better exploit market power and the market is also relatively concentrated ($n + m = 3$).

3.5.3 The Stackelberg solution and symmetry of the game

In the Stackelberg game presented in Section 2.5, we had to assume that one country plays before the other. Hence the Stackelberg discussion is not in line with the assumption of symmetry of the players that is used in the rest of the paper since one player, even with the same cost functions, is a leader. Furthermore, using equations (10) and (11), one can calculate the profits at the Stackelberg equilibrium:

$$\Pi_1^s = \frac{n_s}{(m_s + n_s + 1 + \frac{2c_2}{b})^2} \frac{(a-c_1)^2 \left(1+\frac{c_2}{b}\right)}{b} \tag{27}$$

$$\Pi_2^s = \frac{m_s}{(m_s + n_s + 1 + \frac{2c_2}{b})^2} \frac{(a-c_1)^2 \left(1+\frac{c_2}{b}\right)}{b} \tag{28}$$

and since using Theorem 7, we know that $m_s = 2n_s$, and we can deduce that:

$$\Pi_2^s = 2\Pi_1^s \tag{29}$$

In other words, the follower earns twice more than the leader, which is due to the free-riding opportunity offered by the game. Having this knowledge, one should wonder whether one country would take the risk to lead the game: its sole benefit would be to avoid the closed loop Nash equilibrium paradox and create the possibility to exert market power, but at the expense of earning less than the follower. Furthermore, if both countries have the same number of firms ($N = M$), there is no economic justification to consider that a country plays before the other since they are assumed to have symmetric firms. Therefore, the Stackelberg outcome is not consistent with the symmetry and hence not likely to occur. Then, one might wonder how market power exercise can appear if the countries do not collude. We will attempt to shed some light on this question in the following section.

4 The leader-follower game between the countries

In this section we attempt to reconcile the insights we got from the Stackelberg game with the basic set-up of symmetric players. We do this by extending the game by one additional choice for the players. Each player has the possibility to choose whether he wants to be (act as) leader or follower in the game. The players choose simultaneously and the choice is discrete. Hence we might find two leaders or two followers meaning that both players might act as if they were to be leaders or followers.

The players' profits are given in equations (10) and (11) and each country has two possible choices: act as a leader or as a follower. The set of joint strategies is therefore (subscripts refer to the players):

$$S = \{(\text{leader}_1,\ \text{leader}_2),\ (\text{leader}_1,\ \text{follower}_2),\ (\text{follower}_1,\ \text{leader}_2),\ (\text{follower}_1,\ \text{follower}_2)\}$$

For the sake of simplicity, we assume that the firms face a constant marginal cost, $c_2 = 0$, and that both countries have the same number of firms $N = M$. Therefore, the game is perfectly symmetric.

The (leader$_1$, follower$_2$) and (follower$_2$, leader$_1$) strategies are typical Stackelberg situations where one country leads and the other follows. Their outcome has already been solved in Theorem 7 and is summarized as follows:

$$(\text{leader}_1,\ \text{follower}_2) \Longrightarrow (n,m) = (1,2), \quad (\Pi_1, \Pi_2) = \frac{(a-c_1)^2}{b}\left(\frac{1}{16}, \frac{1}{8}\right) \tag{30}$$

$$(\text{follower}_1,\ \text{leader}_2) \Longrightarrow (n,m) = (2,1), \quad (\Pi_1, \Pi_2) = \frac{(a-c_1)^2}{b}\left(\frac{1}{8}, \frac{1}{16}\right) \tag{31}$$

When both players are followers, they cannot anticipate the reaction function of the opponent, which implies that the outcome is exactly the closed loop Nash equilibrium calculated in Section 3.1 :

$$(\text{follower}_1,\ \text{follower}_2) \Longrightarrow (n,m) = (N,N), \quad (\Pi_1, \Pi_2) = \frac{(a-c_1)^2}{b}\left(\frac{N}{(2N+1)^2}, \frac{N}{(2N+1)^2}\right) \tag{32}$$

We focus now on the (leader$_1$, leader$_2$) situation. When both players are leaders, they jointly anticipate the reaction function of the opponent. If we write $n(m)$ and $m(n)$ the reaction functions of the first and second country respectively, we will have to solve the following game:

$$\begin{array}{lll} \text{Country 1} & \text{Max} & \Pi_1^{m(n)}(n) = \dfrac{n}{(m(n)+n+1)^2} \quad \dfrac{(a-c_1)^2}{b} \\ & \text{st} & n \in \{1, 2...N\} \end{array} \tag{33}$$

$$\begin{array}{lll} \text{Country 2} & \text{Max} & \Pi_2^{n(m)}(m) = \dfrac{m}{(n(m)+m+1)^2} \quad \dfrac{(a-c_1)^2}{b} \\ & \text{st} & m \in \{1, 2...N\} \end{array} \tag{34}$$

When writing the optimization programs of the countries (33) and (34), we have assumed that when maximizing its payoff, each country anticipates the reaction function of the rival firm. At the equilibrium, the solution is given in the following theorem:

Theorem 9. *When $c_2 = 0$ and both countries are leaders, the equilibrium is reached when $n = 1$ and $m = 1$.*

Theorem 9 implies that when both players lead in the market, they eventually end up acting as a cartel: they both concentrate at maximum the market and allow one firm to sell. It is straightforward to calculate the corresponding payoffs at the equilibrium:

$$(\text{leader}_1, \text{leader}_2) \Longrightarrow (n, m) = (1, 1), \quad (\Pi_1, \Pi_2) = \frac{(a-c_1)^2}{b}\left(\frac{1}{9}, \frac{1}{9}\right) \tag{35}$$

To sum-up, the leader-follower game between the countries can be sumarized in the standard bi-matrix of figure 5.

Country 1

Country 2	*leader$_1$*	*follower$_1$*
leader$_2$	$\frac{(a-c_1)^2}{b}\left(\frac{1}{9}, \frac{1}{9}\right)$	$\frac{(a-c_1)^2}{b}\left(\frac{1}{8}, \frac{1}{16}\right)$
follower$_2$	$\frac{(a-c_1)^2}{b}\left(\frac{1}{16}, \frac{1}{8}\right)$	$\frac{(a-c_1)^2}{b}\left(\frac{N}{(2N+1)^2}, \frac{N}{(2N+1)^2}\right)$

Figure 5:
Summary of the leader-follower game between the countries

This game has two Nash equilibria in pure strategies: (leader$_1$,follower$_2$) and (follower$_1$,leader$_2$) that have already been dismissed using a symmetry argument. To seek a symmetric outcome, let us solve it in mixed strategy: the first (second) player looks for the probability θ_1 (θ_2 respectively) to act a leader and $1-\theta_1$ ($1-\theta_2$ respectively) to act as a follower so that its expected profit is maximal, given the choice of the opponent. The decisions of the players are taken independently.

Theorem 10. *The leader-follower game has a unique symmetric solution:*

$$\theta_1 = \theta_2 = \theta_0 = \frac{\frac{1}{16} - \frac{N}{(2N+1)^2}}{\frac{11}{144} - \frac{N}{(2N+1)^2}} \tag{36}$$

The following table summarizes the issue of the leader-follower game when there is a huge number of potential firms ($N,\ M \longrightarrow \infty$):

Table 1: Results of the leader-follower game.

	(leader$_1$,leader$_2$)	(leader$_1$,follower$_2$)	(follower$_1$,leader$_2$)	(follower$_1$,follower$_2$)
Probability of occurrence (%)	67	15	15	3
Market concentration $(m+n)$	2	3	3	∞

With a 67% probability, both countries act as leaders, the market is the most concentrated and market power is exerted at its maximum level. On the contrary, the Cournot outcome is unlikely to happen (3%). Practically, our result suggests that if the countries can choose the way they behave in the market (leader or follower), market power is expected to be fully exerted (both countries acting as leaders) with a quite high probability. On the contrary, the closed loop Nash equilibrium paradox naturally disappears and will rarely be observed in the market.

5 Conclusion

The economic literature provides many ways to model imperfect competition. However, the extent to which market power should be exerted by a player is not treated in the literature, as the common belief is that if a player has the ability to manipulate prices, he will naturally do it.

In this exercise, we propose a stylized model to demonstrate that this belief is not always justified. Two countries strategically compete to choose the number of their firms allowed to sell in a market. Those firms exert market power. We show that when the countries play simultaneously and if they are symmetric enough, the closed loop Nash equilibrium is reached when the market is the most competitive (i.e. all the firms are allowed to enter the market). We argue that this result, which is similar to the Bertrand paradox, is due to the conjectural inconsistency of the Nash equilibrium. It is also similar to the Allaz & Vila (1993) result but is not driven by the same economic rationale. In our case, a free-riding opportunity emerges in the Nash Cournot solution which makes a player prefer increasing his sales instead of strongly withholding. The market is the most concentrated when the countries collude. The Stackelberg case is intermediary between the cartel and the Nash situations and the paradox disappears in it.

We also demonstrate that when the choice to act as a leader or a followers is given to the players, the closed loop Nash equilibrium is very unlikely to occur. The most probable outcome is a full exercise of market power and both countries concentrate the market at the maximum. All our results hold when the marginal production cost is constant or linear.

Future work could introduce a asymetry between the firms and find the conditions under which the competitive paradox in closed loop disappears. An interesting complex extension could model a continuous and endogenous decision of market power exercise by the players. Taking into account the cartel stability between the countries when studying their possible collusion is also an interesting development.

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Appendix 1

Proof of Theorem 1

We first prove the lemma for Country 1. m is fixed in $\mathbb{N}$ and let us study the function $\Pi_1^m(n)$ with respect to n. First we assume that $n \in \mathbb{R}^+$.
$\Pi_1^m : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is continuously differentiable with respect to n and:

$$\forall n \in \mathbb{R}^+,\ \frac{\partial \Pi_1^m}{\partial n}(n) = \frac{(a-c_1)^2}{b} \frac{(1+\frac{c_2}{b})}{\left(m+n+1+\frac{2c_2}{b}\right)^3} \left(m+1+\frac{2c_2}{b}-n\right) \tag{37}$$

Therefore, Π_1^m is upper bounded and reaches its maximum when $n^*(m) = m+1+\frac{2c_2}{b}$. Since $\frac{2c_2}{b} \in \mathbb{N}$, we can deduce that the optimum is reached on $\mathbb{N}$. In a similar way, we prove that Π_2^n reaches its maximum when $m^*(n) = n+1+\frac{2c_2}{b} \in \mathbb{N}$.
Using equation (37), we can demonstrate that $\frac{\partial \Pi_1^m}{\partial n}(n) \geq 0$ for $n \in [0, n^*(m)]$ and $\frac{\partial \Pi_1^m}{\partial n}(n) \leq 0$ for $n \in [n^*(m), \infty)$. Since $n^*(m) \geq 1$, it is easy to see that on $\{1, 2...N\}$, if $n^*(m) \leq N$, Π_1^m is optimal when $n = n^*(m)$ and if $n^*(m) > N$, Π_1^m is optimal when $n = N$. The same reasoning can be done for Π_2^n.

To simplify the notation, we will now call $n^*(m) = \min\left(1+m+\frac{2c_2}{b}, N\right)$ and $m^*(n) = \min\left(1+n+\frac{2c_2}{b}, M\right)$.

Proof of Theorem 2

If it exists, we will denote by (n_{cl}, m_{cl}) the closed loop Nash equilibrium. If we assume that the equilibrium is not extremal, we would have:

$$n_{cl} < N \tag{38}$$
$$m_{cl} < M \tag{39}$$

By definition of the equilibrium, $\Pi_1^{m_{cl}}$ is optimal for n_{cl}, which means that $n_{cl} \geq 1$. Since $n_{cl} < N$, we would have $n_{cl} = 1 + m_{cl} + \frac{2c_2}{b}$. The reasoning is symmetric for Country 2, which gives:

$$n_{cl} = 1 + m_{cl} + \frac{2c_2}{b} \tag{40}$$
$$m_{cl} = 1 + n_{cl} + \frac{2c_2}{b} \tag{41}$$

which is absurd. Therefore the equilibrium is extremal.

Proof of Theorem 3

We prove now that there exists a unique closed loop Nash equilibrium. By definition, the equilibrium is a couple (n_{cl}, m_{cl}) such that $n_{cl} = n^*(m_{cl})$ and $m_{cl} = m^*(n_{cl})$.
First we assume its existence (n_{cl}, m_{cl}). Let us study two cases:

- We assume that $N > M$.
  Two sub-cases are to be considered:
  1). $N > M + 1 + \frac{2c_2}{b}$.
  Since the equilibrium is extremal, we must have either $n_{cl} = N$ or $m_{cl} = M$. If $n_{cl} = N$, this means, given Theorem 1 that $N \leq m_{cl} + 1 + \frac{2c_2}{b}$. In that case, if $m_{cl} < M$, we would have $M > m_{cl} = n_{cl} + 1 + \frac{2c_2}{b} = N + 1 + \frac{2c_2}{b}$, which is absurd given that $N > M$. Hence, $m_{cl} = M$ and this contradicts the fact that $N \leq m_{cl} + 1 + \frac{2c_2}{b}$. Therefore $n_{cl} < N$ and since the equilibrium is extremal, we need to have $m_{cl} = M$. Then $n_{cl} = M + 1 + \frac{2c_2}{b} < N$. To sum up, we must have $(n_{cl}, m_{cl}) = (M + 1 + \frac{2c_2}{b}, M)$.

Finally we easily check that the point obtained $(n_{cl}, m_{cl}) = (M + 1 + \frac{2c_2}{b}, M)$ is indeed a Nash equilibrium.
2). $N \leq M + 1 + \frac{2c_2}{b}$.
Since the equilibrium is extremal, we must have either $n_{cl} = N$ or $m_{cl} = M$. If $m_{cl} = M$, this means, given Theorem 1 that $M \leq n_{cl} + 1 + \frac{2c_2}{b}$. In that case, if $n_{cl} < N$, we would have $N > n_{cl} = m_{cl} + 1 + \frac{2c_2}{b} = M + 1 + \frac{2c_2}{b}$, which is absurd given that $N \leq M+1+\frac{2c_2}{b}$. Hence, $n_{cl} = N$. To sum up, we must have $(n_{cl}, m_{cl}) = (N, M)$. Finally we easily check that the point obtained $(n_{cl}, m_{cl}) = (N, M)$ is indeed a Nash equilibrium.

- We assume that $N < M$.
  By symmetry it is straightforward to demonstrate that the equilibrium is:
  1). If $M > N + 1 + \frac{2c_2}{b}$, $(n_{cl}, m_{cl}) = (N, N + 1 + \frac{2c_2}{b})$.
  2). If $M \leq N + 1 + \frac{2c_2}{b}$, $(n_{cl}, m_{cl}) = (N, M)$.

- We assume that $N = M$.
  The equilibrium is $(n_{cl}, m_{cl}) = (N, M)$.

Proof of Theorem 4

Under assumption (19), the proof of Theorem 3 indicates that the closed loop Nash equilibrium is $(n_{cl}, m_{cl}) = (N, M)$. Let us prove that this outcome maximizes the social welfare. The social welfare $W(n, m)$ is the sum of the firms' profits and the consumer's surplus:

$$W(n,m) = aQ^{n,m} - \frac{b}{2}(Q^{n,m})^2 - c_1 \left(\sum_i x_i + \sum_j y_j \right) - c_2 \left(\sum_i x_i^2 + \sum_j y_j^2 \right) \tag{42}$$

After some algebra, we get:

$$W(n,m) = \frac{(a-c_1)^2}{b} \left(\frac{m+n}{m+n+1+\frac{2c_2}{b}} - \frac{1}{2}\frac{(m+n)^2}{(m+n+1+\frac{2c_2}{b})^2} - \frac{c_2}{b}\frac{m+n}{(m+n+1+\frac{2c_2}{b})^2} \right) \tag{43}$$

$W(n, m)$ depends on the total number of firms $m + n$. Let us calculate the optimum of $W(n, m)$ over $\{1, 2...N\} \times \{1, 2...M\}$. As done before, we consider $W(n, m)$ as a function of $x = n + m$ and we first optimize it on $\mathbb{R}^+$. After some calculations, we get:

$$\forall x \in \mathbb{R}^+,\ \frac{\partial W}{\partial x}(x) = \frac{(a-c_1)^2}{b(x+1+\frac{2c_2}{b})^3} \left((1+\frac{2c_2}{b})(1+\frac{c_2}{b}) + \frac{c_2}{b}x \right) \geq 0 \tag{44}$$

Hence, the social welfare increases with the number of firms, which is quite intuitive since by increasing the number of firms, we get closer to the pure and perfect competition. Therefore, we can deduce that the situation $(n, m) = (n_{cl}, m_{cl}) = (N, M)$ optimizes the social welfare in the game between the countries.

Proof of Theorem 5

In this theorem, it is assumed that the countries collude. This means that they cooperate to optimize their common profit $\Pi = \Pi_1^m + \Pi_1^n$. This ends up to solving:

$$\begin{array}{ll} \text{Max} & \Pi = \Pi_1^m + \Pi_2^n = \frac{(a-c_1)^2}{b}\left(1+\frac{c_2}{b}\right)\frac{m+n}{\left(m+n+1+\frac{2c_2}{b}\right)^2} \\ \text{st} & n \in \{1, 2...N\}, m \in \{1, 2...M\} \end{array} \tag{45}$$

First we study Π as a function of $x = m + n$ over $\mathbb{R}^+$. Π is continuously differentiable and:

$$\forall x \in \mathbb{R}^+,\ \frac{\partial \Pi}{\partial x}(x) = \frac{(a-c_1)^2}{b}\left(1+\frac{c_2}{b}\right)\frac{1+\frac{2c_2}{b}-x}{\left(x+1+\frac{2c_2}{b}\right)^3} \tag{46}$$

Therefore, the total profit Π decreases on $[1+\frac{2c_2}{b}, +\infty[$. Since $\frac{2c_2}{b} \leq 1$, we have $0 \leq 1+\frac{2c_2}{b} \leq 2$ and we can deduce that the maximization program of the cartel (45) reaches its optimum when $x = m+n = 2$, which implies that $m = n = 1$.

Since the sum of the countries' profits has been maximized, we necessarily reach the Pareto-optimal point of the game between the players.

Proof of Theorem 6

We first concentrate on Country 1's optimization program in open loop given in (15). The decisions of the other country J and y_j are exogenous.
When I is fixed (I has n elements), the optimality conditions with respect to x_i give:

$$\forall i \in I,\ a - c_1 - b\sum_j y_j - 2b\sum_i x_i - 2c_2 x_i = 0 \tag{47}$$

and by summing over $i \in I$, we obtain:

$$n(a - c_1 - b\sum_j y_j) - 2(bn + c_2)\sum_i x_i = 0 \tag{48}$$

or

$$\sum_i x_i = \frac{n(a - c_1 - b\sum_j y_j)}{2(bn + c_2)} \tag{49}$$

Plugging (49) in (47), we get:

$$\forall i \in I,\ x_i = \frac{a - c_1 - b\sum_j y_j}{2(bn + c_2)} \tag{50}$$

and we can calculate the optimal profit of Country 1 (given I, J and y_j) as follows:

$$\Pi_1^{ol}(I) = \Pi_1^{ol}(n) = \frac{(a - c_1 - b\sum_j y_j)^2}{4}\frac{n}{bn + c_2} \tag{51}$$

Π_1^{ol} increases with n and is therefore optimal when $n = N$. The same reasoning holds for the other player (Country 2).
The open loop Nash equilibrium is therefore $(n_{ol}, m_{ol}) = (N, M)$.
If we denote by Q_1 (respectively Q_2) the total volume brought by the firms belonging to Country 1 (respectively Country 2) at the equilibrium, equation (49) gives:

$$Q_1 = \frac{N(a - c_1 - bQ_2)}{2(bN + c_2)} \tag{52}$$

and symmetrically, we obtain:

$$Q_2 = \frac{M(a - c_1 - bQ_1)}{2(bM + c_2)} \tag{53}$$

Equations (52) and (53) constitute a system of two equations and variables that can be solved easily to calculate the total consumption:

$$Q(open\ loop) = Q_1 + Q_2 = \frac{2(a-c_1)}{3b}\frac{MN + \frac{c_2}{b}(M+N)}{MN + \frac{4}{3}\frac{c_2}{b}(M+N) + \frac{4}{3}\frac{c_2^2}{b^2}} \tag{54}$$

Proof of Theorem 7

We already know the reaction function of Country 2 (the follower):

$$m^*(n) = \min(n + 1 + \frac{2c_2}{b}, M) \tag{55}$$

Therefore, the profit of Country 1 (the leader) can be explicitly written as follows:

$$\Pi_1^s(n) = \begin{cases} \dfrac{n}{4(n+1+\frac{2c_2}{b})^2} \dfrac{(a-c_1)^2\left(1+\frac{c_2}{b}\right)}{b} & \text{if } 0 \le n \le M - 1 - \dfrac{2c_2}{b} \quad (56) \\ \dfrac{n}{(M+n+1+\frac{2c_2}{b})^2} \dfrac{(a-c_1)^2\left(1+\frac{c_2}{b}\right)}{b} & \text{if } n \ge M - 1 - \dfrac{2c_2}{b} \quad (57) \end{cases}$$

We will denote by κ the constant $\frac{(a-c_1)^2\left(1+\frac{c_2}{b}\right)}{b}$. It is easy to show that over $[0, M-1-\frac{2c_2}{b}]$, Π_1^s is optimal when $n = n_1 = 1 + \frac{2c_2}{b}$. Over $[M-1-\frac{2c_2}{b}, M+1+\frac{2c_2}{b}]$, Π_1^s increases. Since $N \in [M-1-\frac{2c_2}{b}, M+1+\frac{2c_2}{b}]$ (we recall that $N = M$), we deduce that over $[M-1-\frac{2c_2}{b}, N]$, Π_1^s is maximal when $n = n_2 = N$.
Now, to optimize Π_1^s over $\{1, 2, ...N\}$ we need to compare between $\Pi_1^s(n_1)$ and $\Pi_1^s(n_2)$. We denote by $\alpha = 1 + \frac{2c_2}{b}$ and calculate:

$$\Pi_1^s(n_1) = \frac{\kappa}{16\alpha} \tag{58}$$

$$\Pi_1^s(n_2) = \frac{\kappa N}{(2N+\alpha)^2} \tag{59}$$

After some basic algebra, we deduce that $N \ge (\frac{3}{2} + \sqrt{2})\alpha \implies \Pi_1^s(n_1) \ge \Pi_1^s(n_2)$. Since it has been assumed that $N \ge 5$ and because $\alpha \le 2$ (which is due to the fact that $\frac{2c_2}{b} \le 1$), we deduce that $\Pi_1^s(n_1) \ge \Pi_1^s(n_2)$ and Country 1 should choose $n_s = n_1 = 1 + \frac{2c_2}{b}$. Given the reaction function of the follower, we have $m_s = n_s + 1 + \frac{2c_2}{b} = 2n_s$.

Proof of Theorem 9

Writing the KKT conditions of the optimization program of Country 1 (33), one gets:

$$m(n) + n + 1 = 2n\left(1 + \frac{\partial m}{\partial n}\right) \tag{60}$$

or

$$m(n) - n + 1 = 2n\frac{\partial m}{\partial n} \tag{61}$$

By doing the same calculation for the optimization program of Country 2 (34), one gets:

$$n(m) - m + 1 = 2m\frac{\partial n}{\partial m} \tag{62}$$

By differentiating relation (61) with respect to variable m and keeping in mind that:

$$\frac{\partial^2 m}{\partial m \partial n} = \frac{\partial^2 m}{\partial n \partial m} = \frac{\partial 1}{\partial n} = 0$$

one obtains the following relation:

$$1 - \frac{\partial n}{\partial m} = 2\frac{\partial n}{\partial m}\frac{\partial m}{\partial n} \tag{63}$$

By differentiating relation (62) with respect to variable n and keeping in mind that:

$$\frac{\partial^2 n}{\partial n \partial m} = \frac{\partial^2 n}{\partial m \partial n} = \frac{\partial 1}{\partial m} = 0$$

one obtains the following relation:

$$1 - \frac{\partial m}{\partial n} = 2\frac{\partial n}{\partial m}\frac{\partial m}{\partial n} \tag{64}$$

Combining (63) and (64), we get:

$$\frac{\partial m}{\partial n} = \frac{\partial n}{\partial m} \tag{65}$$

Plugging relation (65) into equation (63), we obtain:

$$1 - \frac{\partial n}{\partial m} = 2\left(\frac{\partial n}{\partial m}\right)^2 \tag{66}$$

In other words, $\frac{\partial n}{\partial m}$ and $\frac{\partial m}{\partial n}$ are solutions to the degree two polynomial equation:

$$2X^2 + X - 1 = 0$$

that has two solutions: $\frac{1}{2}$ and -1.

- If $\frac{\partial n}{\partial m} = \frac{\partial m}{\partial n} = -1$. Using equations (61) and (62), we have:

$$m + n + 1 = 0 \tag{67}$$

  which is impossible since $n \geq 0$ and $m \geq 0$.
- If $\frac{\partial n}{\partial m} = \frac{\partial m}{\partial n} = \frac{1}{2}$. Using equations (61) and (62), we deduce that:

$$m = n = 1 \tag{68}$$

Proof of Theorem 10

Since the players chose independently to act as a leader of follower, one can calculate the probability distribution of the profit of the first country:

- (leader$_1$,leader$_2$) leads to a profit $\Pi_1 = \frac{(a-c_1)^2}{b}\frac{1}{9}$ with a probability $\theta_1\theta_2$.
- (leader$_1$,follower$_2$) leads to a profit $\Pi_1 = \frac{(a-c_1)^2}{b}\frac{1}{16}$ with a probability $\theta_1(1-\theta_2)$.
- (follower$_1$,leader$_2$) leads to a profit $\Pi_1 = \frac{(a-c_1)^2}{b}\frac{1}{8}$ with a probability $(1-\theta_1)\theta_2$.
- (follower$_1$,follower$_2$) leads to a profit $\Pi_1 = \frac{(a-c_1)^2}{b}\frac{N}{(2N+1)^2}$ with a probability $(1-\theta_1)(1-\theta_2)$.

Country 1's expected profit is then:

$$\langle\Pi_1\rangle = \frac{(a-c_1)^2}{b}\left(\theta_1\theta_2\frac{1}{9} + \theta_1(1-\theta_2)\frac{1}{16} + (1-\theta_1)\theta_2\frac{1}{8} + (1-\theta_1)(1-\theta_2)\frac{N}{(2N+1)^2}\right) \tag{69}$$

In mixed strategies, Country 1 now faces the following optimization program:

$$\begin{array}{lll} \text{Country 1} & \text{Max} & \langle\Pi_1\rangle \\ & \text{st} & \theta_1 \in [0,1] \end{array} \tag{70}$$

Constraint $\theta_1 \in [0,1]$ simply means that θ_1 is a probability.
Problem (70) is a linear optimization problem (LP) that can be solved algebraically. If we denote by

$$\theta_0 = \frac{\frac{1}{16} - \frac{N}{(2N+1)^2}}{\frac{11}{144} - \frac{N}{(2N+1)^2}}$$

its solution $\theta_1^*(\theta_2)$ is as follows:

$$\theta_2 < \theta_0 \Longrightarrow \theta_1^*(\theta_2) = 1$$

$$\theta_2 > \theta_0 \Longrightarrow \theta_1^*(\theta_2) = 0$$
$$\theta_2 = \theta_0 \Longrightarrow \theta_1^*(\theta_2) \in [0,1]$$

Similarly, the previous calculation can be done for the Country 2 and we obtain the following:

$$\theta_1 < \theta_0 \Longrightarrow \theta_2^*(\theta_1) = 1$$
$$\theta_1 > \theta_0 \Longrightarrow \theta_2^*(\theta_1) = 0$$
$$\theta_1 = \theta_0 \Longrightarrow \theta_2^*(\theta_1) \in [0,1]$$

The equilibrium is reached for (θ_1, θ_2) when $\theta_1^*(\theta_2) = \theta_1$ and $\theta_2^*(\theta_1) = \theta_2$. We then have three solutions:

- $\theta_1^* = 0$ and $\theta_2^* = 1$. Rejected because not symmetric.
- $\theta_1^* = 1$ and $\theta_2^* = 0$. Rejected because not symmetric.
- $\theta_1^* = \theta_0$ and $\theta_2^* = \theta_0$. Accepted because symmetric.